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Calculation of Cosrnic Ray Proton Flux in the Real Dornain

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Abstract The flux of cosmic ray protons is calculated in the real domain. We discuss the precision of the saddle point method up to the second derivative applied to this problem.

1. INTRODUCTION

The majority of primary cosmic ray particles are known to be protons. Their behaviour takes a leading part in cosmic ray propagation in the atmosphere.

The energy spectrum of protons can be calculated by means of the Mellin transformation and its inverse[1] as

$$n(E_0,E,t) = \frac{1}{2\pi i}\,\frac{1}{E}\int_c ds\left(\frac{E_0}{E}\right)^s \exp\left(-\frac{t}{\lambda_a(s)}\right) \tag{1}$$

where

$$\frac{1}{\lambda_a(s)} = \frac{1-\langle(1-K)^s\rangle}{\lambda}$$

$n(E_0,E,t)$ is the probability that a primary proton of energy E_0 falls down to the atmospheric depth t where it has the energy E. λ and K stand for, respectively, the constant collision mean free path and the inelasticity of protons.

In this paper, we calculate the proton flux in the real domain so that we can compute the numerical values exactly at every stage of calculation.

2. CALCULATION OF PROTON FLUX

We calculate the energy spectrum of protons $n(E_0,E,t)$, assuming that:

a) the primary cosmic rays are all protons;

b) the collision mean free path of protons is constant, and

c) the distribution of proton elasticity is flat.

$n(E_0,E,t)$ can be expressed as a sum of products of $P_n(t)$ and $f_n(E_0,E)$, where $P_n(t)$ is the probability that a proton interacts n-times in its passage in the air and $f_n(E_0,E)$ is the probability that the primary energy E_0 reduces to E after n-interactions;

$$n(E_0,E,t) = \sum_{n=0}^{\infty} P_n(t)\ f_n(E_0,E) \tag{2}$$

The probability $P_n(t)$ follows the Poissonian law under the assumption (b) as

$$P_n(t) = \frac{(t/\lambda)^n}{n!} e^{-t/\lambda} \tag{3}$$

The probability $f_n(E_0,E)$ may be expressed' as

$$\underbrace{\int \cdots \int}_{E_i \leq E_{i-1}}^{n-1} g(1-E_1/E_0)dE_1/E_0 \ \ldots\ g(1-E/E_{n-1})dE/E_{n-1} \tag{4}$$

where E_i stands for the proton energy after the i-th interaction and $g(K)$ for the inelasticity distribution. If we apply the Mellin transformation and its inverse to eq. (4), we reach the solution eq. (1).

If the distribution of elasticity is flat, we can simplify this integration; let us introduce a global elasticity G as

$$G = (1-K_1)(1-K_2)\ \ldots\ (1-K_n)\ , \tag{5}$$

and derive the distribution of G, $f_n(G)$.

For n=0, G has no meaning because there is no interaction; we must treat this case especially. For n=1 (single interaction), the assumption (c) directly indicates that $f_1(G) = 1$. For n = 2, $f_2(G) = -\ln G$. For $n=3$, $f_3(G) = (1/2).(-\ln G)^2$. Finally the distribution of G can be expressed as

$$f_n(G) = \begin{cases} \delta(C-i) & \text{for } n = 0 \\ \frac{1}{(n-1)!}\,(\ln \frac{1}{G})^{n-1} & \text{for } n > 0 \end{cases} \tag{6}$$

Combining eq. (3) and eq. (6) in the way of eq. (2), we reach our solution as

$$n(E_0,E,t) = e^{-t/\lambda}\delta(E-E_0) + \sum_{n=1}^{\infty} P_n(t) \int f_n(G)\, dG \;\; \delta(E-GE)$$

$$= e^{-t/\lambda}\; \delta(E-E_0) + e^{-t/\lambda} \frac{t}{\lambda E_0} \sum_{n=1}^{\infty} \frac{1}{(n-1)!n!} \left(\frac{t}{\lambda} \;\ln \frac{E_0}{E}\right)^{n-1}$$

$$= e^{-t/\lambda}\; \delta(E-E_0) \;\; + e^{-t/\lambda} \;\; \frac{t}{\lambda E_0} \frac{2}{z} \;\; I_1(z) \tag{7}$$

where

$$z = \sqrt{(4t/\lambda)\;\ln(E_0/E)}$$

and $I_1(z)$ is the modified Bessel function of order 1.

Multiplying the energy spectrum of primary protons at $t = 0$, $N_0(E_0)dE_0$, by $n(E_0,E,t)$ and integrating with respect to E_0, we can calculate the differential proton spectrum $N(E,t)$. Here we assume

$$N_0(E_0) = N_0 \; E_0^{-\gamma-1} \tag{8}$$

then

$$N(E,t) = \int_E^{\infty} n(E_0,E,t) \;\; N_0 E_0^{-\gamma-1} \;\; dE_0$$

$$= e^{-t/\lambda} \;\; N_0 E^{-\gamma-1} \left[1 + \int_0^{\infty} I_1(z) e^{-(\gamma+1)\lambda z^2/4t} \;\; dz\right]$$

$$= e^{-t/\lambda} \;\; N_0 E^{-\gamma-1} \left[1 + \frac{1}{\lambda(\gamma+1)} \; {}_1F_1\,(1;2;t/\lambda(\gamma+1))\right]$$

$$= N_0 E^{-\gamma-1} \quad e^{-\frac{t}{\lambda}\frac{\gamma}{\gamma+1}} \tag{9}$$

where ${}_1F_1$ is the Kummer function.

