Revista Brasileira de Física, Vol. 18, n? 2, 1988

Extended General Relativistic Lagrangian for Spinning Fluids

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Recebido em 23 de setembro de 1986; versão revista em 12 de janeiro de 1988

Abstract In general relativity, a new lagrangian functional for uncharged Weyssenhoff fluids is proposed, satisfying two simple requirements: (I) all properties of the field variables occurring in the lagrangian should be derivable from the set of Euler-Lagrange equations alone; and (II) for every field variabie occurring in the lagrangian, the corresponding variational equation should be valid. Einstein's equations and the equations of motion for the spin are derived. Comparison with recent related papers in general relativity and Einstein-Cartan theory is made, and topics for further research are suggested.

1. INTRODUCTION

The lagrangian formalism for spinning fluids in gravitational theories has received considerable attention is recent years[1-12]. Different energy-momentum tensors for the spinning matter havebeen obtained, all approaching the usual energy-momentum tensor for the perfect fluid in general relativity when the spin of the material tends to zero.

It is the aim of the lagrangian method to infuse into a scalar functional, L, all elements necessary to obtain the equations of motion of a given physical system. The lagrangian is expressed in terms offield variables $Q_i(x)$, and satisfies a set of variational equations

$$\delta L/\delta Q_i := \partial L/\partial Q_i - \partial_a(\partial L/\partial Q_{i,a}) + \partial_a\partial_b(\partial L/\partial Q_{i,ab}) = 0 , \qquad (1)$$

where both the ∂ and the subscripted comma symbols denote partial derivative. The variables $Q_i(x)$ are independent in the process of partial derivation. In order to have an intelligent game, the following two simple rules seem essential: (I) all properties of the quantities $Q_i(x)$ present in L can be derived from the set of eqs. (1) alone; (II) for every field variable $Q_i(x)$ which appears in L, the correspondingequation $\delta L/\delta Q_i = 0$ must be valid. As far as we are aware, every report already presented on spinning fluids in curved spacetimes has missed either condition (I), or (II), or both.

In this paper we propose a general relativistic lagrangian for spinning fluids which is ***extended***, in the sense that it satisfies both requirements (I) and (II). We also derive the equation of motion for the gravitational field (Einstein's equation) and the equation for the spin, and show that both are physically plausible.

Further on in our presentation we shall need two simple thermodynamical relations[11-13],

$$(\partial \varepsilon/\partial \rho)_s = P/\rho^2 \ , \qquad (\partial \varepsilon/\partial s)_\rho = T \ , \tag{2}$$

where $\varepsilon(\rho,s)$ is the specific internal rest energy (energy per unit of mass), ρ is the rest matter density (mass per volume), s is the specific rest entropy, P is pressure and T is temperature. Throughout the paper we assume $8\pi G = c = 1$.

To describe the spin of the material, we shall also need an orthonormal tetrad of vectors $h^{\mu a}(x)$, where $\mu = 0,1,2,3$ labels the vector and $a = 0,1,2,3$ labels components[14]. The timelike vector h^{0a} is the eulerian fourvelocity u^a of the fluid, while $h^{1a} = \alpha^a$, $h^{2a} = \beta^a, h^{3a} = \gamma^a$ are spacelike vectors. The specific spin tensor s^{ab} of the fluid is related to α^a and β^c via

$$s^{ab} = - K(\alpha^a\beta^b - \beta^a\alpha^b) \ , \tag{3}$$

where $K(x)$ is the specific spin magnitude in the rest system (intrinsic angular momentum per unit mass). Alternatively, we could define the specific spin vector according to

$$s^d = K\gamma^d = K(-g)^{-1/2}\varepsilon^{abcd}u_a\alpha_b\beta_c \ , \tag{4}$$

$$s^{ab} = (-g)^{-1/2}\varepsilon^{abcd}u_c s_d \ , \tag{5}$$

where ε^{abcd} is totally antisymmetric with $\varepsilon^{0123} = +1$. From the orthogonality of the tetrad vectors it is clear that the fluid satisfies the auxiliary condition

$$s^{ab}u_b = 0 \ . \qquad (6)$$

2. THE LAGRANGIAN DENSITY FUNCTIONAL

We separate the total L into geometric and material components $L = L_G + L_M$, with

$$L_G = -(e/2)\,R^c{}_c \ , \quad e = (-\det g_{ab})^{1/2} \ , \qquad (7)$$

$$R_{bc} = 2\partial_{[a}\Gamma^a{}_{b]c} + 2\Gamma^a{}_{d[a}\Gamma^d{}_{b]c} \ , \qquad (8)$$

