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Is the Classical Electron Really Spherical?

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Abstract Instead of the usual classical spherical model of the electron, we develop a toroidal charge distribution model. It is shown that the latter model of the electron is not subject to instabilities. The model also applies tc magnetic charges, enabling us to consider magnetic monopoles. Some possible connectionç with other current topics in theoretical physics are pointed out.

1. INTRODUCTION

In this paper we will be interested mainly in developing a classical rnodel for the space distribution of charge and current of the electron, aiming to find stable configurations. First of all, using some techniques of exterior algebra, we show that the spherical shape is not a suitable model for the electron. Instead of the usual difficulties encountered in the Lorentz model, we have found some topological reasons to discard the spherical model. The unusual feature of this work is that a simple toroidal charge configuration ensures stability for a given distribution of charges and currents, avoiding the drawbacks of the spherical rnodel.

In section 2 we present the mathematical formalism whichisused in this work. We start from the definition of domains in Minkowski space, define forms and integration of forms and arrive at the first set of Maxwell equations in the context of exterior algebra.

In section 3, aiming to establish the second set of Maxwell equations, we start from a symplectic symmetry argument. With the equations established, we apply them, choosing suitable boundary conditions. Surface discontinuities are introduced.

Next, in section 4, we study the stability of surface discontinuities in the fields and apply the results for a hypothetical charged particle which may be the electron. No numerical comparison are made. In section 5 we particularize the above results for some special shapes. The first example is the spherical one. A detailed discussion on the impossibility of having well defined distributions of charge and current in a spherical model is given in terms of topological inconsistencies. The spherical model is discarded and the toroidal model is introduced. With this choice, the above difficulties disappear.

In section 6, a semi-quantitative result is developed. It is shown that, from the point of view of measuring electric fields. the spherical and toroidal models are almost equivalent. For some reasonable distance, such as twice of the classical electron radius, the deviation between the two fields is negligible. Also, the total energy of a toroidal electron is estimated and a *compacted* torus is constructed.

In section 7, we give some elementary reasons supporting the existence of magnetic monopoles. Basically, symmetry arguments of the Maxwell equations are invoked. The physics of a toroidal magnetic monopole becomes identical with that of the electron,

Finally, in section 8 we present a summary of results and of possibilities of the toroidal model. A few examples of possible connections with other fields of theoretical physics are given. Among these, perhaps the most interesting example consists in replacing a vortex model like Olesen-Nielsen's by a closed charged torus. Other connections with some related topícs like Nambu strings and the implications on elementary particle physics are only mentioned. They will be the subject of forthcoming papers.

2, INTEGRATION AND FORMS

Since in this paper we will be mainly concerned with integration of forms on Minkowski space M^4, let us define our terminology.

A compact 3-dimensional subset Ω of the 4-dimensional manifold M^4 can be decomposed into two disjoined components: int Ω which has dim = 3 and is called interior domain and $\partial\Omega$ which has dim = 2 and is the boundary of R. Both parts are regular domains. If int Ω is a simple manifold, it can be covered by a single coordinate system. In general, int Ω must be covered by an atlas $(\phi_i, U_i)_{i\in I}$ consisting of overlapping coordinate systems, where ϕ_i are applications to R^4 and U_i are open subsets. The index I specifies the family of coordinate systems[1]. Next we propose to construct the integral $\int_R F$ of an arbitrary differential form F over Ω.

Let us suppose that there exists a special coordinate system (ϕ_0, U_0) so that the restriction of F to Ω vanishes outside $\phi_0(U_0)$. Then, of course, we define the integral

$$\int_\Omega F = \int_{U_0} F(\vec{e}_1, \vec{e}_2, \vec{e}_3)\; d\lambda^1\; d\lambda^2\; d\lambda^3 \tag{1}$$

where the $\vec{e}_i$ are canonical basis vectors and the $d\lambda^i$ represent the parametrization.

If we choose an inertial frame S on M^4 with inertial coordinates (x^0, x^1, x^2, x^3), we define a 3-dimensional submanifold ξ as a space slice by

$$\xi = \left\{ x \in M^4 \mid x^0 = c\; t^0 \right\} \tag{2}$$

Thus, ξ consists of all the spatial points at a specific time t^0. We say that (x^1, x^2, x^3) are adapted to ξ. Now let Ω be contained in a space slice relative to S. When integrating over R we must observe that the restriction of dx^0 (considered as a form) to R vanishes and the corresponding integrals involve only the space-ccmponents of the integrands.

At each point of M^4 we define a system of unitary vectors $\{\vec{I}_0, \vec{I}_1, \vec{I}_2, \vec{I}_3\}$ corresponding to the four coordinates. An infinitesimal displacernent of a point P is given by

$$d\vec{P} = \omega^0 \vec{I}_0 + \omega^1 \vec{I}_1 + \omega^2 \vec{I}_2 + \omega^3 \vec{I}_3 = \omega^i \boldsymbol{I}_i \tag{3}$$

with $i = 0,1,2,3$ and where ω^i is the form defined by

$$\omega^i = A^i_0\; dx^0 + A^i_1\; dx^1 + A^i_2\; dx^2 + A^i_3\; dx^3 \tag{4}$$

In general[2], $A^i = p^i$ and A^i_3 are functions of x^k and XO. For brevity we put

$$\omega^i = \bar{\omega}^i + p^i\; dx^0 \tag{5}$$

corresponding to

$$d\vec{P} = (\bar{\omega}^i + p^i\, dx^0)\, \vec{I}_i \tag{6}$$

In a parametric description, when the x^k are constant, a point (Q) with displacement given by $d\vec{Q} = p^i\, dx^0\, \vec{I}_i$ describes a curve $C:R\to M^4$. The totality of these curves constitutes a congruence (C) [3]. The congruence is defined as having only one curve passing through each point. The parameter x^0 (or t^0) fixes the position of the point on (C). On the other hand, when t^0 is constant, as assumed above, the displacement is given by $d\vec{P} = \bar{w}^i\, \vec{I}_i$ and this suffices to determine 5. Of course, an observer placed on ξ cannot directly measure the displacements on the normal to ξ. $\vec{I}_0$ is normal to 5.

