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Exact Calculation of the Devil's Staircase of a Discontinuous Linear Circle Map

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Abstract We analyse a farnily of discontinuous orientation-preserving maps of the circle onto itself. We obtain analytical expression for the location and for the lengthof each stepof the devil's staircase. We show that the staircase is complete and that the set of points corresponding to irrational winding numbers is a Cantor set of fractal dimension $D = 0$.

Mappings have been used extensively to describe dynamic systems that exhibit periodic as well as chaotic behavior. They arealso capable of describing the static properties of systems that possess modulated structures[1-3]. The standard map[4] which is studied in connection with the Frenkel-Kontorowa model of modulated structures is an example of such maps. For maps of the circle onto itself which are continuous and invertible (homeomorphisms) the following results have been obtained[5,6]. The winding number $W(\Omega)$ is well defined and is a continuous function of Ω, the parameter that describes a family of mappings. For homeomorphisms that are not pure rotations, $W(\Omega)$ exhibits a plateau for each rational value of W, that is, the function $W(\Omega)$ is a devil's staircase. If, in addition, the maps and their inverses are differentiable (diffeomorphisms), then the devil's staircase is incomplete: the set of points corresponding to irrational winding numbers has a nonzero measure. Also, the orbit corresponding to an irrational winding number is topologically equivalent to a simple rotation what means that the orbit is one-dimensional.

In this paper we analyse a family of orientation-preservingmaps of the circle onto itself which are not homeomorphisms. Nevertheless, some of the above results still hold. The winding number is well defined and constitutes a devil's staircase. We obtain some exact results concerning the devil's staircase including an analytical expression forthe width of the plateau corresponding to each rational winding number. Using this expression we prove that the devil's staírcase is complete and the set of points corresponding to irrational winding numbers is a Cantor set of fractal dimension $D=0$.

We study a family of maps of a circle onto itself $x \to \tilde{f}(x)$, defined on a unit interval, $0 < x < 1$, and described by two parameters t and Ω (see fig.1). It is given by $\tilde{f}(x) = f(x)$ mod 1 where

$$f(x) = t(x+\Omega-n) + n \quad , \quad n = \left[x + \Omega + \frac{1}{2}\right] \tag{1}$$

$[x]$ denotes the integer part of x. The parameters t is restricted to the interval $[0,1]$ so that the maps are orientation-preserving. When $t = 1$, the maps reduce to pure rotations. For $t \neq 1$, the function $f(x)$ has a discontinuity when $(x + R + \frac{1}{2})$ is an integer. We restrict the parameter Ω to lie in the interval $[0, \frac{1}{2}]$.

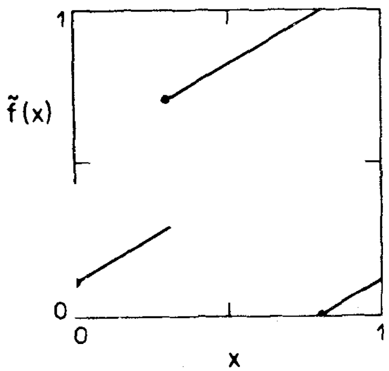

Fig.1 - The map $x \to \tilde{f}(x)$ for a value of R between zero and 1/2. The slope of the segments is equal to t.

The study of such a family of maps is motivated by its relation to a modified Frenkel-Kontorowa model where the cosine periodic potential is replaced by a contiriuous piecewise parabolic periodic potential[7]. Suppose this model is studied by the method due to Griffiths and Chou[8] where a nonlinear eigenvalue equation is set up for an *effective potential*. If the *effective potential* is approximated by the potential itself then the map which gives the position of the atoms becomes identical to the map studied here. Although, this seems at first a not so good approximation the results obtained look very sensible. To make contact with this modified Frenkel-Kontorowa model, the parameter Ω should be interpreted as the misfit parameter and $(1-t)/t = h$ the amplitude of the periodic potential. That is, the modified Frenkel-Kontorowa Hamiltonian should be written in the form

$$H(\{x_j\}) = \sum_j \{\tfrac{1}{2}(x_{j+1} - x_j - \Omega)^2 + \frac{h}{2}(x_j - n_j)^2\} ,$$

where

$$n_j = [x_j + 1/2] .$$

As long as the derivative of $\tilde{f}(x)$ exists, it equals t, so that the Lyapunov exponent of any attractor equals log t, except when the attractor has a point x for which $f(x)$ is discontinuous. This exception, however, will not occur whenever the winding number $W(t,\Omega)$ is rational.

