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On Generalized Elliptic-Type Integrals

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Abstract This work is in continuation of our previous papers [Kalla and Al-Saqabi: this journal, Vol. 16, pp.145-156, 1986; Kalla: Mathematical Structures, Computational Mathematics - Math. Modelling 2, Sofia, 1984, p.216-219]. By using the differencing technique, we express generalized elliptic-type integrals in terms of confluent hypergeometric functions. The method of steepest descent is employed to obtain relations between $K_\mu(k,m)$, $S_\mu(k,\nu)$ and incornplete gamma functions. We tabulate these elliptic-type integrals by using suitable formulae. Some known results follow as particular cases of our formulae established here.

1. INTRODUCTION

In a recent paper[1], the authors have studied a family of integrals

$$K_\mu(k,m) = \int_0^\pi \frac{\cos^{2m}\theta \, d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} \qquad (1)$$

where $0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$ and m is a non-negative integer. For $m = 0$ and $\mu = j$, a positive integer

$$K_j(k,0) = \Omega_j(k) \qquad (2)$$

where

$$\Omega_j(k) = \int_0^\pi (1-k^2\cos\theta)^{-j-\frac{1}{2}} \, d\theta \quad , \qquad (3)$$

$0 \leqslant k < 1$, a family of integrals considered by Epstein and Hubbell[2]. Such integrals are found in the application of the Legendre polynomial expansion method[3] to certain problems involving computation of the radiation field off-axis from a uniform circular disc radiating according to an arbitrary distribution law.

Kalla[4] and Kalla, Conde and Hubbell[5] have defined and studied certain class of generalized elliptic-type integrals. The former are defined as

$$S_\mu(k,\nu) = \int_0^\pi \frac{\sin^{2\nu}\theta \, d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} , \qquad (4)$$

$0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$, $\mathrm{Re}(\nu) > -\frac{1}{2}$.

We observe that,

$$K_0(k,0) = \Omega_0(k) = (\sqrt{2}\,\rho/k)K(\rho) \qquad (5)$$

$$K_1(k,0) = \Omega_1(k) = (\sqrt{2}\,\rho/k(1-k^2)E(\rho) \qquad (6)$$

$$S_0(k,0) = \Omega_0(k) = (\sqrt{2}\,\rho/k)K(\rho) , \qquad (7)$$

and

$$S_1(k,0) = \Omega_1(k) = (\sqrt{2}\,\rho/k(1-k^2))E(\rho), \qquad (8)$$

$$\rho^2 = 2k^2/(1+k^2)$$

where $K(\lambda)$ and $E(\lambda)$ are the complete elliptic integrals of the first and second kind respectively [6, p.295; 7, p.587].

In the previous paper, we obtained a series expansion of $K_\mu(k,m)$ for small values of k, and established its relationship with Gauss' hypergeometric function. Asymptotic expansions, valid in the neighbourhood of $k^2=1$ and some recurrence relations were given.

In this work, we continue our study of the family of integrals eq.(1) and eq.(4). As will also appear in the following sections, the families of integrals $K_\mu(k,m)$ and $S_\mu(k,\nu)$ are related to an interesting class of transcendental functions. By an appeal to the differencing technique, developed for improving the convergence of series[8], we establish their relation with the confluent hypergeometric functions[7,9,10,11]. The method of steepest descent is used to obtain relations with incomplete gamma functions[7,9,10,11]. Some particular cases are mentioned. A number of results obtained earlier by Epstein and Hubbell[2] follow as special cases of our formulae. We tabulate $S_\mu(k,\nu)$ by using its series expansion. We also compute $K_\mu(k,m)$ using two formulae, established in our previous paper.

It is interesting to observe that many formulae established here and in our previous paper for $K_\mu(k,m)$ (m; integer), can be extended to arbitrary values of m.

2. THE DIFFERENCING TECHNIQUE

The behaviour of series expansions can be improved by using a technique developed earlier by Epstein and French[8] for improving the convergence of series. The technique is to express the given integral inthe following form

$$\int_a^b f(\theta)\, d\theta = \int_a^b f^*(\theta)\, d\theta + \int_a^b (f(\theta) - f^*(\theta))\, d\theta \qquad (9)$$

where $f^*(\theta)$ is an integrable function over the interval $a \leqslant \theta \leqslant b$, and is a suitable approximation to $f(\theta)$.