If we use Cartan's *moving frame* point of view[4], we can take the trajectories of the volume elements of R to which we ascribe a density p* as forming a congruence defined by the differential equations

$$\frac{\omega^1}{\dot{j}^1} = \frac{\omega^2}{\dot{j}^2} = \frac{\omega^3}{\dot{j}^3} = \frac{c\ dt}{\dot{j}^4} \tag{7}$$

where $\dot{j}^\alpha = \pi^\alpha \rho^*$ (a = 1,2,3), the π^a, ρ^* being functions of x^0, x^1, x^2, x^3, with $\pi^i = \omega^i/dx^0$.

We define ρ^* by imposing that on R,

$$\iiint_\Omega \rho^* \left[\bar{\omega}^1\ \bar{\omega}^2\ \bar{\omega}^3\right] \tag{8}$$

is an absolute invariant integral in the Poincaré sense[5]. The square brackets correspond to the exterior product in Cartan's notation. An elegant proof that ρ^* can be taken as a Jacobi multiplier of system eq. (7) and that there exists an invariant form depending on three integrals of eq. (7) is developed in Loiseau[6].

An invariant form $[F]$ has the exterior derivative equal to zero, $[F]' = 0$. But $[F]'$ being zero, by Poincaré's theorem, there is an infinity of two-forms Λ such that $[F]' = [\Lambda]$. This is equivalent to saying that

$$\iiint [F] = \iint [\Lambda] \tag{9}$$

One particular expression for $[\Lambda]$ is[6]:

$$[\Lambda] = B^1[\omega^2\omega^3] + B^2[\omega^3\omega^1] + B^3[\omega^1\omega^2]$$
$$+ E^1[\omega^1 dt] + E^2[\omega^2 dt] + E^3[\omega^3 dt] \qquad (10)$$

where (B,E) are the cornponents of the electromagnetic field tensor. The usual Maxwell equations are obtained by putting $j^\alpha = 0$ $(a = 1,2,3)$ and $\rho^* = 0$. This corresponds to having the scalar product $\vec{B}.\vec{E} = 0$ and [A] an exact exterior derivative, i.e., [A] $= [\alpha]'$, with $[\alpha]$, a linear form. The next possibility is to consider $j^a \neq 0$ and $\rho^* = 0$. In this case we must also have $\vec{E}.\vec{B} = 0$. As we know, $\vec{j}$ results perpendicular to $\vec{E}$, with its direction given by the intersection of the two spatial hyperplanes

$$B^3x^2 - B^2x^3 + E^1 = 0 \text{ and } B^1x^3 - B^3x^1 + E^2 = 0 \qquad (11)$$

The possibility of having $\vec{j} \neq 0$ and p* $\neq 0$ leads to the following set of Maxwell equation[7-9]

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} + \text{rot } \vec{E} = 4\pi\frac{\vec{j}}{c}$$

$$\text{div } \vec{B} + 4\pi\, p^* = 0 \qquad (12)$$

$$\text{div } \vec{j} + \frac{\partial\rho^*}{\partial t} = 0$$

It is irnportant to note that, until now, no physical hypothesis has been made on the nature of ρ^*. In the sequel we will develop some models for ρ^* and will consider some physical consequences of $\rho^* \neq 0$.

When the congruence defined by eq. (7) corresponds to trajectories of p*, it is easy to show that this choice corresponds to $\partial\rho^*/\partial t=0$.

3. A SYMMETRY OF THE ELECTROMAGNETIC FIELD

We can give a syrnplectic interpretation to the form $[F]$ defined by $\vec{E}$ and $\vec{B}$ corresponding to the density ρ^* and to the currentdensity $\vec{j}$. First of all we write down the Maxwell equations. Next, we establish the well known property that there exist only the two invariants

$$(\vec{E}/c)^2 + \vec{B}^2 \quad \text{and} \quad (\vec{E}/c.\vec{B})^2 \tag{13}$$

Now, it can easily be shown that if we consider the antisymmetric matrix F which corresponds to the form $\boxed{F}$ as a rotation, only two null straight lines remain fixed. These lines depend only on the above field invariants. But these two invariants also remain unchanged by the transformations

$$(\vec{B}\ ,\ \vec{E}/c) \to (\vec{E}/c\ ,\ \vec{B}) \tag{14}$$

and

$$(\vec{B}\ ,\ \vec{E}/c) \to (-\vec{E}/c,\ \vec{B}) \tag{15}$$

The first transformation is a trivial one. It corresponds to a single axis exchange. The second transformation is very important. From a symplectic point of view it can be written in a 6-dimensional space:

$$\begin{pmatrix} -\dfrac{\vec{E}}{c} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \vec{B} \\ \dfrac{\vec{E}}{c} \end{pmatrix} \tag{16}$$

It is straightforward to write the new invariant form $\boxed{H}$ corresponding to $(-\vec{E}/c,\ \vec{B})$. The Maxwell equations derived from $\boxed{H}$ correspond to a new current density $\vec{i}$ different from $\overset{*}{\vec{j}}$ and to a new charge density ρ different from ρ^*. Explicitly, these eqoations are

$$\begin{aligned} -\frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \text{rot}\ \vec{H} &= \frac{4\pi \vec{i}}{c} \\ \text{div}\ \vec{D} - 4\pi\rho &= 0 \\ \text{div}\ \vec{i} + \frac{\partial \rho}{\partial t} &= 0 \end{aligned} \tag{17}$$

The solutions of the complete set of Maxwell equations are obtained as usual, defining the potentials $(\vec{A},\vec{A}')$ and (V,V'). Defining the dependence of $\vec{i}$ and $\vec{j}$ on the space coordinate and the boundary conditions, the solutions are written in terms of retarded potentials [10]. The symmetry exhibited in the derivation of the Maxwell equations for the fields and particles is very peculiar. It also applies to the two quantities (ρ, ρ^*), showing that at the classical level it is completely impossible to distinguish between electric and magnetic charges. We will make use of this property later.