For a fixed value of t, let $(\Omega^-(L,N,t), \Omega^+(L,N,t))$ be the open interval in the Ω-axis for which the map has a winding number $W(t,\Omega) = L/N$ where L is prime relative to N and L < N. A necessary and sufficient condition[3] that the map has this rational winding number is that the equation $f^N(x_0) = x_0 + L$ has a solution x_0 in the interval $[0,1)$. If we define x_j by $x_j = f(x_{j-1})$, this equation is equivalent to the set of equations

$$x_j = t(x_{j-1} + \Omega - n_j) + n_j , \quad n_j = \left[x_{j-1} + \Omega + \tfrac{1}{2}\right] , \qquad (2)$$

for $j = 1,2,\ldots,N$, with the boundary condition $x_N = x_0 + L$. If the points $x_0, x_1, \ldots, x_{N-1}$ are obtained then the N points of the limit cycle $\tilde{x}_0, \tilde{x}_1, \ldots, \tilde{x}_{N-1}$ are given by $\tilde{x}_{j_{N-j}} = x_j$ mod 1.

Multiplying eq. (2) by t^{N-j} and summing over j from 1 till N, we get the following equation for x_0

$$(x_0 + \Omega)(1 - t^N) + L = \Omega\,\frac{1-t^N}{1-t} - \sum_{j=0}^{N} (n_{j+1} - n_j)t^{N-j} ,$$

where $n_{N+1} = n_0 = 0$. This equation, however, is formal since the integers n_1, n_2, $\dots, n_N$ are yet unknown. It can also be interpreted as an equation for Ω if x_0 is given in the interval $[0,1)$. By taking the limits $x_0 \to (\frac{1}{2}-\Omega)^-$ and $x_0 \to (\frac{1}{2}-\Omega)^+$ we get Ω^+ and Ω^-, rerpectively. Taking into account that $n_1 = 0$ if $x_0 < \frac{1}{2} - \Omega$ and that $n_1 = 1$ if $x_0 \geq \frac{1}{2} - \Omega$, and that in both limits $n_N = L$ and $n_2 = 1$ (as long as $0 \leq \Omega < \frac{1}{2}$) we have

$$\Omega^{\pm} = \left(\frac{1-t}{1-t^N}\right)\left[\frac{1}{2}\,(1+t^{N-1}) \pm \frac{t^{N-1}}{2}\,(1-t) + \sum_{j=2}^{N-1} (n_{j+1} - n_j)\, t^{N-j}\right] . \qquad (3)$$

From this expression we see that the length $\omega = \Omega^+ - \Omega^-$ of the interval (Ω^-, Ω^+) can be obtained independetly of the sequence $n_1, n_2, \dots n_N$, since both Ω^- and Ω^+ are given by the same sequence as shown in the Appendix. It is given by

$$\omega(N,t) = \frac{t^{N-1}(1-t)^2}{1-t^N} \quad , \qquad (4)$$

and is independent of L.

The sum of the lengths of all plateaus are given by

$$S = \sum_{N=1}^{\infty} \phi(N)\omega(N,t) \quad ,$$

where $\phi(N)$ is the number of irreducible fraction with denominator N (the Euler totient function). Using the generating function expansion for $\phi(N)$ [10]

$$\frac{t}{(1-t)^2} = \sum_{N=1}^{\infty} \phi(N)\, \frac{t^N}{1-t^N} \quad ,$$

valid for $t < 1$, we obtain the result $S = 1$, so that the devil's staircase is complete.

