

Geometric Phases and Gravitational Anomalies

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Abstract An analysis is made of the motion of a quantum particle in a gravitational field. and it is shown that a geometric phase arises under certain conditions. The connection of this result with the emergence of gravitational anomalies is described.

1. INTRODUCTION

Classical symmetries which do not exist at all at the quantum-mechanical level occur naturally in field theory, but are rare in systems with a finite number of degrees of freedom. These *anomalous symmetries* are always broken, because the introduction of a regulator, indispensable for renormalization, either spoils some symmetry or violates some other requisite, such as unitarity. Anomalies appear under a variety of guises¹: gauge and non-gauge, Abelian and non-Abelian, perturbative and non-perturbative. Some, such as the axial $U(1)$ in QCD and certain non-perturbative anomalies in string theory⁴, are welcome, while others, like the axial gauge anomalies in the standard model of electroweak interactions, are truly *undesirable*. Anomalies in the latter category render the theory inconsistent, so their absence can be turned into a *powerful* criterion in reducing the collection of renormalizable theories: indeed, the condition of anomaly-freeness usually fixes, or at least restricts, the value of an otherwise arbitrary parameter in the theory². On the other hand, there exists a class of model theories (so far all bidimensional) that can be consistently quantized in the presence of anomalies, at the expense of gauge invariance² or canonical quantization rules³. Probably the last word has not been said yet as to when it is inevitable to free oneself from anomalies.

In view of this prominent role played by anomalies in quantum field theory, it is essential to understand them from a variety of viewpoints. Perhaps the most subtle and least explored are the anomalies which arise in theories with gravitational

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couplings. These (gravitational) anomalies appear in two versions - perturbative⁴ and global⁵ - obstructing the realization of spacetime symmetries. In refs. 4 and 5 a number of interesting results concerning gravitational anomalies were obtained with the Lagrangian (or path-integral) formalism. In this approach one examines the effective interaction induced by integration over the fermionic sector, looking for explicit symmetry breaking or ambiguities under global transformations.

An alternative approach, applied to chiral anomalies⁵, is the Hamiltonian or Hilbert space formulation. In this language an anomaly occurs whenever the symmetry cannot be implemented as a true representation, but only as a **projective**, or ray representation: such cases involve topologically unremovable phases, which break the gauge invariance. Here, when one examines the effective Hamiltonian for the bosonic fields one **finds**^{6,7} that the anomalous content of the theory **is** concisely comprised in a phase whose origin is purely geometric: the phase acquired by the wavefunction of a system subject to (an adiabatic) periodic **potential**⁸.

In this note a study is begun of gravitational anomalies which **makes** use of the Hamiltonian interpretation. It is shown that a quantum mechanical particle with spin and interacting only with a gravitational field can develop a geometric phase. The next section reviews Berry's results⁸ on adiabatic phases and a generalization due to Aharonov and Anandan⁹. In the **following** section an analysis is **made** of the motion of a spinning particle in curved space. Some other related issues are touched upon in the concluding section.

2. GEOMETRIC PHASES

Let **us** review how a geometric phase arises in quantum mechanics, beginning with the case of adiabatic evolutions. Consider a system **subject** to a **slowly** changing **external** field $\vec{Q}(t)$, so that it is meaningful to describe the state of the system in terms of the eigenfunctions of the **momentary** Hamiltonian, defined by

$$H(Q(t))|\psi_n(Q(t))\rangle = E_n(Q(t))|\psi_n(Q(t))\rangle$$

$$\langle \psi_n(Q(t)) | \psi_m(Q(t)) \rangle = \delta_{mn} \tag{2.1}$$

As **first** pointed out by Berry⁸, the adiabatic **theorem**¹⁰ is modified if the motion of the background field $\vec{Q}(t)$ is periodic. The conventional version of the

theorem asserts that if the initial state vector is $|\psi_n(Q(0))\rangle$, then at a later time t it will have evolved into

$$|\psi_n(Q(t))\rangle = e^{i\varphi_n(t)}$$

where

$$\varphi_n = - \int_n^t dt' E_n(Q(t'))$$

provided the background changes **slowly** enough* that transitions do not take place. The phase φ_n is called *dynamical*, because it depends on the functional form of H . **In words**, the system remains in the *same* eigenstate **labelled** by $E_n(Q(t))$, riding along with the background.