It is well known[10] that when $\vec{E}$, $\vec{H}$ or $\vec{D}$, $\vec{B}$ have discontinuities on passing through some two-dimensional rnoving manifold σ in ξ, the Maxwell equations can be written:

$$\frac{\Delta \vec{j}}{c} = \frac{1}{4\pi} (\vec{n} \times \delta \vec{E}) \qquad \Delta\rho^* = -\frac{1}{4\pi} (\vec{n} \cdot \delta \vec{B})$$
$$\frac{\Delta \vec{i}}{c} = \frac{1}{4\pi} (n \times \delta \vec{B}) \qquad \Delta\rho = \frac{1}{4\pi} (\vec{n} \cdot \delta \vec{E}) \tag{18}$$

where $\vec{n}$ is the exterior normal, $\delta \vec{E} = \vec{E}_e - \vec{E}_i$ and $\delta \vec{H} = \vec{H}_e - \vec{H}_i$, with the suffixes e and i denoting exterior and interior regions of σ. From these equations we see that there will always be a magnetic density if $\delta \vec{E}$ is not tangent to o. We also see that if we consider a continuous medium like a conductor or a dielectric, supposing that its exterior surface is a discontinuity surface, the discontinuity can be supressed at the expenses of adding some electric and magnetic currents to its surface. Of course, the values of these currents are given by the above Maxwell equations. In general, these currents $\Delta\vec{i}$ and $\Delta\vec{j}$ are not tangent to a. In the following section we will apply this reasoning to the case of a closed surface embedded in ξ: in particular, to the electron surface.

4. EQUILIBRIUM OF A LAYER OF CURREMTS: THE ELECTRON

In this section we follow the detailed calculation for the charge distribution on the electron surface as given by Loiseau[6].

In his first theory[11], Lorentz proposed the charge of the electron as distrlbuted over a certain space, say over the whole volume occupied by the electron, and considered the volume density as a continuous function of the coordinates, so that the charged particle has no sharp boundary. Indeed, it is surrounded by a thin layer in which the density gradually sinks from the value within the electron to zero, But this hypothesis was not sufficient to eliminate the infinite electron self-energy and the consequent electron dissociation.

After the theory of relativity, several theories of the electron have been formulated. Dirac has amended the situation with his hypothesis of the point electron. But the infinity has survived at the classical level.

Here, we mean to show that the Maxwell equations, supplemented with the terms corresponding to magnetic densities of charge and current, can supply a hint to solve at least the infinite self-energy problem. These supplernentary terms could be responsible for the appearance of Poincare's negative pressure, leading thus to an equilibrium configuration.

Let us take $\partial\Omega$ as the boundary surface of the electron. The extericr applied field is null. The only field values to be considered, namely $\vec{E}_e$, $\vec{H}_e$, $\vec{i}_e$ and $\vec{j}_e$ are to be created by the electron itself. In the external region to $\partial\Omega$, we have $\vec{H}_e = 0$ and $\vec{E}_e \neq 0$. We have also $\vec{E}_e // \vec{n}$ to $\partial\Omega$. The Maxwell equations can be read as

i) external region to $\partial\Omega$

$$\text{rot } \vec{E}_e = 0, \quad \text{div } \vec{E}_e = 0, \quad \vec{H}_e = 0, \quad \vec{i}_e = 0, \quad \vec{j}_e = 0 \tag{19}$$

ii) interior region (int Ω)

$$\text{rot } \vec{H}_i = 0, \quad \text{div } \vec{H}_i = 0, \quad \vec{E}_i = 0, \quad \vec{i}_i = 0, \quad \vec{j}_i = 0 \tag{20}$$

These equations mean that $\vec{E}_e$ and $\vec{H}_i$ are conservative.
On $\partial\Omega$ we have $\vec{E}_e - \delta\vec{E} = 0$ and $\vec{H}_i + \delta\vec{H} = 0$. Of course, $\vec{n}.\delta\vec{H}=0$.
The charges and currents on $\partial\Omega$ are given by

$$\begin{aligned} \frac{\delta\vec{i}}{c} &= -\frac{1}{4\pi}(\vec{n}\times\vec{H}_i), \quad & \delta\rho &= \frac{1}{4\pi}(\vec{n}\cdot\vec{E}_e) \\ \frac{\delta\vec{i}}{c} &= 0 & \delta\rho^* &= 0 \end{aligned} \tag{21}$$

Since the electron is strictly electric we have

$$\iint_{\partial\Omega} \frac{1}{4\pi}\vec{n}\cdot\vec{E}_e \, d\sigma = e \tag{22}$$

and define the potential V:

$$\vec{E}_e = \frac{1}{4\pi}\left(\frac{\partial V}{\partial \vec{n}}\right) \text{ on } \partial\Omega \tag{23}$$

The equilibriurn condition on $\partial\Omega$ is $|\vec{E}_e| = |\vec{H}_i|$.

The current density $\delta\vec{i}$ and the field $\vec{H}_i$ are tangent to $\partial\Omega$ and $\delta\vec{i} \perp \vec{H}_i$. Applying a $\pi/2$ rotation around $\vec{n}$, we get

$$\frac{\delta\vec{i}}{c} = \frac{\vec{E}_e}{4\pi} \quad , \quad \delta\rho = \frac{|\vec{H}_i|}{4\pi} = \frac{|\vec{E}_e|}{4\pi} \quad , \quad |\delta\vec{i}| = c\delta\rho \tag{24}$$

The most rernarkable fact about these equations is that the speed corresponding to the currents on $\partial\Omega$ is equal to c. Also, we conclude that $\vec{i}$ and $\vec{H}$ are orthogonal vectors lying on the envelopes of the orthogonal congruence of curves C_1 and C_2 on $\partial\Omega$.

