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A Massive Scalar Field as a Source of Gravity

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Abstract: We suggest a power series method of solution of Einstein's equations including the cosmological term when a neutral massive scalar field is the only external source of gravity.

1. INTRODUCTION

General Relativity establishes that in the presence of a gravitational field there are irnportant changes in the description of physical phenomena. Particularly. in this theory all the material fields are considered as sources of gravity and, following the general covariance principle. the variables of the induced gravitational fields have to appear in the equations of the material fields themselves. resulting in some modifications in the behavior of the sources. We should note that this coupling between gravity and its sources usually implies a difficult handling of mathematical problems due to the highly non-linear character of Einstein's equations.

One of the examples of gravity-source coupling found in the literature of General Relativity concerns a neutral scalar field acting as the external source of gravitation. When the scalar field is massless, the arising system of coupled equations allows some analytical handling of possible solutions (see Frøyland[1] for important references on this subject). Nevertheless, when the scalar field has a non-zero mass the above mentioned analytical approaches collapse. In this case the complication of the coupled equations suggests the necessity of different techniques for the study of possible solutions.

In this paper we consider the coupling of a scalar neutral massive field with gravity. For this purpose we develop a power series method of solution for the coupled equations. A similar methodology was first used by Wyman[2] in a particular case of a massless scalar field coupled to gravity. More recently, Varela[3] has shown the natural extensions of the Wyman's methodology to the most general case of Einstein's equations, including the cosmological term.

The plan of this work is as follows:

In Section 2 we formulate the field equations for the coupled systern. In Section 3 we search for Schwartzschild-like solutions for the system of coupled equations. For this purpose we develop a power series method and discuss some important aspects of the results. Some comments on the results are presented in Section 4.

2. THE FIELD EQUATIONS

In Einstein's theory of gravitation the metric tensor $g_{\mu\nu}$ may be obtained in principle as the solution of the field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \tag{2.1}$$

where $R_{\mu\nu}$ denotes the Ricci tensor: R. the curvature scalar, is the trace of $R_{\mu\nu}$. The cosmological constant Λ has a non-zero value in the most general case of the theory. κ denotes the gravitational coupling constant and $T_{\mu\nu}$ is the energy-momentum tensor representing the external sources of gravitation.

Now we suppose ϕ (the neutral massive scalar field) is the only external source of the gravitational field. Then $T_{\mu\nu}$ depends only on the variables associated to the field ϕ. This tensor. minimally coupled to gravity, is defined as

$$\kappa T_{\mu\nu} = \Omega[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}(g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2)] \tag{2.2}$$

Here a comma (,) denotes partial derivatives. m is the mass of the field and Ω is a positive constant. The field ϕ satisfies the minimally coupled equation of motion given by

$$\phi_{;\alpha}^{;\alpha} + m^2\phi = 0 \tag{2.3}$$

where semicolon (;) denotes a covariant derivative.

From the field eq. (2.1) we can express R in terms of Λ and T^{α}_{α}, the trace of $T_{\mu\nu}$. Eq.(2.2) allows T^{α}_{α} to be expressed as a function of ϕ, its mass and derivatives. Introducing this result in eq.(2.1) we rewrite Einstein's equations in the equivalent form

$$R_{\mu\nu} = -\Omega\phi_{,\mu}\phi_{,\nu} + \frac{\Omega}{2} g_{\mu\nu} m^2\phi^2 + \Lambda g_{\mu\nu} \tag{2.4}$$

In Section 3 we face the problern of solving the coupled differential eqs.(2.3) and (2.4). For this purpose we impose suitable simplifying conditions on the field ϕ and on the components of the metric tensor.,g

3. SOLUTION OF THE EQUATIONS

Let us consider the static case with spherical symmetry. Then the Schwartzschild-like line element. with signature (+ - - -), is given by

