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Exact Solution for the Ising Model in a Strip with Random Distribution of Bonds

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Abstract We obtain an exact analytical expression for the free-energy of a system of Ising variables on a 2 x N strip with a random distribution of positive and negative bonds.

1. INTRODUCTION

A random distribution of inhomogeneities is essential for the macroscopic characterization of many physical systems. Among magnetic materials, spin glasses are an example of this. There. the random distribution of negative and positive exchange coupling constants between ions sets up a competition between ferro and antiferromagnetic ordering with results not yet fully understood. In the laboratory the non-homogeneity arises from the quenching of magnetic ions in random positions in a non-magnetic matrix. Theoretically we can model the fundamentals of the problem by placing the magnetic ions (spins variables) in a regular lattice and allowing a random distribution of positive and negative bonds.

The model hamiltonian for short-range interactions can be written as

$$H = -\sum_{i,j} J_{ij} S_i S_j \tag{1}$$

with S_i the spin at sites i and J_{ij} the coupling constant between spins at i and j. The summation will be restricted to the pairs $<i, j>$ of nearest neighbors. For the Ising model $S_i = \pm 1$. Onsager[1] obtained the exact thermodynamic functions for the uniform $(J_{ij} = J)$ tws-dimensional version of model eq.(1) through a formally complex model. simplified afterwards by others (Kaufmann[2]. Kac and Ward[3]. Vdovichenco[4]. Schultz, Mattis and Lieb[5]).

The model described by the Hamiltonian eq.(1) shows the important property of invariance of the partition function under the local transformation

$$J_{ij} \rightarrow -J_{ij} \tag{2}$$

of all bonds joining the site i to its neighbors.

This shows that if it is possible to go from one distribution $\{J_{ij}\}$ of bonds to another $\{J'_{ij}\}$, through local transformations as described above, the thermodynamics of the system will be the same. That equivalence is better shown if we introduce the concept of ***frustration*** (Toulouse[6]): on planar lattice with interactions only between nearest neighbors, one says that a plaquette (a primitive cell) is frustrated if the number of negative bonds in its perimeter is odd; otherwise the plaquette is non-frustrated. It is easily shown that a configuration of frustration is not altered by the local transformation eq.(2). Then all distributions of frustrations have the same partition function.

Exact solutions for planar models with J_{ij} assuming values $+J$ and $-J$, with frustrations, are known only for a few regular distributions (fully frustrated model (Villain[7]), layered model (Hoever, Wolff and Zittartz[8]). It will be desirable to have the solutions for a random distribution of frustrations, in order to get a better understanding of the role of the inhomogeneities in the behavior of such systems, and also to test the validity of the approximation methods. Solutions for those models are important for the study of spin glasses. The lack of any general solution makes exact solutions welcome, even for very simplified models.

2. THE SOLUTION FOR A STRIP WITH RANDOM DISTRIBUTION OF BONDS

In this work we find an exact expression for the free-energy per cell of a system of Ising variables in a 2 x N strip with a random distribution of $\pm J$ bonds between nearest neighbors, in the absence of external field. The basic assumption is that the free energy of the infinite strip shows a gaussian probability distribution. Because the interactions are random and short ranged, the total energy can be thought of as the sum of random energies of a large number of subsystems (neglecting interface effects) and therefore has a gaussian distribution. Strips with different distributions of bonds have been studied by other authors (Huse and Morgenstern[9]; Derrida, Vannimenus and Pomeau[10]).

The system is shown schematically in fig. 1. S_n is the spin at site n of the upper row, S'_n the corresponding spin in the lower row. The coupling constants between neighbor spins in the same row were taken to be positive (J). The bonds between neighbors in different lines can be negative or positive $(J_n = \pm J)$. The invariance of the partition function under local transformations eq.(2) assures that there is no lack of generality if we take all horizontal bonds as positive.

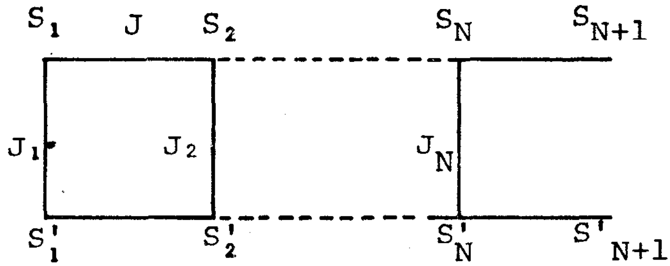

Fig. 1 - Schematic representation of a strip with vertical coupling constants $J_n = \pm J$ randomly chosen.

