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A Membrane Wave Equation for Q.C.D. (SU (∞))

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Abstract We propose a quantum membrane wave functional describing the interaction between a colored $SU(N_c)$ membrane and a quantized Yang-Mills field. Additionally, we deduce its associated wave equation in the t'Hooft $N_c \to \infty$ limit.

1. INTRODUCTION

In recent years representations for Quantum Chromodynamics as extended objects have been pursued by several authors[1,2,3,4]. Among these, the representation of the meson wave functional by the quantum amplitude of a closed trajectory of a colored particle in the vacuum of a pure Yang-Mills field has strongly suggested the equivalence between QCD $(SU(\infty))$ and a dynamic of strings[1,2].

In this brief report, we propose to replace the one-dimensional closed trajectory in the above quantum amplitude by a two-dimensional membrane (surface) possessing color degrees. Thus, we deduce (formally) its associated membrane wave equation in the t'Hooft topological limit of large number of colors $N_c = +\infty$.

2. THE MEMBRANE WAVE FUNCTIONAL

Let us start our analysis by considering the problem of associating a wave functional for a membrane C possessing $SU(N)$ color degrees of freedom interacting with an external quantized Yang-Mills field $A_\mu(X)$.

The colored membrane is characterized by two fields: first, by the usual (bosonic) vector position $X_\mu(\xi)$, $\xi \epsilon D$ ($\mu = 1,\ldots,D$, where D is the space-time dimension), and second, by the membrane color variable $g(\xi)$ which is an element in the fundamental representation of the $SU(N)$ group. Here, we have fixed the two-dimensional flat domain D to be the rectangle

$$D_{|0,2\pi|\times|0,T|} = \{(\xi_0,\xi_1);\quad 0 \leqslant \xi_0 \leqslant 2\pi \quad \text{and} \quad 0 \leqslant \xi_1 \leqslant T\}.$$

The classical action for this membrane is naturally given by[5]

$$S = S_0 + S_1^{(B)} \tag{1}$$

with

$$S_0 = \frac{1}{2}\int_D d^2\xi(\partial_a X^\mu \partial_a X^\mu)(\xi) \tag{1a}$$

$$S_1^{(B)} = \frac{1}{4\pi m}\int_D T_R^{(c)}(g^{-1}\partial_a g)^2(\xi)d^2\xi + 4\pi i\Gamma_{WZ}[g] \ , \tag{1b}$$

where $\Gamma_{WZ}[g]$ denotes the two-dimensional Wess-Zumino functional. Its existence, together with the integer m in the above written σ-model, the action of $g(\xi)$'s afflord us to consider the more suitable fermionic equivalent action

$$S_1^{(F)} = \int_D \bar\psi(\xi)(i\gamma_a\partial_a)\psi(\xi)d^2\xi \ , \tag{2}$$

where the two-dimensional Dirac field $\psi(\xi)$ belongs to the fermionic fundamental $SU(N)$ representation.

At this point, the simplest action taking into account the interaction with the external non-Abelian field is given by*

$$S^{\text{int}}[\psi(\xi);A_\mu(X)] = \int_D \bar\psi(\xi)(\gamma_a\partial^a X^\mu(\xi)A_\mu(X(\xi))\psi(\xi)d^2\xi \tag{3}$$

The complete classical interacring action (eqs. (1a), (2) and (3)) is invariant under the gauge transformations

$$\begin{aligned} A_\mu(X_\mu(\xi)) &\to (h^{-1}A_\mu h + h^{-1}\partial_\mu h)(X_\mu(\xi)) \\ \psi(\xi) &\to h(X_\mu(\xi))\psi(\xi) \\ \bar\psi(\xi) &\to \bar\psi(\xi)h^{-1}(X_\mu(\xi)) \ . \end{aligned} \tag{4}$$

Before turning to the conçtruction of a quantum wave functional for the above system, it is instructive to remark that eqs. (1a),

*(for a different framework see ref.7)

(2) and (3) are the membrane generalizations of the analogous formulae in the one-dimensional string case, where the colored string is described by the position vector $X_\mu(\sigma)$ and the one-dimensional complex fermion (Grassmanian) field $\{\theta(\sigma), \theta^*(\sigma)\}$ in the $SU(N)$ fundamental representation. The associated action is

$$S[X_\mu(\sigma),\theta(\sigma),\theta^*(\sigma),A_\mu(X_\mu(\sigma))] = \tag{5}$$
$$= \int_0^T \frac{1}{2}\dot{X}_\mu(\sigma)^2 d\sigma + \int_0^T \theta^*(\sigma)\dot{\theta}(\sigma) + \int_0^T \dot{X}^\mu(\sigma)A^I_\mu(X(\sigma))(\theta(\sigma)\lambda_I\theta^*(\sigma))d\sigma$$

where $\{\lambda_I\}$ denotes the Hermitian generators of the $SU(N)$ Lie algebra.