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Sigma^2 \tag{3.1}$$

where

$$d\Sigma^2 = d\theta^2 + sin^2\theta d^2\varphi \tag{3.2}$$

both v and λ being only functions of r. Here we consider 4 depending at most on r. Let ϕ' denote the derivative of ϕ with respect to r. Then $\phi_{,\mu}$ takes the form

$$\phi_{,\mu} = (0, \phi', 0, 0) \tag{3.3}$$

Under our symmetry assurnptions. the only non-null cornponents of R,, are[4]:

$$\begin{aligned} R_{00} &= -e^{\nu-\lambda} Big[\frac{\nu''}{2} - \frac{v'\lambda'}{4} + \frac{v'^2}{4} + \frac{v'}{r}\Big] \\ R_{11} &= \frac{\nu''}{2} - \frac{v'\lambda'}{4} + \frac{v'^2}{4} - \frac{\lambda'}{r} \\ R_{22} &= e^{-\lambda}\left[1 + \frac{v'r}{2} - \frac{\lambda' r}{2}\right] - 1 \\ R_{33} &= R_{22}\, sin^2\theta \end{aligned} \tag{3.4}$$

Frorn the known structure of R,, and the definitions of $\phi_{,\mu}$ and,g given in eqs. (3.3) and (3.1). respectively. we obtain the nontrivial field equations

$$-e^{\nu-\lambda}\left[\frac{\nu''}{2} - \frac{v'\lambda'}{4} + \frac{v'^2}{4} + \frac{v'}{r}\right] = \frac{\Omega}{2} m^2 e^{\nu} \phi^2 + \Lambda e^{\nu} \tag{3.5}$$

$$\frac{\nu''}{2} - \frac{v'\lambda'}{4} + \frac{v'^2}{4} - \frac{\lambda'}{r} = -\frac{\Omega}{2} m^2 e^{\lambda} \phi^2 - \Lambda e^{\lambda} - \Omega \phi'^2 \tag{3.6}$$

$$e^{-\lambda}\left[1+\frac{v'r}{2}-\frac{\lambda' r}{2}\right]-1=-\frac{\Omega}{2}m^2r^2\phi^2-\Lambda r^2 \qquad (3.7)$$

The covariant **d'Alambertian** in **eq.(2.3)** is expressed as

$$\phi^{,\alpha}{}_{;\alpha}=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\sqrt{-g}\frac{\partial\phi}{\partial x^\beta}\right) \qquad (3.8)$$

where g is the deterrninant of the rnetric $g_{\mu\nu}$. Then. considering eqs. (3.1) and (3.3). **eq.(2.3)** can be put in tke forrn

$$\phi''+\frac{2}{r}\phi'+\frac{v'-A'}{2}\phi'=m^2e^\lambda\phi \qquad (3.9)$$

where ϕ'' denotes the second derivative of ϕ with respect to r. We **should** note that although **eq.(3.9)** is derived for points placed out of the origin of coordinates $(r \neq 0)$. it rnay give us valuable inforrnation about the solutions at $r=0$ by **using** a suitable lirniting process.

Eqs. (3.5) to (3.7) imply

$$v'+\lambda'=\Omega\phi'^2r \qquad (3.10)$$

$$v'-\lambda'=2\{\{1-(\frac{\Omega}{2}m^2\phi^2+\Lambda)r^2\}e^\lambda-1\}/r \qquad (3.11)$$

When we develop the power series method of solution. the basic fact to consider is that in the absence of **explicit** solutions for $g\mu\nu$ and 4 in this systern of coordinates. the **coupled eqs.** (3.9). (3.10) and (3.11) **allow** the **calculation** of the Taylor expansions for **the** solutions at any point of space. **In** particular. we shall develop the procedure **at** $r=0$.

In order to guarantee a regular behavior of the solutions at $r=0$ we shall impose $\lambda(0)=0$, $\phi(0)=\beta$ and $\phi'(0)=0$, where β **is** an arbitrary real number. **In** particular. let $v(0)=0$. Under these conditisns. eqs. (3.10) and (3.11) **become**

$$v'(0)+\lambda'(0)=0 \qquad (3.12)$$

$$v'(0)-\lambda'(0)=\lim_{r\to 0}\left[2\frac{\{1-(\frac{\Omega}{2}m^2\phi^2+\Lambda)r^2\}e^\lambda-1}{r}\right]=2X^1(0) \qquad (3.13)$$

