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Curvature Copies in Metric and Semi-Metric Manifolds

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Abstract We present examples of curvature copies in metric and semi-metric spaces. These mappings involve geometrical objects of the manifolds and have some physical implications in the context of gravitational theories.

I. INTRODUCTION

In gravitational theories based on curved space-time the field equations are obtained from invariants of the curvature tensor and/or its contractions (Ricci tensor and scalar curvature). The usual arena for these theories is a metric 4-dimensional manifold provided with a linear connection (Riemann-Cartan space U_4) which is called Riemann space (V_4) when the ccnnection is symmetric. In this latter case the most remarkable theory of gravitation is, of course. General Relativity.

An interesting question concerning these geometrical structures is the possibility of the existence of different connections corresponding to the same curvature tensor. Thus different connections can generate the same curvature invariants which determine the field equations. Connections with this property will be called curvature copies here. in analogy with the field strength copies in non-abelian gauge theories[1]. Field copies have been studied also for the case of internal curvatures related to internal gauge transformations in the framework of Einstein theory of gravitation[2].

The investigation of non-metric space-times has attracted less attention in recent years, since theories based on such manifolds usually reveal scarce physical meaning. In spite of this. some aspects of invariance properties and curvature copies in non-metric spaces seem to deserve consideration. The projective, or Einstein λ-transformation[3], for instance. can be used in the construction of metrizable theories of gravitation[4]. or can be theories of gravitation[4], or can be associated

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with gauge mappings for Yang-Mills fields[6]. The relation between projective mappings and conservation laws in quantum physics[e] has benn suggested as well.

In this note we shall present some new examples of copies in the context of metric and semi-metric n-dimensional spoce-times. The applications to 4-dimensional gravity may be specially interesting. We shall follow here the definitions and notations of ref.7.

2. PRELIMINARIES

Let us considera general n-dimensional manifold L_n in which exist: a rotation curvature. a torsion field and a curvature of segmentation. Let $\Gamma^{\gamma}_{\alpha\beta}$ be the affine connection of L_n. By adding a tensor $P_{\alpha\beta.}{}^{\gamma}$ to $\Gamma^{\gamma}_{\alpha\beta}$ we obtain an object that transforms like a connection. This new connection

$$\bar{\Gamma}^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta} + P_{\alpha\beta.}^{\gamma} \tag{1}$$

belongs to a manifold $\bar{L}_n$. From $\bar{\Gamma}_{\alpha\beta.}{}^{\gamma}$ we derive the curvature $\bar{R}_{\alpha\beta\gamma.}{}^{\delta}$ of $\bar{L}_n$. Thus we can impose the condition of invariance of curvature'.

$$\bar{R}_{\alpha\beta\gamma.}{}^{\delta}(\bar{\Gamma}) = R_{\alpha\beta\gamma.}{}^{\delta}(\Gamma) \tag{2}$$

for $\bar{L}_n$ and L_n.

Def. 2.1 A set of connections in $\bar{L}_n$ constructed by the addition of a tensor of third rank $P_{\alpha\beta}^{\gamma}$ to a connection $\Gamma_{\alpha\beta}^{\gamma}$ in L_n and satisfying he condition of invariance of curvature for $\bar{L}_n$ and L_n (ref. eq.(2)) will be called a set of curvature field copies of $\Gamma_{\alpha\beta}^{\gamma}$ in $\bar{L}_n$. The tensor $P_{\alpha\beta.}{}^{\gamma}$ will be named P-field.

From eq.(2) we get for the P-field the condition

$$\nabla_{[\alpha}P_{\beta]\gamma.}{}^{\delta} + S_{\alpha\beta.}{}^{\lambda} + P_{[\beta|\gamma|.}{}^{\lambda}P_{\alpha]\lambda.}{}^{\delta} = 0 \tag{3}$$

where

$$S_{\alpha\beta.}{}^{\lambda} \equiv \Gamma^{\lambda}_{[\alpha\beta]} \qquad \text{and} \qquad \nabla,$$

is the covariant derivative associated with the connection $\Gamma_{\alpha\beta}{}^{\lambda}$.