In this string case, a quantum wave functional is given by the following path integral[8]

$$W[X_\mu(\sigma),A_\mu(X)] = \tag{6}$$
$$= \int d[\theta(\sigma)]d[\theta^*(\sigma)] \sum_{\alpha=1}^{N^2-1} \theta_\alpha(0)\theta_\alpha(T) \exp\{-S[X_\mu(\sigma),\theta(\sigma),\theta^*(\sigma),A_\mu(X_\mu(\sigma))]\}$$

which leads to the well-known Wilson Loop factor defined by the closed string $\{X_\mu(\sigma)\}$. The complete quantum wave functional is defined by the average $\langle W[X_\mu(\sigma),A_\mu(X)]\rangle$ where $\langle\rangle$ denotes the partition functional of the pure Yang-Mills theory.

We shall now use eq. (6) to propose the following functional integral as a quantum wave functional for a $SU(N)$ colored membrane C interacting with the quantum vacuum of a $SU(N)$ Yang-Mills theory:

$$\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma]) \overset{\mathrm{DEF}}{=} \sum_{R=1}^{N^2-1} \int \partial[\psi(\xi)]\,\partial[\bar{\psi}(\xi)]\,(\bar{\psi}(0,0)\,\frac{\lambda^R}{N_c}\,\psi(2\;,0))$$
$$\exp\{-S[X_\mu(\xi),A_\mu(X(\xi),\psi(\xi)]\}\ . \tag{7}$$

Notice that our above proposed memb'rane phase factor $\mathrm{Tr}^{\mathrm{color}}(\psi[\Sigma])$ is a 2×2 matrix in the flat domain D (a = 1,2).

In order to deduce a closed wave functional for the quantum average $<Tr^{color}(\psi[\Sigma])>$ in the limit $N_c = +\infty$, we proceed as in the string case[1,2] by shifting the $A_\mu(X)$ field variable, which by its turn, produces the following result $(\lambda_0^2 = \lim_{N^c \to \infty} (g_0^2 N_c) < \infty)$

$$\frac{1}{4\lambda_0^2} <Tr^{color}\{(D_\mu F_{\mu\nu})(X)\psi[\Sigma]\}> =$$

$$= \int_D \delta^{(D)}(X - X_\mu(\sigma,\xi))\, \partial_c X^\mu(\sigma,\tau) <Tr^{color}\psi[\Sigma_1]> <Tr^{color}\gamma^{(c)}\psi[\Sigma_2]>, \qquad (8)$$

where the split membranes $\Sigma_{(1)}$ and $\Sigma_{(2)}$ are respectively defined by the restriction of the mapping $X_\mu(\xi_1,\xi_2)$ for the (split) domains

$$D_{(1)} = \{(\xi_0,\xi_1);\ \ 0 \leqslant \xi_0 \leqslant \sigma\ ;\ \ 0 \leqslant \xi_1 \leqslant \tau\}$$

and

$$D_{(2)} = \{(\xi_0,\xi_1)\,|\sigma \leqslant \xi_0 \leqslant 2\pi\ ;\ \tau \leqslant \xi_1 \leqslant T\}\ .$$

It is now convenient to multiply both sides of eq.(8) by the membrane current density

$$J_{(a)}(X) = \delta^{(D)}(X - X_\mu(\bar\sigma,\bar\tau)\,\partial_a X^\mu(\bar\sigma,\bar\tau)$$

and integrate out the result relative to the space-time variable X. So, we get the result

$$<Tr^{color}\{(D_\mu F_{\mu\nu})(X^\mu(\bar\sigma,\bar\tau))\,\partial_a X^\mu(\bar\sigma,\bar\tau)\psi[\Sigma]\}> =$$

$$= 4\lambda_0^2 \int_D \delta^{(D)}(X_\mu(\bar\sigma,\bar\tau) - X_\mu(\sigma,\tau))\,\partial_c X^\mu(\sigma,\tau)\,\partial_a X^\mu(\bar\sigma,\tau) \qquad (9)$$

$$<Tr^{color}\psi[\bar\Sigma_{(1)}]> <Tr^{color}\ \gamma^{(c)}\psi[\Sigma_{(2)}]>\ .$$