where the Γ's are the Christoffel symbols and the square brackets denote antisymmetrization. The *extended* material component of L is

$$\begin{aligned} L_M = e\{&-F(\rho,s) + K\rho(\alpha^a\dot{\beta}_a - \beta^a\dot{\alpha}_a) - 2nK^2\rho^2\alpha_{[a}\beta_{b]}\alpha^{[a}\beta^{b]} \\ &+ \lambda_\rho(\rho u^a)_{;a} + \lambda_s\dot{s} + \lambda_{00}(u^a u_a - 1) + \lambda_{11}(\alpha^a\alpha_a + 1) \\ &+ \lambda_{22}(\beta^a\beta_a + 1) + \lambda_{33}(\gamma^a\gamma_a + 1) + 2\lambda_{01}u^a\alpha_a + 2\lambda_{02}u^a\beta_a \\ &+ 2\lambda_{03}u^a\gamma_a + 2\lambda_{12}\alpha^a\beta_a + 2\lambda_{13}\alpha^a\gamma_a + 2\lambda_{23}\beta^a\gamma_a\} \ , \end{aligned} \qquad (9)$$

where the subscripted semicolon means covariant derivative, $\dot{Q}$ means $u^a Q_{;a}$ for Q an arbitrary scalar, vector or tensor quantity, and where the functions λ_ρ, λ_s and $\lambda_{\mu\nu}$ are lagrangian multipliers.

All terms in eq. (9) are already known in the literature, except perhaps the third one (which we discuss later). The functional $F(\rho,s)$ is related to the specific internal energy $\varepsilon(\rho,s)$ according to

$$F(\rho,s) = \rho[1 + \varepsilon(\rho,s)] \ , \qquad (10)$$

while the term $K\rho(\alpha^a\dot{\beta}_a - \beta^a\dot{\alpha}_a)$ is the spin kinetic energy density of the fluid[8,12]. The terms containing the various λ's express geometrical, kinematical and dynamical constraints imposed on the system: those containing $\lambda_{\mu\nu}$ imply that u^a, α^a, β^a, γ^a form an orthonormal tetrad (u^a timelike, the others spacelike), while that with λ_ρ reflects the matter conservation[2,13,15], and that with λ_s prohibits heat exchange between different parts of the fluid.

Finally, the term $2nK^2\rho^2\alpha_{[a}\beta_{b]}\alpha^{[a}\beta^{b]}$ seems to be novel in the literature, and is interpreted as the potential energy density of spin of the fluid. The value of the number n is to be eventually fixed by experimentalists. For simplicity, a term corresponding to the particle identity[16] is omitted in our presentation.

3. EULER-LAGRANGE EQUATIONS

The independent quantities $Q_i(x)$ in L are taken as the twelve λ's, the three scalars K, ρ, s, the contravariant vector components u^a, α^a, β^a, γ^a of the tetrad and the covariant components g_{ab} of the metric.

We start by considering the variational eq. (1) as applied to the lagrangian multipliers. For $Q = A_\rho$ and λ_s they clearly give

$$(\rho u^a)_{;a} = 0 \ , \tag{11}$$

$$u^a s_{,a} = 0 \ , \tag{12}$$

where the subscripted comma means partial derivative, while for $Q = \lambda_{\mu\nu}$ they give

$$g_{ab}h^{\mu a}h^{\nu b} = \eta^{\mu\nu} \tag{13}$$

or equivalently $\eta_{\mu\nu}h^{\mu a}h^{\nu b} = g^{ab}$, where both $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are the constant Minkowski metric diag(+1, -1, -1, -1). A few simple relations derived from eq. (13) are largely used, such as

$$\alpha_{[a}\beta_{b]}\alpha^{[a}\beta^{b]} = 1/2 \quad , \quad \alpha^{[a}\beta^{b]}\alpha_{[a}\beta^{c]} = -\frac{1}{4}(\alpha^b\alpha^c + \beta^b\beta^c) \ ,$$
$$\alpha^a\beta_a = -\beta^a\dot{\alpha}_a \ , \quad u^a\dot{u}_a = \alpha^a\dot{\alpha}_a = \beta^a\dot{\beta}_a = 0 \tag{14}$$

The equations $\delta L/\delta\gamma^a = 0$ is also trivial, giving $\lambda_{P3}h^{\mu a} = 0$; since the vectors $h^{\mu a}$ are linearly independent, it follows that the four coefficients λ_{P3} are null.