We shall analyse specific examples of copies for which the P-field is generated by the mathernatical structure of the initial (base) manifold. that is, copies which involve quantities as the non-metricity tensor $Q_{\alpha\beta\gamma} = -\nabla_\alpha g_{\beta\gamma}$ and the torsion tensor $S\alpha\beta\gamma$ and their respective contractions. We notice that this requirement is not necessarily fulfilled, for example. in the case of general projective copies

$$\bar{\Gamma}_{\alpha\beta}{}^{\lambda} = \Gamma_{\alpha\beta}{}^{\lambda} + \delta^{\lambda}_{\beta}\psi_\alpha \tag{4}$$

for an arbitrary vector field ψ_α.

Def. 2.2 A curvature field copy which is generated by tensorial objects of the manifold [torsion and non-metricitytensors and their contractions) is called here an intrinsic geometrical curvature copy (IGCC. for short).

3. CURVATURE COPIES

As a first example we take a tensor in a semi-metric manifold L_n

$$P_{\alpha\beta}{}^{\lambda}{}_{\cdot} = 2\,a\,g_{\alpha[\beta}Q_{\sigma]}g^{\sigma\lambda} + b\,Q_\alpha\delta^{\lambda}_{\beta} + c\,S_\alpha\delta^{\lambda}_{\beta} \tag{5}$$

where a, b, c are real numbers and

$$Q_\gamma g_{\alpha\beta} = Q_{\gamma\alpha\beta} \qquad \text{(semi-metricity condition)}, \tag{6}$$

$$S_\alpha = \frac{2}{n-1} S_{\alpha\beta}{}^{\beta}\,. \qquad \text{(torsion vector)}. \tag{7}$$

Let us also assume the following restrictions for S_α and $Q\alpha$ in eq.(5)

$$\nabla_\beta Q^\alpha = 0 \tag{8}$$

$$Q_\alpha Q^\alpha = 0 \tag{9}$$

$$S_{\alpha\beta\gamma} = (a-1)g_{\gamma[\alpha}Q_{\beta]} \tag{10}$$

Theor. 3.1. - The P-field expressed by eq.(5) is a P-field of an IGCC if the conditions (8) - (10) are satisfied, with Q_α and S_α defined by eqs. (6) and (7). respectively.

The proof of this theorem follows immediately from the substitution of eq.(5) into eq.(3), with the conditions (8) - (10). We remark that in this case the original manifold L_n is semi-metric with non-symmetric connection, the torsion being determined by the null constant vector Q_α, which expresses the existence of a curvature of segmentation.

Another example of a semi-metric curvature copy is provided by the P-field

$$P_{\alpha\beta.}{}^{\gamma} = g_{\alpha\beta} Q^{\gamma} \tag{11}$$

with the assumptions

$$\nabla_\beta Q^\alpha = 0 = S_{\alpha\beta\gamma} \tag{12}$$

Theor. 3.2. - The P-field given by eq.(11) corresponds to an IGCC if the condition eqs.(6) and (12) are imposed.

Again the proof follows from direct substitution in to eq.(3). Here we have a copy in Weyl's geometry W_n, since

$$\nabla_\gamma g_{\alpha\beta} = -Q_\gamma g_{\alpha\beta}$$

and

$$S_{\alpha\beta\gamma} = 0$$

By considering now the P-field

$$P_{\alpha\beta.}{}^{\gamma} = \delta^{\gamma}_{\alpha} S_\beta \tag{13}$$

with

$$S_{\alpha\beta\gamma} = S_{[\alpha} g_{\beta]\gamma} \qquad \text{(vectorial irreducible component of the torsion)} \tag{14}$$

and

$$\nabla_\alpha S_\beta = 0 \tag{15}$$

we obtain a different copy in a semi-metric L_n

Theor. 3.3. - The P-field introduced by eq.(13) is connected with an IGCC if the assumptions eqs. (6). (14) and (15) are taken into account.