For a fixed t, let C be the Cantor set of points of the devil's staircase in the R-axis corresponding to irrational winding numbers. The fractal dimension D of the set C can be calculated as follows. Let $\varepsilon(N)$ be a certain scale and $S(N)$ the sum of the widths of the plateaus whose lengths are larger than $\varepsilon(N)$, and $\tilde{S}(N) = 1 - S(N)$. The fractal dimension D is given by

$$D = \lim_{N\to\infty} \frac{\log \tilde{S}/\varepsilon}{\log 1/\varepsilon}$$

By choosing $\varepsilon(N) = \omega(N,t)$ we have

$$\tilde{S} = \sum_{n=N}^{\infty} \phi(n)\omega(n,t) \quad .$$

An upper bound for $\tilde{S}(N)$ is obtained by observing that $\phi(n) \leqslant n-1$ from which we get $D \leqslant 0$. Therefore, $D = 0$.

In the Appendix we show that the sequence $n_2, n_3, \ldots, n_N$ is given by

$$n_{j+1} = \left[j \frac{L}{N}\right] + 1 \quad ,$$

for $j = 1,2,\ldots,N-1$, so that the expression (3) gives actually the limits of the plateau corresponding to the winding number L/N. Defining the set of numbers σ_1, $\sigma_2, \ldots, \sigma_N$, which take only the values 0 and 1, by

$$\sigma_j = \left[j \frac{L}{N}\right] - \left[(j-1) \frac{L}{N}\right] \quad ,$$

for $j = 1,2,\ldots, N$, we have the following expression for Ω^- and Ω^+

$$\Omega^{\pm} = \left(\frac{1-t}{1-t^N}\right) \left[\frac{1}{2} (1+t^{N-1}) \pm \frac{t^{N-1}}{2} (1-t) \sum_{j=2}^{N-1} \sigma_j \, t^{N-j}\right] \quad .$$

As an example we have $(\sigma_1,\sigma_2,\ldots,\sigma_N)$ = (0101101011011) for the case $L=8$ and $N=13$. Since σ_j has the property $\sigma_{N-j+1} = \sigma_j$ for $j = 2,3,\ldots, N-1$ we can write also

$$\Omega^{\pm} = \left(\frac{1-t}{1-t^N}\right) \left[\frac{1}{2} (1+t^{N-1}) \pm \frac{t^{N-1}}{2} (1-t) + \sum_{j=2}^{N-1} \sigma_j \, t^{j-1}\right] \quad . \qquad (5)$$

In the limit $t\to 1$ of pure rotation we obtain $\Omega^{\pm} \to L/N$ by using the property $\sum_{j=1}^{N} \sigma_j = L$.

Figure 2 shows the phase diagram in the variables Ω and $h = (1-t)/t$. Regions corresponding to several rational winding numbers are shown. The last enlargement of the figure presents regions with winding