In our case,

$$f(\theta) = \frac{\cos^{2m}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} \qquad (10)$$

Let

$$\begin{aligned} g(\theta) &= (1-k^2\cos\theta)^{-\mu-\frac{1}{2}} \\ &= (1-k^2\cos\theta)^{-\lambda}, \quad \lambda = \mu + \tfrac{1}{2} \\ &= \sum_{r=0}^{\infty} \binom{\lambda}{r} (k^2\cos\theta)^r \\ &\simeq e^{\lambda(k^2\cos\theta)} + \frac{\lambda}{2}(k^2\cos\theta)^2\, e^{(\lambda+\frac{2}{3})(k^2\cos\theta)} \end{aligned} \qquad (11)$$

Hence a good approximation for $f(\theta)$ can be

$$f^*(\theta) = \cos^{2m}\theta \left[e^{\lambda k^2\cos\theta} + \frac{\lambda}{2}(k^2\cos\theta)^2\, e^{(\lambda+\frac{2}{3})(k^2\cos\theta)}\right] \tag{12}$$

and consequently,

$$\int_0^\pi f^*(\theta)\,d\theta = \int_0^\pi \cos^{2m}\theta . e^{(\mu+\frac{1}{2})(k^2\cos\theta)}\,d\theta + \frac{1}{2}(\mu+\frac{1}{2})k^4\int_0^\pi \cos\theta^{2m+2}\, e^{(\mu+\frac{7}{6})(k^2\cos\theta)}\,d\theta \tag{13}$$

Using the transformation $\cos\theta = 1-2w$ and a binomial expansion, we obtain after some simplification,

$$\int_0^\pi f^*(\theta)d\theta = \sum_{n=0}^{2m} 2^n \binom{2m}{n}(-1)^n \int_0^1 w^{n-\frac{1}{2}}(1-w)^{-\frac{1}{2}}\, e^{(\mu+\frac{1}{2})k^2(1-2w)}\,dw + (\mu+\frac{1}{2})k^4 \sum_{n=0}^{2m}(-1)^n\binom{2m+2}{n}2^{n-1}\int_0^1 w^{n-\frac{1}{2}}(1-w)^{-\frac{1}{2}}\, e^{(\mu+\frac{7}{6})k^2(1-2w)}\,dw \tag{14}$$

$$= \sum_{n=0}^{2m}(-1)^n 2^n\binom{2m}{n} e^{k^2(\mu+\frac{1}{2})}\,\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})\,[\Gamma(n+1)]^{-1}\,.$$
$$\phi(n+\frac{1}{2};\ n+1;\ -2k^2(\mu+\frac{1}{2})) + (\mu+\frac{1}{2})k^4\sum_{n=0}^{2m+2}(-1)^n\, 2^{n-1}\binom{2m+2}{n}e^{k^2(\mu+\frac{7}{6})}$$
$$\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})\,[\Gamma(n+1)]^{-1}\ \phi(n+\frac{1}{2},\ n+1;\ -2(\mu+\frac{7}{6})k^2)$$
$$\equiv R(m,k,\mu) \tag{15}$$

by an appeal to the integral representation for the confluent hypergeometric function[9,10,11]

$$\phi(\alpha;\gamma;z) = {}_1F_1(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 e^{zt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}\,dt \tag{16}$$

$$\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0\ .$$

Now, let's consider

$$\int_0^{\pi} (f(\theta) - f^{(*)}(\theta))d\theta = \int_0^{\pi} \frac{\cos^{2m}\theta \, d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} - \int_0^{\pi} \cos^{2m}\theta . e^{(\mu+\frac{1}{2})(k^2\cos\theta)} d\theta$$

$$- \frac{1}{2}(\mu+\frac{1}{2})k^4 \int_0^{\pi} \cos^{2m+2}\theta \; e^{(\mu+\frac{7}{6})(k^2\cos\theta)} d\theta$$

$$= I_1 - I_2 - I_3 \quad , \text{ say} . \tag{17}$$

Further,

$$I_1 = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})_{2n} \sqrt{\pi}\,\Gamma(m+n+\frac{1}{2}) \; k^{4n}}{(2n)! \; \Gamma(m+n+1)} \quad , \tag{18}$$

a result given earlier by Kalla and Al-Saqabi[1].

$$I_2 = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})^n \; k^{2n}}{n!} \int_0^{\pi} \cos^{2m+n}\theta \; d\theta \quad . \tag{19}$$

As

$$\int_0^{\pi} \cos^{i}\theta \; d\theta = \frac{[1 + (-1)^{i}](\sqrt{\pi}/2) \; \Gamma(\frac{i+1}{2})}{\Gamma(\frac{i+2}{2})}$$

we get

$$I_2 = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})^{2n} \; k^{4n} \; (2m+2n)!\pi}{(2n)! \; [(m-n)!]^2 \; 2^{2m+2n}} \quad . \tag{20}$$