Solving eqs. (3.12) and (3.13) for $\nu'(0)$ and $\lambda'(0)$ we obtain

$$v'(0) = 0 \tag{3.14}$$

$$\lambda'(0) = 0 \tag{3.15}$$

Taking eq.(3.9) to the limit $r \rightarrow 0$ and recalling the conditions imposed on $\lambda(0), \lambda'(0), \phi(0)$ and $\phi'(0)$, we find

$$\phi''(0) = \frac{1}{3} m^2 \beta \tag{3.16}$$

Differentiating eqs.(3.10) and (3.11) with respect to r we find, in the limit $r \rightarrow 0$, the equations for $\nu''(0)$ and $\lambda''(0)$. From the values of the functions, their first derivatives at $r = 0$ and eq.(3.16), we obtain

$$\nu''(0) + \lambda''(0) = 0 \tag{3.17}$$

$$\nu''(0) - \lambda''(0) = \lambda''(0) - \Omega m^2 \beta^2 - 2\Lambda \tag{3.18}$$

Then, the values of $\nu''(0)$ and $\lambda''(0)$ are given by

$$v''(0) = -\frac{1}{3}\Omega m^2 \beta^2 - \frac{2}{3}\Lambda \tag{3.19}$$

$$\lambda''(0) = \frac{1}{3}\Omega m^2 \beta^2 - \frac{2}{3}\Lambda \tag{3.20}$$

If we differentiate eq.(3.9) with respect to r and evaluate the resulting expression in the limit $r \rightarrow 0$ we find

$$\phi'''(0) = 0 \tag{3.21}$$

In the limit $r \rightarrow 0$, the second derivatives of eqs.(3.10) and (3.11) lead to the equations for $v'''(0)$ and $\lambda'''(0)$, resulting, after some manipulations,

$$v'''(0) + \lambda'''(0) = 0 \tag{3.22}$$

$$v'''(0) - \lambda'''(0) = \frac{2}{3}\lambda'''(0) \qquad (3.23)$$

From eqs. *(3.22)* and *(3.23)* we obtain

$$v'''(0) = 0 \qquad (3.24)$$

$$\lambda'''(0) = 0 \qquad (3.25)$$

Taking the second derivative of eq.(3.9) to the limit $r \rightarrow 0$. we find the value of the fourth derivative of $\phi(r)$ at $r = 0$. The result is:

$$\phi^{(4)}(0) = \frac{1}{3}\Omega m^4\beta^3 + \frac{1}{5}m^4\beta + \frac{2}{3}\Lambda m^2\beta \qquad (3.26)$$

Finally, evaluating the third derivatives of eqs. *(3.10)* and *(3.11)* in the limit $r \rightarrow 0$ we obtain. after some involved algebra. the system of equations whose solutions are $v^{(4)}(0)$ and $\lambda^{(4)}(0)$. This is given by

$$v^{(4)}(0) + \lambda^{(4)}(0) = \frac{2}{3}\Omega m^4\beta^2 \qquad (3.27)$$

$$v^{(4)}(0) - \lambda^{(4)}(0) = \frac{1}{2}\lambda^{(4)}(0) - \frac{5}{6}\Omega^2 m^4\beta^4 - \frac{10}{3}\Omega\Lambda m^2\beta^2 - \frac{10}{3}\Lambda^2 - 2\Omega m^4\beta^2 \qquad (3.28)$$

Solving eqs. *(3.27)* and *(3.28)* we find

$$v^{(4)}(0) = -\frac{2}{5}\Omega m^4\beta^2 - \frac{1}{3}\Omega^2 m^4\beta^4 - \frac{4}{3}\Omega\Lambda m^2\beta^2 - \frac{4}{3}\Lambda^2 \qquad (3.29)$$

$$\lambda^{(4)}(0) = \frac{16}{15}\Omega m^4\beta^2 + \frac{1}{3}\Omega^2 m^4\beta^4 + \frac{4}{3}\Omega\Lambda m^2\beta^2 + \frac{4}{3}\Lambda^2 \qquad (3.30)$$

From eqs. *(3.16).* (3.21) and *(3.26).* and recalling that $\phi(0) = \beta$ and $\phi'(0) = 0$. we can develop the first terms of the Taylor expansion for $\phi(r)$. around $r = 0$. We obtain

$$\phi(r) = \beta + \frac{1}{6}m^2\beta r^2 + \frac{1}{24}\left[\frac{1}{3}\Omega m^4\beta^3 + \frac{1}{5}m^4\beta + \frac{2}{3}\Lambda m^2\beta\right]r^4 + \dots \tag{3.31}$$