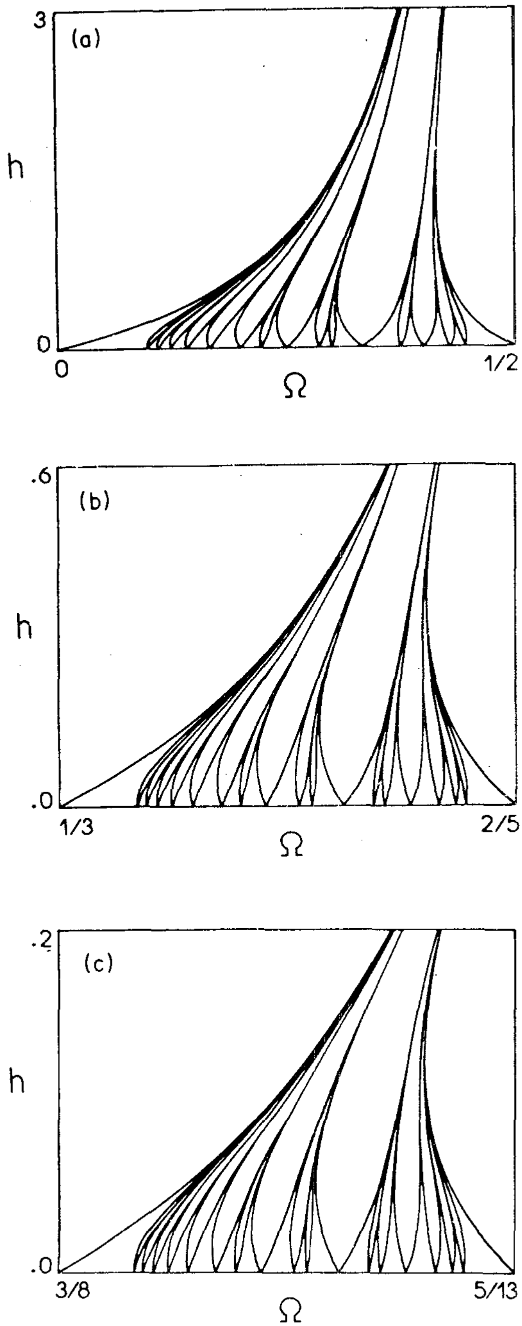

Fig.2 - Phase diagram in the variables Ω and $h = (1-t)/t$ where (c) and (b) are two successive enlargments of (a). Regions with rational winding numbers L/N are shown in (a), (b), and (c) with N up to 10, 29 and 77, respectively.

number L/N up to N = 77. Notice that the widths of plateaus behave as h/N when $h \to 0$.

Let us consider the case of an irrational winding number α. The locus of points in the phase diagram such that $W(t,\Omega) = \alpha$ is obtained from eq. (5) by taking the limit $N \to \infty$ with σ_j now given by

$$\sigma_j = [j\alpha] - [(j-1)\alpha] ,$$

for $j = 1,2,3,\ldots$. In this limit both Ω^- and Ω^+ approach $\Omega^*(\alpha,t)$ given by

$$\Omega = (1-t) \left\{ \frac{1}{2} + \sum_{j=2}^{\infty} \sigma_j \, t^{j-1} \right\} .$$

We discuss now the local scaling property of the phase diagram around the point $h = 0$ and $\Omega = \alpha$ where α is an irrational number. We will consider only the case where $\alpha = (3-\sqrt{5})/2$, the square of the golden mean $\gamma = (\sqrt{5}-1)/2$. The number α is the limit of the sequence F_1/F_3, F_2/F_4, $\ldots, F_i/F_{i+2}, \ldots$ where F_i are the Fibonacci numbers defined by $F_{i+2} = F_{i+1} + F_i$ with $F_1 = 0$ and $F_2 = 1$. If i is odd then $F_i/F_{i+2} < \alpha < F_{i+1}/F_{i+3}$. Given an odd i we define the following change of scales

$$h \to X = \frac{1}{2} F_{i+3} \, h ,$$

$$\Omega \to Y = F_{i+2} \, F_{i+3} (\Omega - \alpha) ,$$

and call this transformation T_i. For sufficient large i, this transformation leads to a phase diagram in the XY plane, showm in fig.3, whose pattern is independent of i.

We have found numerically that the regions of the plane $h\Omega$ corresponding to the winding numbers $(\ell F_{i+1} + mF_i)/(\ell F_{i+3} + mF_{i+2})$ and $(\ell F_{k+1} + mF_k)/(\ell F_{k+3} + mF_{k+2})$ with ℓ and m nonnegative integers and relative primes, will map in one and the same region of the XY plane by the transformations T_i and T_k, respectively, for sufficient large i and k. This region will be labeled (ℓ,m).