The equation $\delta L/\delta K = 0$ brings a relation between kineticandpotential energy density of spin,

$$K\rho\alpha^{a}\dot{\beta}_{a} = nK^{2}\rho^{2} \tag{15}$$

(hence only $n>0$ gives positive kinetic energy), while the variational equation for the specific entropy s gives the evolution of the thermasy[8];

$$(\rho^{-1}\lambda_{s})^{\cdot} = -T \ . \tag{16}$$

To obtain eq. (16), use is made of eqs. (10), (2) and (8).

The equation corresponding to $\delta\alpha$ is a good deal more laborious, and gives

$$\lambda_{01}u_{a} + (\lambda_{11} + nK^{2}\rho^{2})\alpha_{a} + (\lambda_{12} + \frac{1}{2}\dot{K}\rho)\beta_{a} + K\rho\dot{\beta}_{a} = 0 \ ; \tag{17}$$

from the independence of u_a, α_a, β_a, it then follows that

$$\lambda_{11} = -nK^{2}\rho^{2} + K\rho\alpha^{a}\dot{\beta}_{a} = 0 \ , \quad 2\lambda_{12} = -\dot{K}\rho \ , \tag{18}$$

and eq. (17) simplifies to

$$\lambda_{01}u_{a} + nK^{2}\rho^{2}\alpha_{a} + K\rho\dot{\beta}_{a} = 0 \ . \tag{19}$$

Instead of working out the equation $\delta L/\delta\beta^{a} = 0$ by brute force (as was done for $\delta\alpha^{a}$), we note that the lagrangian (9) is invariant under the simultaneous interchanges $\alpha \leftrightarrow \beta$, $1 \leftrightarrow 2$, $K \to -K$; from eqs. (18)-(19) we then have

$$\lambda_{22} = \lambda_{12} = \dot{K} = 0 \ , \tag{20}$$

$$\lambda_{02}u_{a} + nK^{2}\rho^{2}\beta_{a} - K\rho\dot{\alpha}_{a} = 0 \ . \tag{21}$$

The calculations concerning $\delta L/\delta u^{a} = 0$ are again long, and give

$$2K\rho\alpha^{b}\beta_{b;a} - \rho\lambda_{\rho,a} + \lambda_{s}s_{,a} + 2(\lambda_{00}u_{a} + \lambda_{01}\alpha_{a} + \lambda_{02}\beta_{a}) = 0 \ , \tag{22}$$

while the equation $\delta L/\delta\rho = 0$ is trivial and gives

$$\lambda_{\rho} = -F_{,\rho} \ ; \tag{23}$$

if we now take the inner product of eq. (22) with u^a and use eqs. (23) and (15) we find the expression of λ_{00},

$$2\lambda_{00} = -\rho F_{,\rho} - 2nK^2\rho^2 . \qquad (24)$$

4. EINSTEIN'S EQUATION; SPIN EQUATION

The calculation involving the equation $\delta L/\delta g_{ab} = 0$ are considerably long, so a compromise conciseness/clarity must be set in our presentation. We transcribe below only the final expression of $\delta/\delta g_{ab}$, as applied separately to each relevant term in eq. (9). One result is widely known, and gives the Einstein tensor

$$\delta L_G/\delta g_{ab} = \frac{1}{2}\, eG^{ab} . \qquad (25)$$

The other partial results are

$$(\delta/\delta g_{ab})\,[eF(\rho,s)] = \frac{1}{2}\, eg^{ab}F(\rho,s) , \qquad (26)$$

$$(\delta/\delta g_{ab})\,[eK\rho(\alpha^c\dot{\beta}_c - \beta^c\dot{\alpha}_c)] = e\{nK^2\rho^2 g^{ab} + u^{(a}s^{b)c}\dot{u}_c$$
$$+ [\rho u^{(a}s^{b)c}]_{;c} + n\rho^2 s^{ac}s^b{}_c\} , \qquad (27)$$

$$(\delta/\delta g_{ab})\,[2neK^2\rho^2\alpha_{[c}\beta_{d]}\alpha^{[c}\beta^{d]}] = ne(\frac{1}{2}\, K^2\rho^2 g^{ab} + \rho^2 s^{ac}s^b{}_c) , \qquad (28)$$

$$(\delta/\delta g_{ab})\,[e\lambda_\rho(\rho u^c)_{;c}] = \frac{e}{2}\,\rho F_{,\rho}\, g^{ab} , \qquad (29)$$

$$(\delta/\delta g_{ab})\, e[\lambda_{00}(u^c u_c - 1) + 2\lambda_{01}u^c\alpha_c + 2\lambda_{02}u^c\beta_c] =$$
$$-\frac{e}{2}\,(\rho F_{,\rho} + 2nK^2\rho^2)u^a u^b - 2e\rho u^{(a}s^{b)c}\dot{u}_c . \qquad (30)$$

