Revista Brasileira de Física, Vol. 18, nº 4, 1988

Einstein Equation and Yang-Mills Theory of Gravitation

E. STEDILE*

Departamento de Física, Universidade Federal do Paraná, Caixa Postal 19081, Curitiba, 80000, PR, Brasil

Recebido em 24 de junho de 1988

Abstract We point out the possibility of Yang Mills theory of gravitation being a candidate as a gauge model for the Poincaré group. If we accept the arguments favoring this theory then Einstein's equations can be derived by a different method in which they arise from a dynamical equation for the torsion field. in a particular case.

1. INTRODUCTION

Analogies between the Yang Mills (YM) theory at the classical level and general relativity (GR), under their common geometrical basic setting. have long been noticed. However. GR is not beyond criticism from a theoretical point of view. Reviews on this subject have been written by Hehl[1]. Zhenlong[2] and others. and from a different standpoint by Lugonov et al[3]. (see also references therein). A very general point frequently made is that GR does not have the entire Poincaré local symmetry of spacetime.

Stelle and West[4] analyzed in detail the local geometricai structure of GR. as a gauge theory of the de Sitter group $SO(3,2)$. To reproduce the structure of Einstein-Cartan theory the $SO(3,2)$ gauge symmetry is spontaneously broken down to the Lorentz group. In this approach the gravitational vierbein and spin connections can be derived from their original $SO(3,2)$ gauge fields. by passing over to a set of nonlinearly-transforming fields. through a redefinition involving a Goldstone field. The original $SO(3,2)$ gauge fields generate pseudo-translations and rotations in the so-called interna1 anti-de Sitter space. under a kind of parallel transport.

Norris et al[5] proposed an underlying fibre bundle structure for gauge theories of gravitation. and an extension to an affine structure group. They have shown that the decomposition of a generalized affine connection contains more than curvature and torsion.

* Work supported by CNPq (Brazilian Government Agency).

This should affect also the field equations. They considered an extension of the linear frame bundle to the affine frame bundle and pointed out that the torsion is just one parte of the *total curvature* of a generalized afine connection.

Mielke[6], within the framework of differential geometry. considered a Yang's parallel displacement gauge theory with respect to pure gravitational fields. He showed that, in a four-dimensional Riemannian manifold, double self-dual solutions obey Einstein's vacuum equation with a cosmological term, whereas the double anti-self dual configurations satisfy the Raynich conditions of geometrodynamics. Under duality conditions the Stephenson-Kilmister-Yang theory not only embraces $R_{\mu\nu} = 0$, but also Nordstrom's vacuum theory.

The lagrangian structure of the Poincart! gauge field equations for gravitation. and their Einsteinian content. under duality condition. is already known[7]. Here we will be concerned with the field equations only.

2. FIELD EQUATIONS

In a previous paper[8] we exhibited a Poincart! gauge model for gravitation. under a Inonu-Wigner contraction of the gauge fields. This approach is built up in a Poincart! P-bundle. In this case, by duality symmetry. Bianchi identities lead to Yang-Mills equations

$$d * F + [\Gamma, *F] = 0 \tag{1}$$

$$d * T + [\Gamma, *T] + [S * F] = 0 \tag{2}$$

for curvature and torsion. respectively. In the above equations Γ stands for the connection form valued in the Lorentz algebra and S is the solder form. valued in the algebra of the group of translations.

Popov and Daikhin [9], on the basis of a heuristic argument, have pointed out that for a Levi-Civitá connection eqs. (1) and (2) reduce to and

$$R_{\mu\nu;\lambda} - R_{\mu\lambda;\nu} \tag{3}$$

and

$$R_{,,} = 0 \tag{4}$$

respectively. So,in the sourceless case,this approach includes Yang[10] and Einstein theories of gravitation,and they are redundant. The sources of $Y-M$ equations are the Noether current densities whose charges are the generators of the gauge group. Therefore,they will be the density of relativistic angular mornemtum M and the energy-stress tensor θ,and the field equations (1) and (2) become

$$d * F + [\Gamma, *F] = *M \tag{5}$$

$$d * T + [\Gamma, *T] + [S, *F] + *\theta \tag{6}$$

in the P-bundle space. This latter equation suggests a propagation for the torsion field,and is different from the Einstein-Cartan equation which is not dynarnic.

To write it in spacetime conponents we have to project it down by means of the vierbein fields h^a_μ. If we choose a basis where the solder form can be identified with those h^a_μ and consider a Riernannian case (Levi Civitá connection i.e. $T = 0$). eq.(6) reduces to

$$[S, *F] = *0 \tag{7}$$

Now we take Kibble's prescription eq.(11) for the vierbein fields

$$h^a_\mu = \delta^a_\mu + \epsilon B^a_\mu \tag{8}$$

where $\epsilon << 1$ is a coupling ccnstant and B^a_μ is a gauge potential. These fields must satisfy the condition

$$h^a_\mu h^\mu_b = \delta^a_b \tag{9}$$

therefore

$$h^\mu_a = \delta^\mu_a - \epsilon B^\mu_a \tag{10}$$

and so $\epsilon^2 B^a_\mu B^\mu_a$ has to be negligible. There is a compeling reason to choose Kibble's prescription instead of $h^a_\mu = \delta^a_\mu$: otherwise the components $\Gamma^a_{b\mu}$ are null.