This follows by using the same substitutions referred to above.

We shall hereafter analyse the case of metric n-dimensional spaces. in which

$$\nabla_\gamma g_{\alpha\beta} = 0 \qquad \text{(metricity condition)} \tag{16}$$

We point out that in this situation the imposition of a metricity condition for the copy manifold $\bar{L}_n$, as well as for the base manifold L_n introduces a symmetry restriction for the corresponding P-field.

Lemma 3.1 - In a curvature copy mapping of a metric manifold M_n into a metric manifold $\bar{M}_n$ the P-field has the symmetry property $P_{\alpha\beta\gamma} = P_{\alpha[\beta\gamma]}$.

The proof follows from

$$\bar{\nabla}_\gamma g_{\alpha\beta} = \nabla_\gamma g_{\alpha\beta} = 0 \tag{17}$$

where $\bar{\nabla}_\lambda$ is associated with the connection $\bar{\Gamma}_{\alpha\beta}{}^\gamma$ in $\bar{M}_n$.

Incidentally we also observe that if in this last case we further make the restriction $S_{\alpha\beta\gamma} = 0$, reducing the manifold to a Riemann space V_n then the P-field vanishes.

Theor. 3.4. - A curvature copy mapping of a Riemmanian n-dimensional manifold into itself is necessarily the identity mapping.

To prove this theorem we use the Lemma and the fact that the connection $\Gamma_{\alpha\beta}^{\gamma}$ and its copy are symmetric. Thus in this case $P_{\alpha\beta\gamma} = P_{(\alpha\beta)\gamma}$, and once we have from the Lemma $P_{\alpha\beta\gamma} = P_{\alpha[\beta\gamma]}$, we are led to $P_{\alpha\beta\gamma} = 0$, which implies $\bar{\Gamma}_{\alpha\beta}{}^{\gamma} = \Gamma_{\alpha\beta}{}^{\gamma}$, i.e., the identity copy mapping.

We take now as an example of P-field in a metric n-dimensional space with torsion (U_n) the tensor

$$P_{\alpha\beta\gamma} = 2\, g^{\lambda\gamma} g_{\alpha[\lambda} S_{\beta]} \tag{18}$$

with S_λ defined by eq.(7), plus the condition

$$S_\alpha S^\alpha = 0 \tag{19}$$

which gives. in view of the metricity,

$$\nabla_\beta S_\alpha = 0.$$

Theor. 3.5. - The P-field in **eq.(18)** is related to an **IGCC if** we impose eqs. (16) and (19).

This particular copy has an specialization, **derived** by imposing on the vielbein field e^j_α in U_n the condition

$$\nabla_\beta e^j_\alpha = 0 \tag{20}$$

where $j = 0, \ldots, n-1$ **is** an **internal** index. From **eq.(20)** we have

$$\Gamma_{,,}^{;} \;= e_j^{\ \gamma}\partial_\alpha e^j_\beta \tag{21}$$

which is the connection of the Weitzenbock space A_n Relations (18) and (21) lead to

$$\Gamma_{\alpha\beta}^{\ \ \gamma} = e^\gamma_j \partial_\alpha e^j_\beta + \frac{2}{n-1}(\delta^\sigma_\beta \delta^\gamma_\alpha - g_{\alpha\beta} g^{\sigma\gamma}) e^\lambda_j \partial_{[\sigma} e^j_{\lambda]} \tag{22}$$

This copy is entirely determined by the vielbein **parallel** fields of A_n We notice that not only the U_4 space has been used in the formulation of alternative theories of the gravitational field, but also the 4-dimensional Weitzenbock space A_4 has been adopted in this context[8]. **In** A_n the curvature tenspr vanishes. so that. **according** to **eq.(2)**. the curvature of the copy space $\bar{A}_n$ **also** vanishes.