Analogously, our results concerning the values of $v(r)$, $\lambda(r)$ and their **first derivatives** enable **us** to express the first terms of the Taylor expansions for e^{ν} and e^{λ} around $r = 0$. These terms are given by

$$e^{\nu} = 1 - \frac{1}{6}[\Omega m^2\beta^2 + 2\Lambda]r^2 - \frac{1}{60}\Omega m^4\beta^2 r^4 + \dots \tag{3.32}$$

$$e^{\lambda} = 1 + \frac{1}{6}\left[\Omega m^2\beta^2 + 2\Lambda\right]r^2 + \frac{1}{24}\left[\frac{16}{15}\Omega m^4\beta^2 + \frac{2}{3}\Omega^2 m^4\beta^4 + \frac{8}{3}\Omega\Lambda m^2\beta^2 + \frac{8}{3}\Lambda^2\right]r^4 + \dots \tag{3.33}$$

Expansions *(3.32)* and *(3.33)* **provide us** with some degree of information about the metric structure of space-time. **If** we restrict our attention to **small** r, these equations enable **us** to make approximate predictions on the behavior of clocks and measuring rods and the motion of single test particles. However, the calculation of the tidal forces acting on systems of test particles **may** only be achieved when the components of the Riemann tensor are known. The next step is to show how our previous results make the derivation of the curvature at $r = 0$ a straightforward matter.

Within the framework of the assumed space-time symmetries and considering the intrinsic symmetries of the Riemann tensor it can be shown that **all** its **non-vanishing** components may be computed from the knowledge of only six of **them**. These are given by[4]

$$\begin{aligned}
R^0_{101} &= -\frac{1}{2}\nu'' + \frac{1}{4}\lambda' v' - \frac{1}{4}\nu'^2 \\
R^0_{202} &= -\frac{1}{2}re^{-\lambda}v' \\
R^0_{303} &= -\frac{1}{2}re^{-\lambda}v' \sin^2\theta \\
R^1_{212} &= \frac{1}{2}re^{-\lambda}\lambda' \\
R^1_{313} &= \frac{1}{2}re^{-\lambda}\lambda' \sin^2\theta \\
R^2_{323} &= (1 - e^{-\lambda})\ \sin^2\theta
\end{aligned} \tag{3.34}$$

Then if we introduce in eq.(3.34) the values of λ and v at $\mathbf{r}=0$, and those derived for $\nu'(0)$, $\lambda'(0)$ and $\nu''(0)$, as given by eqs. (3.14), (3.15) and (3.19), respectively, we see that the only independent component that does not vanish is R^0_{101}, the value of which is found to be

$$R^0_{101}=\frac{1}{6}\Omega m^2\beta^2+\frac{1}{3}\Lambda \tag{3.35}$$

Then. in the particular case when the cosmological constant takes the negative value given by

$$\Lambda=-\frac{1}{2}\Omega m^2\beta^2 \tag{3.36}$$

the value of R^0_{101} also vanishes and the space-time manifold may be considered flat at those points sufficiently close to $\mathbf{r}=0$. This in turn implies the vanishing of every locally mearurable effect of gravity (including tidal forces) at $\mathbf{r}=0$. Furthermore. since in this case the Einstein tensor also vanishes. the condition given by eq.(3.36) may be associated to an *effective vacuum* at $\mathbf{r}=0$.

4. CONCLUSION

We have proposed a power series method of solution of the field equations with cosmological term and a massive scalar source. The derivation of the first four terms of the Taylor expansions about $\mathbf{r}=0$ of the metric and the scalar field has been shown explicitly. Also. we have calcuiated the componentes of the curvature tensor at $\mathbf{r}=0$ and discussed some implications in a particular case. A further development would be to introduce a similar technique in the study of the asyrnptotic behavior of the solutions.

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Resumo

Sugerimos um método de solução em série de potências para as equações de Einstein incluindo o termo cosmológico, quando um campo escalar massivo neutro é q única fonte externa de gravidade.