However, if the background field resumes its initial configuration after a time T , i.e., $\vec{Q}(T) = \vec{Q}(0)$, the state vector is instead

$$|\psi_n(Q(0))\rangle = e^{i\varphi_n + i\gamma_n}$$

where the extra phase γ_n , differently from φ_n , depends only on the geometry of the surface $E_n(Q)$ in Q -space^S. From the time-dependent Schrodinger equation, one finds

$$\begin{aligned} \gamma_n &= \int_0^T dt \langle \psi_n(Q(t)) | i \frac{d}{dt} | \psi_n(Q(t)) \rangle \\ &= \oint_C d\vec{Q} \cdot \langle \psi_n(Q) | i \vec{\nabla}_Q | \psi_n(Q) \rangle \end{aligned} \quad (2.2)$$

where C is the **closed** circuit in Q -space generated by the background motion. The last expression shows that γ is a function of C , i.e., independent of the parametrization t .

When the circuit is open, it is possible to absorb the extra phase in ψ_n , since eqs. (2.1) fix it only up to a phase factor; this is no **longer** possible once the circuit closes, because γ is non-integrable.

* Or rather

$$\Delta H = \frac{|\langle \psi_n | \frac{\partial H}{\partial t} | \psi_m \rangle|}{E_m - E_n} \ll E_m - E_n.$$

From the viewpoint of parameter space this means that there is a connection*

$$\vec{A}_n(Q) = \langle \psi_n(Q) | i \vec{\nabla}_Q | \psi_n(Q) \rangle$$

in terms of which

$$\gamma_n[C] = \oint_C d\vec{Q} \cdot \vec{A}_n(Q)$$

An alternative expression follows from eq.(2.1) and eq.(2.2)

$$\gamma_n = - \int_S d\vec{S} \cdot \text{Im} \sum_{m \neq n} \frac{\langle \psi_n | \vec{\nabla}_Q H | \psi_m \rangle \times \langle \psi_m | \vec{\nabla}_Q H | \psi_n \rangle}{[E_m(Q) - E_n(Q)]^2}$$

The denominator provides a hint as to why a degeneracy in Q-space is a common feature of a large number of systems with non-vanishing γ_n .

The paradigm of such systems is a degenerate **two-level** system subject to an interaction linear in the **external** field, namely $H_{int} \propto \vec{\sigma} \cdot \vec{Q}(t)$. This class is **very** large indeed, including certain triatomic molecules, the quantum **Hall** effects, chiral anomalies, and **Skyrmions**¹². Perhaps its simplest version is a **spin-1/2** particle in a cyclic magnetic field. See fig. 1. (Here, $\vec{Q}(t)$ is the magnetic field vector.)

For this class of systems, it can be shown that $\gamma_n[C]$ is proportional to the solid angle Ω subtended by C from the origin $Q = 0$ (fig.1). Since γ is the flux of \vec{A} through C, this means that there is a monopolar magnetic flux in parameter (Q) space, as if there were a monopole sitting at the origin. In particular, γ is unchanged by circuit deformations which preserve the solid angle. This property and the fact that γ vanishes for an open circuit are what make it a topological quantity.

In principle, the only restriction on the value of γ is that it must be a real number (this follows from $\langle \psi_n | \psi_n \rangle = 1$). Under certain conditions even a system made out of bosons can have $\gamma = \pi$, in which case its wave function changes sign upon a full rotation in coordinate space. Perhaps the most familiar

* The phase change $|\psi_n\rangle \rightarrow e^{i\chi(Q)}|\psi_n\rangle$ induces $A_n \rightarrow A_n - \nabla_Q \chi$. \vec{A} becomes non-Abelian if the level n is degenerate¹¹.