To solve the problern completely, we must calculate the density ρ on $\partial\Omega$. It is clear that when passing through $\partial\Omega$ the potential V is continuous with discontinuous normal derivative. At a point M on $\partial\Omega$ we have

$$\left(\frac{\partial V}{\partial \vec{n}}\right)_e = -4\pi\rho(M) = 2\iint_{\partial\Omega} \rho(p)\frac{\cos\psi}{r^2_{PM}}\, d\sigma p \tag{25}$$

thus

$$\rho(M) - \frac{1}{2\pi}\iint_{\partial\Omega} \rho(p)\frac{\cos\psi}{r^2_{PM}}\, d\sigma_p = 0 \tag{26}$$

where $\vec{n}$ is the interior normal to $\partial\Omega$, ψ is the angle between $\vec{n}$ and $\vec{r}_{PM}$ which is the distance vector PM.

Expression *(26)* is a homogeneous Fredholm equation which has a unique solution[12] apart from a multiplicative constant factor. The solution is obtained if the total charge is given:

$$e = \iint_{\partial\Omega} \delta\rho(P)\, d\sigma_p \tag{27}$$

The boundary $\partial\Omega$ can be parametrized by two families of orthogonal curves C_1 and C_2, tangent to the unitary vectors $\vec{I}_1$ and $\vec{I}_2$. $\vec{I}_3$ is parallel to $\vec{n}$. Thus, the infinitesimal displacements of the point M are

$$d\vec{M} = \omega^1\, \vec{I}_1 \quad \text{on} \quad C_1 \tag{28}$$

$$d\vec{M} = \omega^2 \vec{I}_2 \quad \text{on} \quad C_2 \tag{29}$$

But $\omega^1 \vec{I}_1$ and $\omega^2 \vec{I}_2$ are intcgrable[2] $\omega^1 = \nu^1 dx^1$ and $\omega^2 = \nu^2 dx^2$, with (V', ν^2) functions of (x^1, x^2). Along C_2 the current is constant $= c\delta\rho$. Consequently, the current is orthogonal to the lines of equidensity of the electric charge of C_2 on $\partial\Omega$ and is tangent to their orthogonal trajectories $C_1.C_2$ are electric equipotential lines. Thus

$$W_{i\ 12} = H_i = 4\pi\delta\rho(M) \tag{30}$$

and

$$dW_i = 4\pi\ \delta\rho\ \omega^2 \tag{31}$$

Integrating, we get

$$W_i = 4\pi \int_{C_2} \nu^2\ \delta\rho dx^2 = 4\pi\ \delta\bar{\rho}(M) + \text{constant} \tag{32}$$

With x^1 = constant, this results in

$$\delta\bar{\rho}(M) = \int_{C_2} \nu^2\ \delta\rho\ dx^2 + \text{constant}/4\pi \tag{33}$$

Finally,

$$\begin{aligned}(W_i)_Q &= \text{constant} + \iint_{\partial\Omega} \frac{\delta\bar{\rho}(P)}{r_{PQ}} \cos\phi\ d\sigma_P = \\ &= \text{constant} + \iint_{\partial\Omega} \frac{d(1/r)}{d\vec{n}}\ \delta(P) d\sigma_P = \\ &= \text{constant} - \iint_{\partial\Omega} \frac{d(1/r)}{d\vec{n}}\ \delta\rho(P) d\sigma_P \end{aligned} \tag{34}$$

where $\vec{n}$ is the exterior normal, Q is a point on $\partial\Omega$ and ϕ is the angle between the normal and the vector PQ. For charges of opposite sign, the normals are reversed, giving rise to positive or negative charges on $\partial\Omega$.

5. THE ELECTRON SHAPE: SPHERE OR TORUS?

The classical picture of an electron is a negatively charged small sphere. The charge is uniformly distributed throughout the sphere The current function is easily calculated by assuming a polar coordinate system and selecting an arbitrary system of orthogonal curves C_1

and C_2. As stated in section 2, our 3-dimensional subset Ω is a regular domain. When the domain int Ω is a simple manifold, it can be covered by a single coordinate system. Also, by Cartan's lemma, 3R is a regular 2-dimensional domain without boundary: $\partial\partial\Omega = 0$. But this is not thecase for a spherical electron. The sphere is a compact topological space, since it is a bounded closed subset of R^3. It is impossible to cover the sphere with a single coordinate system since $S^2 \equiv \partial\Omega$ is not a simple manifold.

In terms of the curves C_1 and C_2, this topological condition implies in a serious drawback. C_1 and C_2 are defined so that at all points on $\partial\Omega$ the fields $\vec{E}_i$ and $\vec{H}_i$ are uniform. We can not have any singularity. If, for instance, we take as current lines the orthogonal circles to a diameter PP', the field $\vec{H}_i$ which is tangent to thermeridian through M will not be determined at the poles PP'. This fact shows the impossibility of having a layer of spherical currents in equilibrium.

The spherical distribution is subject to a second difficulty. Taking a closed curve γ on $\partial\Omega$, γ will divide $\partial\Omega$ into two distinct domains D_1 and D_2. If γ is taken as a line of magnetic field C_2, the current flowing from D_1 to D_2 can not return to D_1. If γ is not closed, giving thus a uniform field $\vec{H}_i$, it must have two limit points. The orthogonal trajectories must be closed: this implies that the limit points are points of indeterminacy of the current.

We conclude that it is impossible to have a uniform spherical layer of electric currents in equilibriurn, with a stationary motion on it. Our spherical model of the electron, although differing drastically from the Lorentz model in two aspects: existence of a layer of discontinuity and moving charges, also is unstable.