Eq.(7), in components in the P-bundle space, is

$$S^{b}_{\lambda}\, F^{a\ \lambda}_{b\ \mu} = k\,\theta^{a}_{\mu} \tag{11}$$

where k **is** a constant. **Projecting eq.(11)** onto the spacetime and considering **eq.(8)** we are led to

$$\delta^{b}_{\lambda}\, R^{\alpha\ \lambda}_{b\ \mu} + \epsilon\, B^{b}_{\lambda}\, R^{\alpha\ \lambda}_{b\ \mu} = k\,\theta^{\alpha}_{\mu} \tag{12}$$

or

$$R_{\alpha\mu} - \epsilon\, B^{\beta}_{\lambda}\, R^{\ \lambda}_{\alpha\beta\ \mu} = -k\,\theta_{\alpha\mu} \tag{13}$$

where $R^{\alpha}_{\beta\lambda\mu}$ are the components of the Riemann tensor.

Taking into account the canservation law

$$\nabla^{\alpha}_{;,}0 \quad = 0 \tag{14}$$

we are led to

$$\nabla^{\alpha}(R \quad - \epsilon B^{\beta}_{\lambda} R^{\ \lambda}_{\alpha\beta\ \mu}) = 0 \tag{15}$$

or

$$\epsilon\nabla^{\alpha}(B^{\beta}_{\lambda} R^{\ \lambda}_{\alpha\beta\mu}) = \nabla^{\alpha} R_{\alpha\mu} = \frac{1}{2}\nabla^{\alpha}(g_{\alpha\mu}R) \tag{16}$$

hence

$$\epsilon B^{\beta}_{\lambda} R^{\ \lambda}_{\alpha\beta\ \mu} = \frac{1}{2} g_{\alpha\mu} \mathrm{R} + \Lambda C_{\alpha\mu} \tag{17}$$

A being *a* constant.

Here Ç are **the** components of a tensor C. which satisfy the condition

$$\nabla^{\alpha} C_{\alpha\mu} = 0 \tag{18}$$

From the geometrical **point** of view C is **parallely** transported by the **connection** Γ and in the **language** of gauge fields **it** is a conserved current.

With the result given in **eq.(17)**, **eq.(13)** becomes

$$R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R + \Lambda C_{\alpha\mu} = -k\,\theta_{\alpha\mu} \tag{19}$$

About this equation we have some comments to make. concerning to the C term. and supposing condition **eq.(18)** already satisfied: **i)** if $\Lambda C_{\alpha\mu} = -k\, g_{\alpha\mu}$, eq.(19) becomes

$$R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R - k\, g_{\alpha\mu} = -k\, \theta_{\alpha\mu} \tag{20}$$

which is Einstein's equation with a cosmological constant k; **ii) if** A = 1 and

$$C_{\alpha\mu} = R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R$$

(the usual Einstein tensor), **eq.(19)** becomes

$$R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R = -\frac{k}{2} \theta_{\alpha\mu} \tag{21}$$

which is Einstein's equation; **iii)** in the absence of sources. $\theta_{\alpha\mu} = 0$, one must have from **eq.(19)** $R_{\alpha\mu} = 0$ and also R = 0, which gives the intrinsic curvature of spacetime.

Henceforth, the simplest form of eq.(19) which generalizes Poisson's equation is for A = 0, and so we are led to Einstein's equation

$$R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R = -k\, \theta_{\alpha\mu} \tag{22}$$

3.CONCLUSION

The scenario here developed suggests the possibility of Einstein **model** being a sub-theory of a gravitational Y-M theory. This seems to be important to build **up** a gauge model for gravitation, in the same way that electromagnetism and $SU(2)$ theories have been constructed. The use of Kibble's prescription above **is** something that can be viewed as a gauge transformation, if we interpret h^{α}_{μ} as a gauge potential. This transformation is important to obtain Einstein's equation.

The author is **very** grateful to the referee of this paper for many **helpful** suggestions.

REFERENCES

1. **F.H.Hehl**, P. von der Heyde. G.D. Kerlick and J.M. Nester. Reviews of Modern Physics, *48*, 393 (1976). **F.H.Hehl**, Four Lectures on Poincaré Gauge field

theory, in *Spin, Torsion, Rotation and Supergravity* in 6th Course of the International School of Cosmology and Gravitation. Erice, Italy (preprint) (1979).
2. Z.Zhenlong *et al.*. Scientia Sinica. **22,** 628 (1979).
3. A.A. Logunov and V.N. Folomeskin. Theoretical and Mathematical Physics. **82.** 653 (1978).
4. K.S. Stelle and P.C. West. Physical Revíew *D21*, 6, 1466 (1980).
5. L.K. Norris *et al.*, Physics Letters *79*A, 278 (1980).
6. E.W. Mielke, General Relativity and Gravítation, *13*, 175 (1981).
6. E.W. Mielke, General Relativity and Gravitaticn. *13*A, 471 (1986).
7. P. Baekler and E.W. Mielke, Physics Letters *113*A, 471 (1986)
8. R. Aldrovandi and E.Stedile, International Journal of Theoretícal Physics, 23 301 (1984).
9. D.A. Popov and L.I. Daikhin, Soviet Physics Doklady. 20. 818 (1976).
10. C.N. Yang. Physical Review Letters. 33, 445 (1974).
11. T.W.B. Kibble. Journal of Mathematical Physics. 2, 212 (1961).

Resumo

Salientamos a possibilidade de se ter uma teoria de Yang-Mills para a gravitação, como candidata a um modelo de gauge do grupo de Poincard. Se aceitarmos os argumentos favoráveis a esta teoria. então as equações de Einstein podem ser obtidas por um mdtodo diferente. Elas surgem de uma equação dinâmica para a torção em um caso particular.