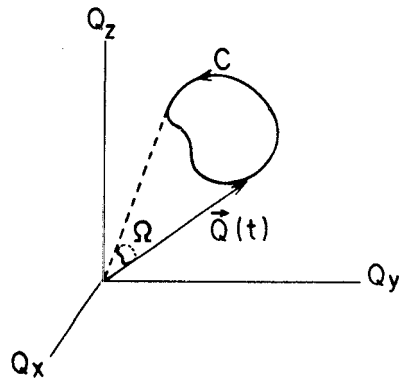


Fig.1 - The vector $\vec{Q}(t)$ traces a curve C in Q -space which subtends a solid angle Ω from the origin.

system exhibiting this property is the Skyrmion - a soliton in a theory of purely bosonic fields. which is nevertheless a **fermion**¹².

We conclude this summary on the adiabatic phase with a brief remark on the connection between geometric phases and chiral anomalies. As mentioned in the introductory section, the integration over the fermionic sector of an anomalous theory induces in the bosonic sector an interaction **proportional** to the phase γ . **It** turns out that the origin of this phase is very much the same as in the example just described; it is a property of the degenerate fermionic vacuum in the **presence** of a bosonic background, which can be attributed to a magnetic flux in parameter space. Since the parameters here are the bosonic fields, the anomalous term **will** describe monopoles living in **the** configuration space of bosonic fields. But this is precisely what Wess-Zumino terms **are**¹³. The geometric phase thus provides an interesting standpoint on anomalies. tracing it in the end to the fact that energy eigenfunctions acquire phases during closed loop excursions in parameter space. or rather they form twisted **line** bundles over S^1 (ref.6).

Now let us free the concept of the geometric phase from the restriction to adiabatic evolutions, and consider motions of the parameter \vec{Q} that are **just** periodic. Let $\psi(t)$ be a normalized state vector in Hilbert space H evolving with

the Schrodinger equation

$$i \frac{\partial \psi(t)}{\partial t} = H(t)\psi(t).$$

Suppose further that the motion is periodic, i.e., $\vec{Q}(t) = \vec{Q}(0)$, so that

$$\psi(T) = e^{i\Lambda} \psi(0) \tag{2.3}$$

A real.

Consider now the space of rays $P_{\mathcal{X}}$ associated with H , obtained by identification of all vectors which differ by a complex number. In order to express the results in a concise form it is convenient to define a mapping π which projects a vector ψ onto its ray $\hat{\psi}$, namely

$$\pi(\psi) = \hat{\psi} : \psi = z\hat{\psi}, \quad z \in \mathbb{C}$$

so π projects \mathcal{X} onto $P_{\mathcal{X}}$, $\pi(\mathcal{X}) = P_{\mathcal{X}}$. See fig. 2.

Notice that each periodic evolution defines a sequence of states in H - a curve $I : (0, T) \rightarrow \mathcal{X}$ - that begins and ends on ray $\hat{\psi}$. Through the mapping π it defines also a closed curve Γ in $P_{\mathcal{X}}$, namely the projection $\pi(I) = \Gamma$. See fig. 2.

One defines the geometric phase β as that piece of the total phase Λ which is not dynamical

$$\beta = \Lambda + \int_0^T dt \langle \psi(t) | H | \psi(t) \rangle$$

Although β may seem to depend on the circuit I it is a function of $\hat{\Gamma}$ only. In other words, β is the same for all curves Γ in H which project onto the same curve $\hat{\Gamma} = \pi(\Gamma)$ in $P_{\mathcal{X}}$. This assertion can be easily proved in two short steps.