Let us now dwell on another model which is not subject to such instabilities. Our surface of revolution will be taken as that of the torus $S^1 \times S^1$. In this case, the instabilities noted above disappear.Our torus will have axis Oz and equator plane (Ox, Oy). A point Q on the torus has semi-polar coordinates: α = azimuth of the plane $Q\ Oz$, distance from Q to Oz, z = elevation.

By symmetry, the equicharge lines are parallel on the orthogonal planes to Oz. The current lines are the meridians. This current distribution, although in a quite distinct context, resembles the vortex description of a type II superconductor[13]. Returning to our forms defined in section 2, the infinitesimal displacement of a point Q can be written as

$$d\vec{Q} = d\ell\,\vec{K}_1 + \ell\,d\alpha\,\vec{K}_2 + dz\,\vec{K}_3 = \omega^1\,\vec{K}_1 + \omega^2\,\vec{K}_2 + \omega^3\,\vec{K}_3 \tag{35}$$

where $(\vec{K}_1,\vec{K}_2,\vec{K}_3)$ are orthogonal unit vectors. With these notations, the Poisson equation for the potential becomes

$$\Delta W = \frac{\partial^2 W}{\partial \ell^2} + \frac{1}{\ell}\frac{\partial W}{\partial \ell} + \frac{1}{\ell^2}\frac{\partial^2 W}{\partial \alpha^2} + \frac{\partial^2 W}{\partial z^2} = 0 \tag{36}$$

with $W(\infty) = 0$ and W = constant in the external region. On the torus, if we take $d\alpha$ in the positive sense of $\vec{K}_2$, we get for the current flowing through an element $w^2 = R\,da$ of a parallel of radius R (length = 2π R)

$$\frac{\mu}{2\pi\ell}\,\omega^2 = \frac{\mu}{2\pi\ell}\,\ell d\alpha = d\bar{\ell}M \tag{37}$$

with M taken on the parallel. We see that

$$\rho_M = \frac{\mu}{2\pi}\,\alpha_M + \text{constant} \tag{38}$$

On the other hand, the gradient of W is

$$\text{grad } W = \frac{\partial W}{\partial \ell}\vec{K}_1 + \frac{1}{\ell}\frac{\partial W}{\partial \alpha}\vec{K}_2 + \frac{\partial W}{\partial z}\vec{K}_3 \tag{39}$$

Combining this result with the corresponding value of ρ_M, we get the following value for the field strength H_i on the boundary $\partial\Omega$:

$$\begin{aligned} H_i &= \frac{4\pi}{\ell}\frac{\partial\bar{\rho}_M}{\partial\alpha} = \frac{2\mu}{\ell} \\ W_i &= 2\mu\alpha + \text{constant} \\ W_e &= 0 \end{aligned} \tag{40}$$

Obviously, at a point Q belonging to int Ω, H_i will be orthogonal to the meridian plane $Q0z$. On $\partial\Omega$

$$\delta\rho = \frac{|\delta\vec{i}|}{c} = -\frac{1}{4\pi}|\vec{n}\times\vec{H}_i| = \frac{2\mu}{4\pi\ell} = \frac{\mu}{2\pi\ell} \tag{41}$$

For the charge distribution on $\partial\Omega$ when the exterior field is null, we get

$$\rho_M = -\frac{1}{2\pi}\frac{\mu}{\ell_M} \tag{42}$$

For the electric field

$$(\vec{E}_e)_{\vec{n}} = \frac{2\vec{\mu}}{\ell} \tag{43}$$

The total electric charge is

$$e = \iint_{\partial\Omega}\frac{|\vec{\mu}|}{2\pi}\frac{d\sigma_M}{\ell_M} \tag{44}$$

To calculate $d\sigma_M$ we change the coordinate system. On $\partial\Omega$ we use as co-ordinate lines the meridians and the parallels. With this procedure, care must be taken to avoid having the inner radius of the torus greater than the outer radius. This possibility, although interesting, could lead to serious topological difficulties for our purposes. The meridian equations $\ell = f(s)$ and $z = z(s)$, with s taken as an arc element on the closed meridian. The surface element on R is $\ell\, d\alpha\, ds$. Thus:

$$e = \frac{|\vec{\mu}|}{2\pi}\iint_{\partial\Omega} d\alpha ds = S_0\,|\vec{\mu}| \tag{45}$$

with S equal to the perimeter of the rneridian.

From the charge density

$$\bar{\rho}_M = \frac{e}{2\pi\, S_0}\,a + \text{constant} \tag{46}$$

we deduce

$$H_i = \frac{2e}{S_0\ell_M}\ , \quad \delta\rho = \frac{e}{2\pi\, S_0\Omega}\ , \quad |\delta\vec{i}| = \delta\rho, \quad |\vec{E}|_{\partial\Omega} = \frac{|\vec{\mu}|}{S_0\ell} \tag{47}$$

and

$$W_i = \frac{2e}{S_0}\,a + \text{constant} \tag{48}$$

At the exterior of the torus we get:

$$W_e = \frac{e}{2\pi S_0} \iint_{\partial\Omega} \frac{d\alpha\ dS_p}{r_{pQ}} \tag{49}$$

With this equation our problem is completely solved with several interesting physical properties to be uncovered in the model. Thus, a surface of revolution $\partial\Omega$ of closed meridian section can provide a simple example of a layer with electric (or magnetic) currents in equilibrium. At the external region there is an electric field. This layer can represent an electron.

6. THE EXTERNAL ELECTRIC FIELD: COMPARISON BETWEEN THE TOROIDAL ELECTRON AND THE POINT ELECTRON

Unlike the early theories involving spherical symmetry, here we calculate the external electric field of a toroidal electron. The electric potential at a point Q external to $\partial\Omega$ is

$$V_Q = \frac{e}{2\pi S_0} \int_0^{2\pi} d\alpha \int_0^{S} \frac{dS_p}{r_{PQ}} \tag{50}$$

If we take Q on the circular meridian with center A

$$V_Q = -\frac{e}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{d\alpha_p\ d\phi_p}{r_{PQ}} \tag{51}$$

with ϕ being the angle (Ay,AP).