Collecting terms, Einstein's equation for the spinning Weyssenhoff fluid finally emerges:

$$G^{ab} = [\rho(1+\varepsilon) + P + 2nK^2\rho^2]u^a u^b - (P + nK^2\rho^2)g^{ab}$$
$$- 2(\delta^m_n + u^m u_n)\ [\rho u^{(a}s^{b)n}]_{;m} . \qquad (31)$$

It is instructive to compare eq. (31) with similar gravitational field equations encountered in the literature. In studying spinning fluids in the general relativity, Ray and Smalley[12] found a result similar to our eq. (31), but in their paper the terms with $K^2\rho^2$ are lacking; this is not surprising, since their lagrangian also lacks terms in $K^2\rho^2$.

In the Einstein-Cartan theory (an alternative theory of gravitation with nonsymmetric connection)[17], the spinning fluid studied in the majority of works is that of the Weyssenhoff type. In those works, a canonical energy-momentum tensor for the fluid is developed, instead of the metrical energy-momentum as is done in this paper. Interestingly enough, the gravitational field equations obtained in those spacetimes with torsion closely resemble our eq. (31) obtained in the riemannian spacetime; the equations would coincide if n were equal to -1, which however would imply a negative kinetic energy of spin according to eq. (15).

Another coincidence deserves mention, now concerning the equation of motion for the specific spin s^{ab}. To obtain it, we initially take the dot derivative of eq. (3) and use $\dot{K} = 0$ (eq. (20)); we next use eqs. (19) and (21) rewritten in the form (for $K\rho \neq 0$)

$$\dot{\beta}^a = -K\rho\alpha^a - \beta^c \dot{u}_c u^a \quad , \quad \dot{\alpha}^a = K\rho\beta^a - \alpha^c \dot{u}_c u^a \quad , \tag{32}$$

and finally obtain

$$\dot{s}^{ab} = 2u^{[a}s^{b]c}\dot{u}_c \tag{33}$$

This general relativistic equation of motion is identical to its counterparts encountered in Riemann-Cartan-Weyssenhoff formulations[4,5,8,9,19].

Since eq. (33) does not depend on n, measurements of the time development of s^{ab} are useless to fix the value of n. However, the motion of a particle is sensitive to the energy-momentum T^{ab} (source of gravitation), which is seen from eq. (31) to depend on n; an appropriate investigation of geodesic motions can then determine the value of n, an interesting topic which nevertheless lies beyond the scope of the present article.

5. WHAT IS LEFT OPEN

The purpose of this paper has been accomplished, in that an extended general relativistic lagrangian treatment for a Weyssenhoff fluid was presented. The paper permits further extension in various directions:

First, since spinning fluids of non-Weyssenhoff types are also considered in the literature[17] (sometimes electrically charged and magnetized[4-6]), they are also awaiting for an extended lagrangian treatment.

Second, gravitational theories other than general relativity have been alternative nests for spinning fluids, so they also deserve a completely metrical lagrangian approach (as opposite to a composite metric-canonical lagrangian approach[17]).

Finally, the outcome $K=0$ of the present paper (eq. (20)) implies that every element of fluid maintains unaltered the magnitude of the specific spin along its worldline, which is a rather restrictive kind of motion for the fluid. As a matter of fact, the constraint $K = 0$ seems to be a commonplace in the literature, e.g. in refs.4-6, 8, 9, 12. Lagrangians predicting motions with K not necessarily null are thus highly desirable.

Thanks are due to the referees, for a number of suggestions which helped to generalize the original typescript

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Resumo

Propomos uma nova funcional lagrangeana para fluidos de Weyssenhoff com spin, em relatividade geral, satisfazendo duas exigências simples: (I) a de que todas as propriedades das variáveis de campo que ocorram na lagrangeana possam ser obtidas exclusivamente a partir das equações de Euler-Lagrange, e (II) a de que seja válida a equação variacional correspondente a cada variável de campo que ocorra na lagrangeana. As equações de Einstein e equações de movimento para o spin são obtidas. Comparamos nossos resultados com similares em relatividade geral e em Einstein-Cartan, e sugerimos tópicos merecedores de pesquisas adicionais.