First express $\beta(\Gamma)$ in terms of

$$\tilde{\psi}(t) = e^{if(t)} \psi(t),$$

with f so chosen that $f(T) - f(0) = \Lambda$, to find

$$\beta(\Gamma) = \int_0^T dt \langle \tilde{\psi} | i \frac{d}{dt} | \tilde{\psi} \rangle$$

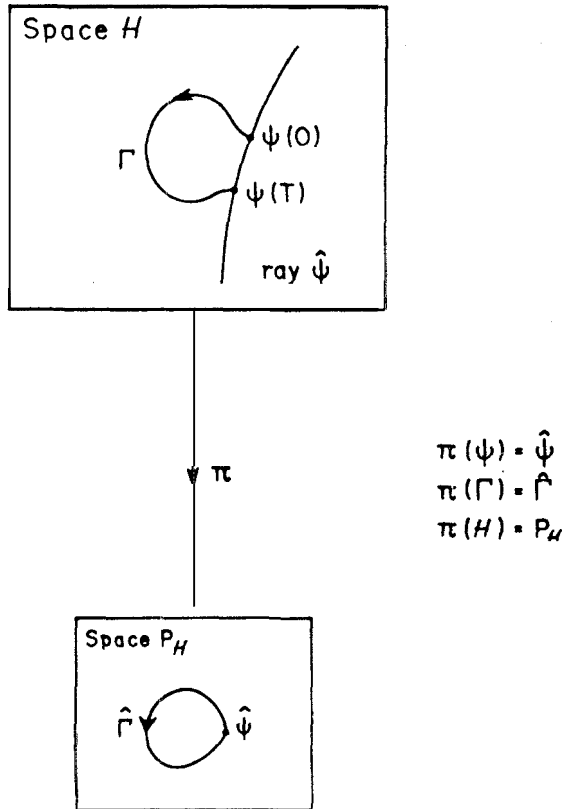


Fig. 2 - H is the Hilbert space of the wave-function ψ , P_H the space of rays $\hat{\psi}$, and π is the projection map $\pi(\psi) = \hat{\psi}$. Γ is a curve in H , while $\hat{\Gamma}$ is a closed curve in P_H .

Next we consider another curve Γ' such that $\pi(\Gamma') = \pi(\Gamma) = \{\hat{\Gamma}\}$. This means that along Γ' there exists a phase $\alpha(t)$ such that

$$\psi'(t) = e^{i\alpha(t)} \psi(t) = e^{i\alpha(t) + if(t)} \tilde{\psi}(t)$$

Now choosing $\alpha(t)$ so that $\alpha(T) = \alpha(0)$, we have

$$\psi'(t) = e^{if(t)} \tilde{\psi}(t)$$

with

$$f'(T) - f'(0) = A$$

and

$$\beta(\Gamma') = A + \int_0^T dt \langle \psi'(t) | H | \psi'(t) \rangle = \int_0^T dt \langle \tilde{\psi} | i \frac{d}{dt} | \tilde{\psi} \rangle = \beta(\Gamma)$$

Hence, $\beta[\hat{\Gamma}]$ is independent of the Hamiltonian H , the phase A , and the parametrization t : it is a geometric property of closed loops in P_X . An explicit calculation in the next section will substantiate this feature of β .

Notwithstanding this distinction, it is easy to verify that β reduces to γ when adiabatic conditions prevail⁹. Moreover, the value of β associated with a given evolution can be calculated with the formula for γ in terms of eigenfunctions. Since an adiabatic Hamiltonian can always be found which generates the given motion. As our purpose is just to show that a non-vanishing phase arises, the phase β will prove more convenient.

3. SPINNING PARTICLE IN A GRAVITATIONAL FIELD

As will become clear soon, it is interesting to consider the motion of a particle with non-vanishing angular momentum subject to a *stationary* gravitational field, that is to say, one described by a metric which is z^0 (time)-independent, and an interval which is not invariant under time reversal ($x^0 \rightarrow -z^0$).