Let us call $R = 0Q$, $R' = 0P$ and γ the angle $(0P, 0Q)$. With this notation

$$\frac{1}{r_{pQ}} = \frac{1}{R} \left[- \frac{2R'}{R} \cos\gamma + \left(\frac{R'}{R}\right)^2 \right]^{-\frac{1}{2}} \tag{52}$$

We also assume R > R'_0, with R'_0 being the greatest possible value of R'. Oeveloping $1/r_{PQ} = 1/r$, we get

$$\frac{1}{r} = \frac{1}{R} + \frac{R'}{R^2} \cos\gamma + \frac{(R')^2}{R^3} \frac{1}{2} (3\cos^2\gamma - 1) + \dots$$

$$= \frac{1}{R} \left[1 + \frac{R'}{R} X_1(\cos\gamma) + \frac{(R')^2}{R^3} X_2\ (\cos\gamma) + \dots \right] \tag{53}$$

where the X_m are Legendre polynomials. Substituting into the integral, we get

$$V_Q = -\frac{e}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{d\alpha\ d\phi}{R}\left(1+\frac{R^1}{R}\cos\gamma+\frac{(R^1)^2}{R^2}\frac{1}{2}(3\cos^2\gamma-1)+\ldots\right)=$$

$$= \iint_{\partial\Omega}\frac{\delta\rho\ d\alpha}{r} \qquad (54)$$

Performing the integration

$$V_Q = -\iint_{\partial\Omega}\delta\rho\,\frac{d\alpha\ d\phi}{R} - \frac{1}{R^2}\iint_{\partial\Omega} R^1(P)\,\frac{d\tau}{\ell}\,\delta\rho\cos\gamma -$$

$$-\frac{1}{R}\iint_{\partial\Omega}\delta\rho\ (3\cos^2-1)\ (R^1)^2\ d\sigma+\ldots \qquad (55)$$

Since the origin A is at the centre of gravity

$$\iint_{\partial\Omega} R' \cos y\ d\sigma = 0$$

Then

$$V_Q = -\frac{e}{R}-\frac{1}{R^3}\iint_{\partial\Omega}\delta\rho\left(\frac{3}{2}\cos^2\gamma-1\right)(R^1)^2 d\sigma+\ldots \qquad (56)$$

The corresponding field is

$$E_Q = \frac{e}{R^2}+\frac{3}{R^4}\iint_{\partial\Omega}\delta\rho\left(\frac{3}{2}\cos^2\gamma-1\right)(R^1)^2\ d\sigma+\ldots \qquad (57)$$

By choosing a meridian symmetric relative to the plane $0z = 0$ and calling

$$A = \iint_{\partial\Omega}\delta(p)(z^2+y^2)d\sigma\ ,\qquad C = \iint_{\partial\Omega}\delta\rho\ell^2\ d\sigma \qquad (58)$$

we get

$$E_Q = \frac{e}{R^2}-\frac{c-A}{R^4}\left(\frac{3}{2}\cos^2\Gamma-\frac{1}{2}\right)+\frac{1}{R^6}(\ldots) \qquad (59)$$

where Γ is the corresponding value of γ $(\Gamma = 0z,\ 0Q)$. For a circular meridian (circular torus) we have

$$A = \frac{e}{4}(2a^2+3b^2)\quad ,\quad C = \frac{e}{2}(2a^2+b^2)$$

and

$$V = -\frac{e}{R} + \frac{e}{8}\left\{\frac{(2a^2 - b^2)(3\cos^2\Gamma - 1)}{R^3}\right\} + \ldots$$

$$E = \frac{e}{R^2} - \frac{3}{8}\,e\,\frac{(2a^2 - b^2)\ (3\cos^2\Gamma - 1)}{R^4} + \ldots \tag{60}$$

where a and b are the outsr and the inner radii, respectively.

The field $\vec{E}$ is on the meridian plane. Its component norma? to $R = 0Q$ is easily calculated by taking the derivative with respect to Γ of the potential V

$$E_N = -\frac{3}{8}\,e\,\frac{2a^2 - b^2}{R^4}\sin^2\Gamma \tag{61}$$

The tangent of the angle between the field $\vec{E}$ and $0Q$ is

$$\tan\varepsilon = -\frac{3}{8}\,\frac{(2a^2 - b^2)\ \sin^2\Gamma}{R^2\left\{1 \qquad \dfrac{}{R^2} \qquad (3\cos^2\Gamma - 1)\right\}} \tag{62}$$

Taking into account all calculated terns, the externa1 electric field of a toroidal electron can be written as

$$E = \frac{e}{R^2}\left[1 - \frac{3}{16}\,\frac{1}{R^2}\,(2a^2 - b^2)\ \sin 2\sigma\right] \tag{63}$$

It is easily seen that for $1/R$ << an arbitrary small number which can be, for instance, the inverse of the classical radius of the electron, it becornes hard to distinguish experimentally (only measuring the field) the toroidal electron froni a point electron located at the origin, The greatest deviation for the field is found when $\Gamma = \pi/4$

$$E = \frac{e}{R^2}\left[1 - \frac{3}{16}\,\frac{1}{R^2}\,(2a^2 - b^2)\right] \tag{64}$$

If $a = b$, this equation results

$$E = \frac{e}{R^2}\left[1 - \frac{3}{16}\,\frac{b^2}{R^2}\right] \tag{65}$$

For $b = R_0/2$, where R_0 is the classical electron radius[10], the field measured at a distance, say $R = 2R_0$, the deviation of theobserved field from that of the point electron amounts to less than 1%.