The phase diagram in the XY plane, shown in fig.3, has the following properties. The width of the region (ℓ,m) vanishes as $2\gamma X(\ell+\gamma m)$ when $X \to 0$, and touches the Y-axis at the point

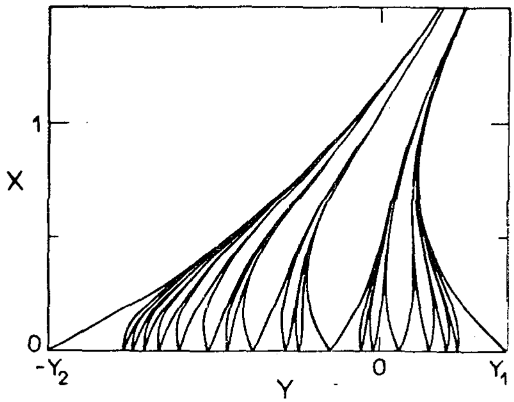

Fig.3 - Phase diagram in the plane XY defined by changing the original Ω and h scales around the point $h = 0$ and $\Omega = \alpha = (3-\sqrt{5})/2 = 0.3819\ldots$. The values of Y_1 and Y_2 are $Y_1 = \alpha/(1+\alpha) = (5-\sqrt{5})/10 = 0.2763\ldots$ and $Y_2 = 1/(1+\alpha) = (5+\sqrt{5})/10 = 0.7236\ldots$.

$$Y = \frac{\gamma}{1-\gamma^2}\,\frac{\gamma\ell - m}{\ell + \gamma m} .$$

The pattern of the phase diagram is invariant by the enlargement $X \to \alpha^{-1}X$ and $Y \to \alpha^{-2}Y$ in which case the region (ℓ,m) goes into the region $(\ell - m, 2m-\ell)$.

We have analysed here the family of maps defined by eq. (1) only for $0 \leq t \leq 1$ in which case the mappings are orientation-preserving. When $t > 1$ it is not orientation-preserving anymore and the Lyapunov exponent $\log t$ is larger than zero. We expect, therefore, the occurrence, in this case, of a chaotic behavior. This, however, will be the subject of a further investigation.

APPENDIX

In this appendix we prove the existence of a solution of the system of eqs. (2) for which the integers n_j are given by

$$n_1 = \begin{cases} 1 & \text{for } \Omega^- < \Omega \leq \bar{\Omega} , \\ 0 & \text{for } \bar{\Omega} < \Omega \leq \Omega^+ , \end{cases} \tag{A1}$$

$$n_j = \left[(j-1)\frac{L}{N}\right] + 1 \quad \text{for} \quad j = 2,\ldots,N ,$$

where $\Omega^{\pm}$ are the stability limits given by eq. (3), and

$$\bar{\Omega} = \frac{1}{2}\,(\Omega^- + \Omega^+) \quad . \tag{A2}$$

If the values of n_j for $j = 1,\ldots,N$ are given, it follows from eq. (2) that the values of x_j for $j = 0,1,\ldots,N-1$ are uniquely determined. Therefore what we have to prove is the validity of the relation

$$n_{j+1} = \left[x_j + \Omega + \frac{1}{2}\right]$$

for $j = 0,1,\ldots,N-1$, or equivalently,

$$-\frac{1}{2} \leqslant x_j + \Omega - n_{j+1} < \frac{1}{2} \quad , \text{ for } j = 0,\ldots,N-1 \, . \tag{A3}$$

Solving eq. (2) for x_0 we obtain

$$x_0 + \Omega - n_1 = F(\Omega) - \frac{n_1 + (1-n_1)t^{N-1}}{1 - t^N} \quad , \tag{A4}$$

where we have defined

$$F(\Omega) = \frac{\Omega}{1-t} - \frac{1}{1-t^N} \sum_{j=2}^{N-1} \sigma_j \, t^{N-j} \; . \tag{A5}$$