Another subspace of U_n. besides A_n is the Riemann space V_n. which results from the assumption $S_{\alpha\beta\gamma} = 0$ in U_n.

Copies in U_4. **such** as that given by eq. (18). with $n = 4$, **rise** the question of gravitational theories with identical free field equations but different couplings with matter. We can take. for instance. the interaction of spin **-1/2** particle with the **gravitational** field described by[9]

$$\mathcal{L}_M = \frac{i}{2} e^\alpha_j (\bar{\Psi}\gamma^j D_\alpha - \Psi\gamma^j D_\alpha \bar{\Psi}) - m\Psi\bar{\Psi} \tag{23}$$

where

$$D_\alpha = \partial_\alpha - \frac{i}{2} A_{ij\alpha} S^{ij}$$

$$S^{ij} = \frac{i}{4}\left[\gamma^i, \gamma^j\right]$$

is the infinitesimal generator of the internal Lorentz group: $A_{ij\mu}$ is the Lorentz gauge field associated with the affine connection of U_4; $\gamma^i, \gamma^j = 2\eta^{ij}$; $j = 0, \ldots, 3$. The copy $\bar{\Gamma}_{\alpha\beta}^{\ \ \gamma}$ originates a field $\bar{A}_{ij\mu} \neq A_{ij\mu}$. and consequently a modified matter Lagrangian $\bar{\mathcal{L}}_M$. In the 4-dimensional teleparallel space A_4 the internal connection $A_{ij\mu}$ vanishes. so that in the minimal coupling instance we have $\bar{\mathcal{L}}_M = \mathcal{L}_M$. and the copy does not alter the interaction. However, if we take a non-minimal coupling Lagrangian density like

$$\mathcal{L}'_M = iK\sqrt{-g}\Psi\gamma^k\Psi e^{\mu}_k S_m u \tag{24}$$

where K is a constant and

$$S_\mu = \frac{2}{3} e^{\nu}_{k} \partial_{[\mu} e^{k}_{\nu]}$$

we obtain $\bar{\mathcal{L}}'_M = 2\mathcal{L}'_M$. since in the copy manifold $\bar{S}_\mu = 2S_\mu$. Thus even in the A_4 case the gravitational interaction of fermions may be affected by the copies.

Finally we want to mention the effect of those mappings on trajectories in U_4. Autoparallelel curves[10] in the copy space differ in general from those in the original one, even if the gravitational equations do coincide. This deviation depends. of course, on the symmetric part of the field copy, since the symmetric part of the connection appears in the equation that defines autoparallel. Extremal curves are derived directly from the metric and do not undergo any copy effect. Concerning the motion of spin test particles outside of the matter. it is reasonable to expect some influence of copies on the trajectories in U_4 theories which have propagation of torsion.

RFERENCES

1. T.T. Wu and C.N. Yang, Phys.Rev. D12, 3843 (1975).
2. C.G. Oliveira and P.S. Letelier, Preprint. Universidade de Brasíiia.
3. A. Einstein. *The Meaning of Relativity* (Princeton University Press. N.J., 1956).
4. V.D. Sandberg, Phys. Rev. D12, 3013 (1975).
5. W.R. Davis, Lett. Nuovo Cim.. 22. 101 (1978).
6. W.R. Davis, Lett. Nuovo Cim.. *18*, 319 (1977).

7. J.A. Schouten. *Ricci - Calculus* (Springer-Verlag, Berlin. 1954).
8. K. Hayashi and T. Shirafuji. Phys. Rev. D19, 3524 (1979). and references therein.
9. K. Hayashi and T. Shirafuji. Prog. Theor. Phys. 64, 866 (1980).
10. F.W. Hehl, P. von der Heyde. G.D. Kerlick and J.M. Nester. Rev. Mod. Phys.. 48, 393 (1976).

Resumo

Apresentamos exemplos de cópias de curvatura em espaços mdtricos e semi-métricos. Tais mapeamentos envolvem objetos geomdtricos das variedades. e têm implicações físicas no contexto de teorias gravitacionais.