The centre of gravity can be taken at rest. We note that choosing this particular reference frame, the electron energy m_0c^2 can be written

$$m_0c^2 = \frac{1}{8\pi} \int_R (E^2 + H^2)\, d\tau \tag{66}$$

In the internal region $\vec{E}_i = 0$

$$m_0c^2 = \frac{1}{8\pi} \iiint_\Omega H^2\, d\tau \tag{67}$$

But the electron surface is a circular torus. We assume its section as having radius r at a distance a from the origin. Thus, since $|\vec{H}_i| = 2e/S_0\ell$

$$m_0c^2 = \frac{e^2}{2a} \left[\frac{1 - \sqrt{1 - r^2/a^2}}{r^2/a^2}\right] \tag{68}$$

According to electrodynamics, the electron would have to have an infinite self-energy since the potential $\Phi = e/R$ of its field becomes infinite at the point $R = 0$. But the occurrence of the physically meaningless infinite self-energy of the elementary particle is related to the fact that such a particle must be considered as pointlike. Thus, when we go to sufficiently small distances we get internal contradictions. Formula (68) shows that, at least formally, the electromagnetic self-energy of the toroidal particle can be equated to the rest energy m_0c^2 without any serious divergence.

If, on the other hand, we consider a spherical electron as possessing a certain radius R_0, then its self-potential energy would be of order e^2/R_0. Thus, from

$$e^2/R_0 - m\,c^2$$

we get

$$R_0 \sim e^2/m_0c^2$$

This ratio determines the limit of applicability of electrodynamics to the classical electron[10].

Now, eq. (68) can be written as

$$\frac{m_0c^2}{e^2} = \frac{1}{R_0} = \frac{1}{2a}\,\frac{1-\sqrt{1-x^2}}{x^2}$$

where $x = r/a$. Let us find the two parameters of the torus, r and a, with the obvious condition $R_0 = a + r$. This amounts to say that $a = r = R_0/2$.

In all meridian planes. each circular meridian of radius $R_0/2$ has the origin as a common point with the opposite meridian. We can say that we have *compacted* the torus to have it contained in a sphere of radius R_0.

7. A POSSIBLE CLASSICAL MAGNETIC MONOPOLE

In section 3 we have shown that it is possible to pass from one set of Maxwell equations to the other, only by applying the transformation $(\vec{E},\vec{H}) \to (-\vec{H},\vec{E})$. In our toroidal model this is equivalent to studying a current layer where $\vec{E}_e = 0$ in the external region and $\vec{H}_i = 0$ in the internal region. $\vec{E}_i$ gets tangent to $\partial\Omega$ and $\vec{H}_e$ is orthogonal to $\partial\Omega$. Taking into account these restrictions, we arrive at

$$\delta\rho' = \frac{\vec{n}\cdot\delta\vec{E}}{4\pi} = 0 \quad , \quad \delta\vec{i} = 0 \tag{69}$$

The layer becomes of pure magnetic origin. For the potentials, Maxwell's equations are reduced to

$$\vec{E} = \text{grad } V \quad \text{and} \quad \vec{H} = \text{grad } W \tag{70}$$

with corresponding expressions at the external region

$$\Delta V = 0 \quad \text{and} \quad \Delta W = 0 \ .$$

On $\partial\Omega$ we have $\delta\vec{E} = -\vec{E}_i$ and $\delta\vec{H} = \vec{H}_e$. With these conventions, we see that $(\vec{H},\vec{E},\vec{j})$ can be taken as an orthogocal inverse frame of reference. We pass from the direction of $\vec{E}_i$ to that of $\vec{j}$ by a negative rotation.

It is easy to see that on $\partial\Omega$ the equilibrium condition results in

$$|\vec{H}_e|^2 = |\vec{E}_i|^2 \qquad \text{and} \qquad |\vec{H}_e| = |\vec{E}_i| \tag{71}$$

The global set of equations to be fulfilled on the boundary will be

$$\delta_{\vec{1}} = - c \frac{E_i}{4\pi} \quad , \quad \delta\rho^* = - \frac{|\vec{H}_e|}{4\pi} = - \frac{1}{4\pi} \left| \frac{d W_e}{d\vec{n}} \right| \tag{72}$$

and, finally

$$\vec{E}_i = \frac{d W_e}{d\vec{n}} \quad , \quad |\delta_{\vec{1}}| = c \ \delta\rho^* \tag{73}$$

We will define the potential correspondíng to a single layerof density $\delta\rho^*$ by

$$e^* = - \frac{1}{4\pi} \iint_\Omega \left| \frac{d W_e}{d\vec{n}} \right| d\sigma \tag{74}$$

To be more specific, we say that the lines of the electric field are lines of equidensity p* = constant and the current lines are their orthogonal trajectories.

The other results are identical with the electric case, with the conditions of exchanging W and V and changing the signofthecharge.

These are the conditions to be satisfied by the charge distributions to arrive at the Kottler-Loiseau classical model for the electron (or magnetic monopole).

8. CONCLUSIONS

Throughout this work we have assurned classical physics, i.e., electromagnetism. We have attempted to give a somewhat qualitative picture of work in which people have been engaged, concerning the implications of describing the classical distributions in terms of some geometrical symmetries. We have been encouraged in this work by the possibility that such a set of models give a natural explanation of the stability of some elementary systems, such as the electron.

Our work has been entirely based on the interchangeability of the fields E and H with the corresponding passage from the electric density ρ to the magnetic density ρ^*. In section 2 we have preferred to use some symplectic geometrical arguments instead of the usual Riemann interpretation of a function of complex variables. Our intention was to show clearly the role played by the Maxwell invariants. Besides this, the exterior calculus employed in matrix notation strongly suggests the symplectic interpretation.