The spin vector S_μ of a particle in free fall obeys the equations¹⁴

$$\left(\frac{d}{d\tau} - \Gamma^\alpha_{\beta\nu} \frac{dx^\nu}{d\tau} g^{\mu\beta} \right) S_\mu = 0$$

It is further constrained by the condition

$$S^\alpha \frac{dx^\alpha}{d\tau} = 0,$$

expressing the fact that only three of its components are independent. [In the particle's rest frame, $S_\mu = (\vec{S}, 0)$, so the spin is always orthogonal to the velocity $\frac{dx^\mu}{d\tau} = \gamma v^\mu$]. Using the constraint to eliminate the component S_0 , we are left with

$$\frac{dS_i}{dt} = (\Gamma_{i0}^j - \Gamma_{i0}^0 v^j + \Gamma_{ik}^j v^k - \Gamma_{ik}^0 v^k v^j) S_j$$

For our purposes it suffices to solve the equations within the Post-Newtonian approximation. Accordingly, the expansion of the metric in powers of the particle velocity v , up to second order, gives¹⁴

$$\frac{d\vec{S}}{dt} = \left[\frac{1}{2}(\vec{\nabla} \times \vec{\xi}) \times + \frac{\partial \phi}{\partial t} + (\vec{v} \cdot \vec{\nabla})\phi + 2\vec{\nabla}\phi(\vec{v} \cdot) - \vec{v}(\vec{\nabla} \cdot) \right] \vec{S} \quad (3.1)$$

where

$$\zeta_i = O(v^3)$$

and

$$4 = O(v^2)$$

are defined by the equations*

$$g_{i0} = \zeta_i + O(v^5),$$

$$g_{00} = -1 - 2\phi + O(v^4).$$

At this point one notices that although the magnitude of \vec{S} is not constant, there is another vector whose motion is a pure precession. Since the constant value for $S^a S_a$ is proportional to

$$(1 + 2\phi)\vec{S}^2 - (\vec{v} \times \vec{S})^2$$

one defines a new spin vector

$$\vec{S}' = (1 + \phi)\vec{S} - \frac{1}{2}(\vec{v} \cdot \vec{S})\vec{S} \quad (3.2)$$

so that

$$\vec{S}' = \text{constant.}$$

The vector \vec{S}' is the one that precesses: differentiating eq.(3.2), we find

* For a sphere of mass M and angular momentum J at rest, $\phi = -GM/r$,
 $\vec{\zeta} = \frac{2G}{r^3} \vec{r} \times \vec{J}$.

$$\frac{d\vec{S}}{dt} = \frac{d\vec{S}}{dt} + \left[\frac{\partial\phi}{\partial t} + \vec{v} \cdot \vec{\nabla}\phi + \frac{1}{2}\vec{\nabla}\phi(\vec{v} \cdot) + \frac{1}{2}\vec{v}(\vec{\nabla}\phi \cdot) \right] \vec{S},$$

and a comparison with eq.(3.1) leads to

$$\frac{d\vec{L}}{dt} = \vec{\Omega} \times \vec{S}$$

with

$$\vec{\Omega} = -\frac{1}{2}\vec{\nabla} \times \vec{\zeta} - \frac{3}{2}\vec{v} \times \vec{\nabla}\phi$$

In order to see that a geometric phase arises in this situation, it is simplest to consider first a spin $\frac{1}{2}$ particle at rest. If the gravitational field is stationary, such as the one generated by a rotating body, then ζ is not equal to zero, and there is spin precession even for a particle at rest. The Hamiltonian becomes $-c\vec{\Omega} \times \vec{S}$ with $c = 1 + \phi$. The system is therefore analogous to a spin precessing in a uniform magnetic field, and the geometric phase can be computed as in ref.9.