Integrating $N(E,t)$ with respect to E, we can calculate the integral proton spectrum $N(>E,t)$ as

$$N(>E,t) = N_0 \frac{e^{-\gamma}}{\gamma} e^{-\frac{t}{\lambda}\frac{\gamma}{\gamma+1}} . \tag{10}$$

3. DISCUSSIONS

As the appearance of the two kinds of results eqs. (1) and (7) is quite different, it may be meaningful to check the equivalence of the two.

The first term of eq. (7) can be expressed by the Mellin transformation and its inverse as

$$\frac{1}{2\pi i} \int ds\ e^{-t/\lambda} \frac{1}{E} \left(\frac{E_0}{E}\right)^s . \tag{11}$$

And the second term of eq. (7) can be expressed as

$$\frac{1}{2\pi i} \int ds\ e^{-t/\lambda} \frac{1}{E} \left(\frac{E_0}{E}\right)^s \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{t}{\lambda(s+1)}\right)^n . \tag{12}$$

Combining eqs. (11) and (12), we can find the equivalence of eqs. (1) and (7). The details of equivalence check are found in ref.2. Starting from eq. (1) under the assumption (c), we also reach the results eqs. (9) and (10).

Here we compare the numerical values at the stages of eqs. (1) and (7). The solution (7) can be calculated exactly. In fig. 1, the quantity $n(E_0,E,t)$ is shown as a function of E directly in log-log scale, in the case of E_0 = 1000 TeV and t/λ = 1, for our result and for the result of eq. (1). The latter was computed approximately by means of the saddle point method up to the second derivative[3]. As the two solutions are equivalent to each other, the difference seen in the two computational results comes from the approximation applied to compute the complex integral. We notice in the figure that the simple-minded application of the saddle point method gives numerical values about 10-20% higher than our exact values.

4. CONCLUSIONS

We have calculated the cosmic ray proton flux analytically in the real domain, so that we can compute the numerical values exactly at

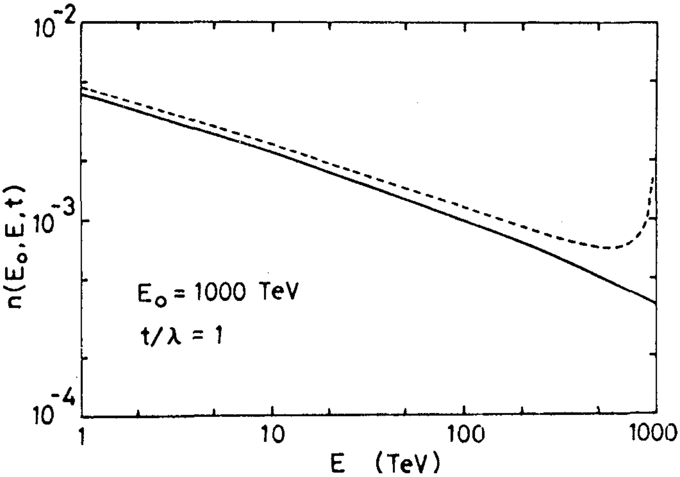

Fig.1 - Grapth of the quantity $n(E_0,E,t)$ versus E in the case E_0 = 1000 TeV and t/h = 1. The solid line is for the result of eq. (7) and the broken one for the result of eq. (1) computed by means of the saddle point method up to the second derivative.

every stage of the calculation. Our result may help in grasping the physical meaning of each term in the soiution. Also, the comparison between the treatments in the real and the complex domains indicates that we should pay an attention to the precision problem in computing some process of the solution of the complex integral.

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Resumo

Calcula-se, no campo real, o fluxo de prótons dos raios cósmicos. Discute-se a precisão do método do ponto de sela até a segunda derivada inclusive aplicado a este problema.

Note Added

After submitting this work for publication, we learnt that N. G.Boyadzhyan, A.P. Garyaka and E.A. Mamidzhanyan had reported, in Sov. J.Nucl.Phys. *34*, 67 (1981), the energy spectrum of protons as

$$n(E_0,E,t) = \delta(E-E_0)e^{-t/\lambda} + \frac{1}{E_0}\left(\frac{E}{E_0}\right)^{\alpha}\sqrt{\frac{t}{\lambda}}\, I_1\left(2\sqrt{\frac{t}{\lambda}\ln\frac{E_0}{E}}\right)\left(\ln\frac{E_0}{E}\right)^{-1/2} \quad (N.1)$$

for the case

$$W(E,E_0) = \frac{1}{E_0}\left(\frac{E}{E_0}\right)^{\alpha} \quad (N.2)$$

where $W(E,E_0)$ is the energy distribution of protons for each collision. However, we found that this result was incorrect. The correct solution for a > -1 should be written in the real domain as

$$n(E_0,E,t) = \delta(E-E_0)\, e^{-t/\lambda}$$
$$+\, e^{-t/\lambda}\frac{1}{E_0}\left(\frac{E}{E_0}\right)^{\alpha}\sqrt{\frac{t}{\lambda}(\alpha+1)}\, I_1\left(2\sqrt{\frac{t}{\lambda}(\alpha+1)\ln\frac{E_0}{E}}\right)\left(\ln\frac{E_0}{E}\right)^{-1/2} \quad (N.3)$$

(N.3) coincides with eq.(7) when we put a = 0.

The equivalence of (N.3) and eq.(1) under the condition of (N.2) can be checked in a similar way as we did in the section 3. We are grateful to Prof. H.M.Portella for having called our attention to the above-mentioned reference.