The hamiltonian of the system is

$$H = -\sum_{n=1}^{N} J_n S_n S'_n - J\sum_{n=1}^{N}(S_n S_{n+1} + S'_n S'_{n+1} \tag{3}$$

Assuming cylindrical boundary conditions.

$$S_{N+1} \equiv S_1, \quad S'_{N+1} \equiv S'_l \tag{4}$$

the transfer matrix method allows us to write the partition function as the trace of a product of matrices $(\beta = l/k_\beta T)$

$$Z = \sum_{S_n, S'_n} exp(-\beta H) = (2\, sinh\, 2\beta J)^N Tr \prod_{n=1} V_n \tag{5}$$

where the 4 x 4 matrices V can be expressed of Pauli spins operators σ_x, σ_z acting on the upper row and σ'_x, σ'_z acting on the lower row

$$V_n = exp(\theta\sigma_x + \theta\sigma'_x)\, exp(\phi_n \sigma_z \sigma'_z) \tag{6}$$

with

$$\theta = tanh^{-1} e^{2\beta J} \tag{7}$$

$$\phi_n = \beta J_n \tag{8}$$

Choosing symmetrical and antisymmetrical combinations of the states of occupation numbers as basis. V_n takes the following explicit form (diagonalized by blocks):

$$V_n = \begin{vmatrix} e^{\phi_n}\cosh 2\theta & e^{-\phi_n}\sinh 2\theta & 0 & 0 \\ e^{\phi_n}\sinh 2\theta & e^{-\phi_n}\cosh 2\theta & 0 & 0 \\ 0 & 0 & e^{\phi_n} & 0 \\ 0 & 0 & 0 & e^{-\phi_n} \end{vmatrix} \tag{9}$$

The difficulty is in calculating the trace of the product of the upper blocks,

$$\begin{vmatrix} e^{\phi_n}\cosh 2\theta & e^{-\phi_n}\sinh 2\theta \\ e^{\phi_n}\sinh 2\theta & e^{-\phi_n}\cosh 2\theta \end{vmatrix} = W_n \tag{10}$$

The product of these matrices is not commutative for different J_n's and its general expression does not display a simple form.

Since $Tr\prod_n W_n$ has to be real and

$$det\, W_n = 1 \tag{11}$$

the eigenvalues of ΠW_n can be written as $e^{\pm\Gamma}$, where Γ is real positive, of order N. Then

$$Tr\Pi W_n = 2\cosh\Gamma \tag{12}$$

The free energy f per cell, averaged over the configurations of J_n, will be

$$-\beta f = \left\langle \frac{1}{N}\ln Z \right\rangle = \ln(2\sinh 2\beta J) + \frac{1}{N} < \Gamma > \tag{13}$$

In the thermodynamic limit most of Γ's are distributed randomly around $<\Gamma>$ with width $<(\Delta\Gamma)^2>^{1/2}$. We can assume a gaussian distribution and prove the following relations

$$<\cosh\Gamma> = e^{\frac{1}{2}<(\Delta\Gamma)^2>}\cosh <\Gamma> \tag{14}$$

$$<(\cosh\Gamma)^2> = e^{2<(\Delta\Gamma)^2>}\cosh^2 <\Gamma> \tag{15}$$

From eq.(14) and eq.(15) we have

$$lncosh < \Gamma >= 2\, ln < cosh\, \Gamma > -\frac{1}{2}\, ln < (cosh\, \Gamma)^2 > \tag{16}$$

$< \Gamma >$ is of order **N.** so in the limit $N \rightarrow \infty$,

$$-\beta f = ln(2\, sinh\, 2\beta J) + \frac{2}{N}\, ln < cosh\, \Gamma > -\frac{1}{2N}\, ln < (cosh \Gamma)^2 > \tag{17}$$

Let the probability of $J_n = -J$ be p and of $J_n = J$ be $1-p$. Then.

$$\begin{aligned} < cosh\, \Gamma > &= \frac{1}{2} < Tr\Pi W_n > = \frac{1}{2} Tr\Big[(1-p)W_+ + pW_-\Big]^N \\ &= \frac{1}{2}\Big[cosh\, \nu + (sinh^2 \nu - 4p(1-p)\, sinh^2 \phi)^{\frac{1}{2}}\Big]^N \\ &+ \frac{1}{2}\Big[cosh\, \nu - (sinh^2 \nu - 4p(1-p)\, sinh^2 \phi)^{\frac{1}{2}}\Big]^N \end{aligned} \tag{18}$$

where we define

$$W_\pm = \begin{vmatrix} e^{\pm\phi} cosh\, 2\theta & e^{\mp\phi} sinh\, 2\theta \\ e^{\pm\phi} sinh\, 2\theta & e^{\mp\phi} cosh\, 2\theta \end{vmatrix} \tag{19}$$

with

$$cosh\, \nu = cosh\, \phi\, cosh\, 28 \tag{20}$$

and

$$\phi = \beta J \tag{21}$$

On the other hand.

$$\begin{aligned} < (cosh\, \Gamma)^2 > &= \frac{1}{4}\Big\langle \Big(Tr \prod_n W_n\Big) \Big(Tr \prod_n W_n\Big) \Big\rangle \\ &= \frac{1}{4}\Big\langle Tr\Big[\Big(\prod_n W_n\Big) \otimes \Big(\prod_n W_n\Big)\Big]\Big\rangle \\ &= \frac{1}{4}\Big\langle Tr \prod_n \Big(W_n \otimes W_n\Big)\Big\rangle \\ &= \frac{1}{4} Tr\Big[pW_- \otimes W_- + (1-p) W_+ \otimes W_+\Big]^N \end{aligned} \tag{22}$$

(The notation $\otimes$ means direct product of matrices).