For convenience, let us choose the coordinate axes so that $\vec{\Omega}$ is along the z-axis, that is,

$$H = -\mu\Omega\sigma_z$$

, with

$$\mu = \text{const.}$$

and

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

If the initial state is

$$|\psi(0)\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix},$$

then

$$|\psi(t)\rangle = e^{i\mu\Omega t\sigma_z} |\psi(0)\rangle = \begin{pmatrix} e^{i\mu\Omega t} \cos\theta/2 \\ e^{-i\mu\Omega t} \sin\theta/2 \end{pmatrix}.$$

The spin vector $\langle \psi | \vec{\sigma} | \psi \rangle$ precesses at an angle θ about the z-axis with a period $T = \pi/\mu\Omega$. (See fig. 3). Insertion of $|\psi(t)\rangle$ in the expression for β gives

$$\beta = ir - \mu\Omega t \cos \theta.$$

After one period. $\beta = (1 - \cos \theta)\pi$, thus confirming that the dynamical dependence drops out of $\beta(T)$.

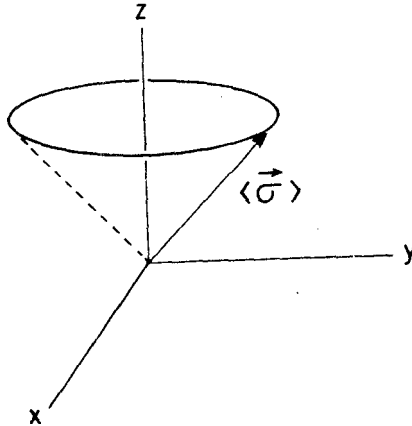


Fig. 3 - The spin vector $\langle \psi(t) | \vec{\sigma} | \psi(t) \rangle$ traces a curve in R^3 which determines a closed curve $\hat{\Gamma}$ in P_{χ} .

Once more the phase equals half the solid angle subtended by the curve traced by a vector; this is very much like the adiabatic case. but there the vector was the magnetic field. whereas here it is the spin.

When the particle is moving, the Hamiltonian becomes $-\vec{\Omega} \times \vec{S}$, where $\vec{\Omega}$ depends on the position and velocity of the particle, and \vec{S} is given by eq.(3.2). As a consequence. the motion is more complex, but the spin still precesses and formula (2.3) remains valid. There is no a priori reason why β should vanish for an arbitrary circuit $\hat{\Gamma}$.

4. CONCLUSIONS

The result of section 3 brings new members - those with purely gravitational interactions - into the class of systems that display a geometric phase. This result is interesting also because it provides a first quantizea framework for one to think about the elusive issue of anomalies. In analogy with chiral anomalies, it suggests that a quantum field theory of fermions in a gravitational field may develop

anomalies - a prediction which turns out to be true under certain **conditions**^{4,5}. In order to avoid misunderstandings we emphasize that this result does not imply symmetry **breaking** in the first-quantized system, which has a finite number of degrees of freedom. As pointed out in **ref.15**, the connection between geometric phases and Wess-Zumino terms in quantum mechanics is valid only under **adiabatic** conditions; if transitions between spin states take place, Berry phases with opposite signs add up to a vanishing net effect.

Several aspects of this work can be pursued further. It would be interesting to treat the particle motion in more generality: for this one needs a Hamiltonian description of a spinning particle in general relativity. In the case of chiral **anomalies**, the geometric phase can be expressed in terms of indices of Dirac operators; what happens when gravitational couplings are present? One can also investigate how the anomalous terms effect the quantization of the gauge sector⁷.

Finally, it is worthwhile to **call** attention to a point which arises in **connection** with the Lense-Thirring effect of a spinning particle in the vicinity of a large rotating mass. This effect indicates that it is the whole mass of distant **matter** that selects, amongst **all** frames, those which are inertial. On the other hand, **short-distance** effects determine whether spacetime symmetries are realized at **all**. Seemingly, these two disparate scales **operate** together - one specifying whether **inertial frames exist**; the other, what they are. It would be interesting to investigate this point further, looking for a possible connection between microscopic and cosmological scales, which is otherwise expected only in the very early stages of the universe.

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Resumo

Analisando-se o movimento de uma partícula quântica em um campo gravitacional, mostra-se, sob certas condições, a existência de uma fase geométrica. Descreve-se a relação deste resultado com o aparecimento de anomalias gravitacionais.