In order to write the left-hand side of the above result in a form similar to the membrane wave equation, we use the relations

$$\left\{\frac{\delta}{\delta X_\mu(\sigma,\tau)}\right\} \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma)) = \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma_1)F_{\mu\nu}(X(\sigma,\tau))\,\partial_e X^\nu(\sigma,\tau)\gamma^{(e)}\psi(\Sigma_2)),$$

(10a)

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\bar\sigma,\bar\tau)\delta X^\mu(\sigma,\tau)}\right\} \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma)) =$$

$$= \mathrm{Tr}^{\mathrm{color}}(D_\mu F_{\mu\nu}(X_\mu(\bar\sigma,\bar\tau))\,\partial_c X^\nu(\sigma,\tau)\gamma^{(c)}\psi(\Sigma)) \quad , \qquad (10b)$$

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\bar\sigma,\bar\tau)\delta X^\mu(\bar\sigma,\bar\tau)}\right\} \equiv$$

$$\equiv \lim_{\varepsilon\to 0^+}\int_{-\varepsilon}^{\varepsilon} d\beta\,\frac{\delta^2}{\delta X_\mu(\bar\sigma+\beta,\bar\tau+\beta)\delta X^\mu(\bar\sigma-\beta,\bar\tau-\beta)} \quad . \qquad (10c)$$

By substituting eq. (10b) into eq.(9), we obtain our proposed membrane version of the string Migdal-Makkenko wave equation (compare with eq.(9), ref. 2, and eq. (7), ref.4).

$$P_F\left\{\frac{\delta^2}{\delta X_\mu(\bar\sigma,\bar\tau)\delta X^\mu(\bar\sigma,\bar\tau)}\right\} \langle \mathrm{Tr}^{\mathrm{color}}(\psi(\Sigma))\rangle =$$

$$= 4\lambda_0^2 \int_D \delta^{(D)}(X_\mu(\bar\sigma,\bar\tau) - X_\mu(\sigma,\tau))\,\partial_b X^\mu(\sigma,\tau)\,\partial_c X^\mu(\bar\sigma,\bar\tau)$$

$$\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(b)}\psi[\Sigma_{(1)}]\rangle\,\langle \mathrm{Tr}^{\mathrm{color}}\gamma^{(c)}\psi[\Sigma_{(2)}]\rangle \qquad (11)$$

To summarize, we propose a membrane version of the string Migdal-Makkenko loop wave equation in $SU(\infty)$,

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REFERENCES

1. A.A.Migdal, Phys. Rev. 102, 199 (1983).
2. Luiz C.L.Botelho, Phys. Lett. *169*B, 428 (1985).
3. P.Hasenfratz and J.Kuti, Phys.Rev. *40*C, 75 (1978); P.A.Collins and R.W.Tucker, Nucl. Phys. B*112*, 150 (1976).
4. I.Ya Aref'eva, Phys. Lett. *95*B, 269 (1980).
5. P.Goddard, Nucl. Phys. B*116*, 157 (1976); P.Freund, Phys. Lett.*151* B, 387 (1985).
6. E.Witten, Nucl. Phys. 1455, 110 (1978).
7. N.K.Nielsen, Nucl. Phys. 7675, 249 (1980).
8. M.B.Halpern, A. Jevicki and P.Senjanovic, Phys. Rev. *16*D, 2476 (1977).
9. C.G.Callan, R.Rhom and E.Martinec, Strings in Background Fields,Princeton Preprint, 1986.

Resumo

Considerando a interação entre uma membrana possuindo carga colorida $SU(N_c)$ e o vácuo quântico na teoria de Yang-Mills $SU(N_c)$, deduzimos uma versão tipo membrana para a equação de Migdal-Makkenko na QCD $(SU(\infty))$.