From eq. (3) we have

$$F(\Omega^+) = \frac{1}{1-t^N}\left[\frac{1}{2}\,(1-t^N) + t^{N-1}\right] \quad , \tag{A6}$$

$$F(\Omega^-) = \frac{1}{2}\,\frac{1+t^N}{1-t^N} \quad , \tag{A7}$$

$$F(\bar{\Omega}) = \frac{1}{2}\,\frac{1+t^{N-1}}{1-t^N} \quad . \tag{A8}$$

Using eqs. (A4)-(A8) and the fact that $F(\Omega)$ is an increasing function of R, it is straightforward to show that the condition (A3) is satisfied for $j=0$. For $j \geqslant 1$, eq. (2) gives for x_j the solution

$$x_j + \Omega - n_{j+1} = F(\Omega) + \frac{G_j}{1-t^N} - \frac{n_1 t^{\ell} + (1-n_1)t^{\ell-1}}{1 - t^N} \quad , \tag{A9}$$

where we have defined

$$G_j = (1-t^{\ell}) \sum_{\ell=2}^{N-1} \sigma_\ell \cdot t^{N-\ell} - (1-t^N) \sum_{\ell=2}^{j} \sigma_\ell \, t^{j-\ell} . \qquad \text{(A10)}$$

Using the property $\sigma_{N-j+1} = a_3$ for $j = 2,\ldots,N-1$, and also the fact that σ_j is 1 if and only if j is of the form $[k\beta] + 1$, where $\beta=N/L$ and k is an integer, we can rewrite eq. (A10) in the form

$$G_j = \sum_{k=1}^{L-1} t^{[k\beta]} - \sum_{k=1}^{m} t^{j-[k\beta]-1} - \sum_{k=m+1}^{L} t^{N+j-[k\beta]-1} , \qquad \text{(A11)}$$

where m is an integer such that

$$[m\beta] + 1 = j_i \leq j \leq j_s = [(m+1)\beta] . \qquad \text{(A12)}$$

Since G_j is an increasing function of j for a fixed m, we have

$$G_j \geq G_{j_i} = -1 + t^{j_i-1} + \sum_{k=1}^{m} \{t^{[k\beta]} - t^{[m\beta]-[k\beta]}\}$$

$$+ \sum_{k=m+1}^{L-1} \{t^{[k\beta]} - t^{N+[m\beta]-[k\beta]}\} \qquad \text{(A13)}$$

Using the property $[x] - [y] \geq [x-y]$, we conclude without difficulty that the two sums above are non-negative. Therefore,

$$G_j \geq G_{j_i} \geq -1 + t^{j_i-1} \geq -1 + t^{j-1} . \qquad \text{(A14)}$$

In quite the same way we can demonstrate that

$$G_j \leq G_{j_s} \leq t^{j_s} - t^{N-1} \leq t^j - t^{N-1} \qquad \text{(A15)}$$

Using the results (A6)-(A8), and (A14)-(A15) in eq.(A9), it is a simple matter to show that the inequalities (A3) are indeed satisfied for $j = 1,2,\ldots,N-1$.

REFERENCES

1. Per Bak, Rep. Prog. Phys. 45, 587 (1982).
2. M.H.Hensen, Per Bak and T.Bohr, Phys.Rev.Lett. 50, 1637 (1983).
3. C.S.O.Yokoi, M.J.de Oliveira and S.R.Salinas, Phys.Rev.Lett. 54,163 (1985).
4. W.Li and Per Bak, Phys.Rev.Lett. 57, 655 (1986).
5. V.I.Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer-Verlag, New York, 1983).
6. M.R.Herman, in *Geometry and Topology*, edited by J.Palis, becture Notes in Mathernatics, Vol. 597 (Springer-Verlag, Berlin, 1977) p. 271.
7. S.Aubry, J.Phys. C*16*, 2497 (1983).
8. R.B.Griffths and W.Chou, Phys.Rev.Lett. *56*, 1929 (1986).
9. D.E.Lanford III, Physics *14*D, 403 (1985).
10. M.Abramowitz and I.A. Stegun *Handbook of Mathematical Functions* (Dover, New York, 1970), p. 826.

Resumo

Analisamos uma família de mapeamentos no circulo sobre si mesmo, que são descontínuos e preservam a orientação. Obtemos expressões analíticas para a localização e para a largura de cada um dos degraus da escada do diabo. Mostramos que a escada é completa e que o conjunto de pontos correspondentes a números de rotação irracionais é um conjunto de Cantor de dimensão fractal D = 0.