The question as to whether the spherical electron or the toroidal electron is the more fundamental model can be formulated in a different way. From the point of view of stability the spherical electron must be rejected. Our spherical electron differs from the Lorentz electron essentially by having electric currents on the surface, moving at the speed of light. Thus, it could be argued that a surface discontinuity replacing a massive distribution must overcome the problern of self-energy divergence. Roughly speaking, this occurs. But another drawback, of a pure geometrical-topological origin arises: it becomes impossible to specify the charges and currents at all pointsonthe surface. This latter difficulty has led us to adopt the toroidal electron of Loiseau, This model has very interesting properties: topologically, the domain int $(S^1 \times S^1)$ is homeomorphic with the point group $SL(2,R)$. This group is singly connected, causing thus the connectedness of the torus. A single coordinate system covers the domain R. With the current flow lines lying in planes containing the axis Oz of the torus, the torus becomes equivalent to an infinite solenoid wrapped round Oz with the ends joined together. Consequently, we have a trapped flux in the interior of the torus. In the exterior region, no magnetic effectcanbe detected, Nevertheless, a large magnetic energy can be stored withinthe toroidal electron. This can also be seen when the inner radius of the torus becomes zero.

There is a divergence in the total energy of the electron whose source is not of Coulomb origin, but it comes from the fact that all lines of the trapped flux are compressed within a vanishing area. This very peculiar highly energetic closed circular string can serve as a starting point for many theoretical speculations,

One possibility is to relate this configuration with the Nambu string assuming that in space-time the surface of discontinuity can be represented by Mercator's projection as a rubber band. If there is a meaning to this analogy, the string should not be thought of as a mathematical line but as an object having some thickness. A specially short string, corresponding to the compacted torus, should have about thesame length and thickness.

In recent years, with the resurrection of the famous early 1931 paper of Dirac[14], the idea of a magnetic monopole reappered. The issue can be raised as to whether the existence of a perfectly localized magnetic monopole can lead to an effect which has passed unnoticed until now[15]. There is no basic objection to the existence of magnetic monopoles; their fields as considered here were deduced from legitimate Maxwell equations. Nevertheless, we have not used quantum mechanics and it has been shown quantum mechanically that if magnetic monopoles exist, the magnitude of the elementary unit pole would have to be related to the inverse of the elementary charge by a constant factor. In the GUT model[16] the monopole charge is 70 times larger than the electric charge and this fact should alter our estimates made in the electron case**.

To close this discussion we would like to point out some features of the toroidal configuration which could serve as a motivation for future investigations.

Consider our toroidal electron as a closed string with flux Φ_0 passing through the meridians. For the currents in dynamical equilibrium, a constant magnetic field H_0 will be created inside the string. Each circular meridian has radius h as above. The flux Φ will be given by

$$\Phi_0 = H_0 \pi b^2 \tag{75}$$

The magnetic energy stored per unit length is

$$J = \frac{1}{2} H_0^2 \pi b^2 = \Phi_0^2 / 2\pi b^2 \tag{76}$$

For the entire torus we get

$$J_{total} = a\Phi_0^2/b^2 \tag{77}$$

Using the results of section 6 we can make an estirnate of the magnetic energy and of the rnagnetic field flux contained in the *compacted* torus

$$J_T \sim m_0 c^2 = \frac{\Phi_0^2}{b^2} \tag{78}$$

But $a = b = R_0/2$. Then

$$J_T \sim 2\Phi_0^2/R_0 \tag{79}$$

Thus, frorn eq. (79) we deduce

$$\Phi_0^2 \sim \frac{1}{2} R_0 m_0 c^2 \sim \frac{1}{2} e^2$$

This flux corresponds to the most stable energy within the limits of classical electromagnetism, Correspondingly, the maximum possible magnetic field for a toroidal electron is

$$H_0 = \frac{2\sqrt{2}}{\pi} \frac{e}{R_0^2} \tag{80}$$

Obviously,the resulting eqs. (79) and (80) are significantlyaltered in the quantum framework.

From eq. *(77)* we see that we can diminish the energy of a magnetic toroidal string with a constant flux by spreading out the string. If our *compacted* torus is the most stable configuration for the electron, we can not alter b without altering a and the condition $R_0 = a+b$. Probably, the spreading does not occur.

In a type II superconductor it is the Meissner effect which prevents the magnetic flux from spreading out. Of course, the magnetic theory of type II superconductors has nothing to do with our model. Nevertheless, some topological ideas extracted from that theory could be useful for suggesting some possibilities in our theory.

Here, we only sketch some ideas on type II superconductors. When we pass a critical applied field strength B, superconductivity will not be cornpletely destroyed but rather, the magnetic field will penetrate into the metal in form of thin magnetic strings, or vortices. The flux being quantized, it is shown that a vortex string with a multiple flux has

$$J_n = n^2 J_0 \tag{81}$$

It is unstable, meaning that there can be no exact ground state. But, as in the sine-Gordon model one can construct approximative ground states consisting of n widely separated Nielsen-Olesen vortices[18]. Nevertheless, a string-like excitation such as the Nielsen-Olesen vortex cannot itself represent a physical particle since it has infinite energy due to its infinite length. So, if the N-O vortex is going to be physically relevant we must find a way to terminate it. N-O includes a monopole at an endpoint of the string and an anti-monopole at the other endpoint. Our model could offer an alternative to this approach: we suggest the hypothesis of considering an infinite solenoid as equivalentto a toroidal configuration. This configuration can be electron-like or monopole-like. The boundary conditions at infinity are replaced by cyclic conditions.

Of course, relativistic effects would be taken into account,

The challenge of the toroidal model for elementary particles in the future would be to find new techniques for relating the above problems to some interesting questions like the confining potential for quarks and the interaction between closed strings[19].

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Resumo

Ao invés do modelo esférico clássico usual do elétron, nós desenvolvemos um modelo com distribuição toroidal de carga. Mostramos que este último modelo não fica sujeito a instabilidades. O modelo também se aplica a cargas magnéticas, o que nos permite a consideração de modelos de monopolos magnéticos. Algumas possíveis conexões com outros tópicos recentes de física teórica são também mencionadas.