The eigenvalues of the matrix $pW_- \otimes W_- + (1-p)W_+ \otimes W_+$ are *1* and the three (real) roots of the cubie equation *

$$x^3 - (4\cosh^2\nu - 1)x^2 + [4\cosh^2\nu - 1 + 4p(1-p)(\sinh^2 2\phi + 2)]x$$
$$- 1 - 4p(1-p)\sinh^2 2\phi = 0 \qquad (23)$$

In the thermodynamic limit we need only the largest **root** of the above equation

$$x_{max} = \frac{1}{3}(4\cosh^2\nu - 1) + 2r\cos(\alpha/3) \qquad (24)$$

where

$$\cosh a = q/r^3 \qquad (25)$$

with

$$q = \left(\cosh^2\nu - 1\right)^2\left(4\cosh^2\nu - 5\right) + \frac{1}{2}$$
$$- 2p(1-p)\left[\frac{1}{3}\left(4\cosh^2\nu - 1\right)\left(\sinh^2 2\phi + 2\right) - \sinh^2 2\phi\right] \qquad (26)$$

and

$$r = \left[\frac{4}{9}\sinh^2\nu(4\cosh^2\nu - 1) + \frac{4}{3}p(1-p)(\sinh^2 24 + 2)\right]^{\frac{1}{2}} \qquad (27)$$

Then.

$$-\beta f = \ln(2\sinh 24) + 2\ln\left[\cosh\nu + \right.$$
$$\left. + \left(\sinh^2\nu - 4p(1-p)\sinh^2\phi\right)^{\frac{1}{2}}\right] - \frac{1}{2}\ln x_{max} \qquad (28)$$

For a random distribution of positive and **negative** bonds between lines with probability p and $1-p$ the strip will have a random distribution of frustrated

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* A simple graphical analysis shows that if the three roots of **eq.(23)** are real for $p = 0$ and $p = 1/2$. then they will be real for any other value of p between 0 and **I.** The solutions x $= 1$, for $p = 0$, and $x = \cosh 2\phi$ for $p = l/2$ are easily obtained by inspections. **In** both cases the remaining quadratic equations have real roots.

plaquettes with probability $\eta = 2p(1-p)$. Notice that with a random distribution of bonds it is not possible to get more than fifty percent of frustrated plaquettes.

Curves of the free-energy per cell for several densities od randomly distributed frustrated plaquettes are shown in fig. 2. The results for $\eta = 2$ (non-frustrated) and $\eta = 1$ (fully frustrated) are obtained directly from eqs.(15) and (9).

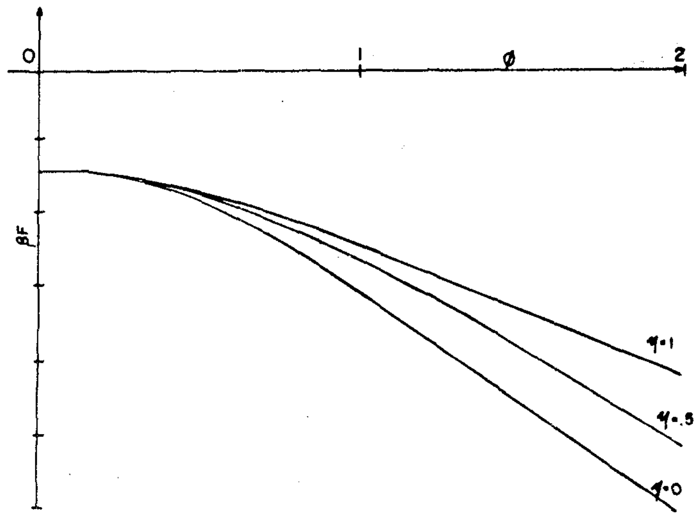

Fig.2 - Free energy per cell as a function of J for several densities of frustrated cells (V).

3. CONCLUSIONS

As expected the behavior of the free-energy function ís analytic over all range of the parameters η (density of frustration) and ϕ (temperature) because the system is effectively unidimensional in the thermodynamic limit and the interactions are short-ranged. We note that the effect of the frustration is only to raise the free energy.

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Resumo

Obtemos uma expressão analítica exata para a energia livre de um sistema de variáveis de Ising em uma tira 2 x N com uma distribuição aleatória de ligações positivas e negativas.