Similarly, we have

$$I_3 = \frac{1}{2}(\mu+\frac{1}{2})k^4 \sum_{n=0}^{\infty} \frac{(\mu+\frac{7}{6})^{2n} \; k^{4n}}{(2n)!} \cdot \frac{(2m+2n+2)!\pi}{2^{2m+2n+2}[(m+n+1)!]^2} \tag{21}$$

Hence

$$\int_0^{\pi} (f(\theta) - f^{(*)}(\theta))d\theta = \pi \sum_{n=0}^{\infty} \frac{(2m+2n)!\ k^{4n}}{(2n)!2^{2m+2n}[(m+n)!]^2}$$

$$\Big[(\mu+\tfrac{1}{2})_{2n} - (\mu+\tfrac{1}{2})^{2n} - \frac{(2m+2n+1)}{2(m+n+1)}$$

$$\times \tfrac{1}{2}(\mu+\tfrac{1}{2})k^4 (\mu+\tfrac{7}{6})^{2n}\Big]$$

$$\equiv S(m,k,\mu) \quad . \tag{22}$$

Thus

$$\int_0^{\pi} \frac{\cos^{2m}\theta\ d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} = R(m,k,\mu) + S(m,k,\mu) \tag{23}$$

where $R(m,k,\mu)$ and $S(m,k,\mu)$ are given by eq.(15) and eq. (22) respectively,

Particular Case: For $m=0$, eq. (15) reduces to

$$\int_0^{\pi} f^{(*)}(\theta)d\theta = \int_0^{\pi} \Big[e^{(\mu+\frac{1}{2})(k^2\cos\theta)} + \tfrac{1}{2}(\mu+\tfrac{1}{2})(k^2\cos\theta)^2 \cdot e^{(\mu+\frac{7}{6})(k^2\cos\theta)}\Big]d\theta$$

$$= \pi\Big[I_0((\mu+\tfrac{1}{2})k^2) + \frac{k^4}{4}(2\mu+1)I_0((\mu+\tfrac{7}{6})k^2)$$

$$- \frac{k^2}{4}\frac{(\mu+1)}{(\mu+\frac{7}{6})} I_1((\mu+\tfrac{7}{6})k^2))\Big] \tag{24}$$

where I_0 and I_1 are the modified Bessel functions of the first kind[10,11]. To obtain eq. (24) from eq. (15), we have used the following results[10]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} e^{-z} \phi(\nu+\tfrac{1}{2}, 2\nu+1, 2z) \tag{25}$$

$$(\gamma-\alpha-1)\Phi + \alpha\Phi(\alpha+1) - (\gamma-1)\Phi(\gamma-1) = 0 \ . \tag{26}$$

Further, we observe from eq.(22), that $m=0$, $V=j$, an integer, leads to

$$\frac{1}{\pi}\int_0^{\pi} (f(\theta) - f^*(\theta))_{m=0} d\theta = \sum_{n=0}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2}\left\{\frac{(2j+4n)!(j!)}{(j+2n)!(2j)!} - 2^{2n}(2j+1)^{2n}\right\}$$

$$- \sum_{n=1}^{\infty} 18(2j+1)\left(\frac{2}{3}\right)^{2n}(6j+7)^{2n} n(2n-1)2^{-6n}(n!)^{-2}k^{4n}$$

$$= \sum_{n=2}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2}\left\{\frac{(2j+4n)!(j)!}{(j+2n)!(2j)!} - 2^{2n}(2j+1)^{2n} - 18\left(\frac{2}{3}\right)^{2n}(2j+1)(6j+7)^{2n-2}\, n(2n-1)\right\} \tag{27}$$

eqs.(24) and (27) are in agreement with the results of Epstein and Hubbell[2].

3. $K_\mu(k,m)$ IN TERMS OF INCOMPLETE GAMMA FUNCTIONS

Let

$$F(\theta) = (\mu + \frac{1}{2}) \log(1 - k^2\cos\theta) \tag{28}$$

hence

$$K_\mu(k,m) = \int_0^{\pi} e^{-F(\theta)} \cos^{2m}\theta \, d\theta \tag{29}$$

Expanding $F(\theta)$ in Maclaurin series, we observe that all the odd derivatives of $F(\theta)$ vanish at the origin and we can write

$$F(\theta) = F(0) + \frac{F^{(2)}(0)}{2!}\theta^2 + \frac{F^{(4)}(0)}{4!}\theta^4 + \frac{F^{(6)}(0)}{6!}\theta^6 + \ldots \tag{30}$$

then

$$K_\mu(k,m) = e^{-F(0)} \int_0^\pi e^{-\frac{F^{(2)}(0)}{2!}\theta^2} \left[1 - \frac{F^{(4)}(0)}{4!}\theta^4 - \frac{F^{(6)}(0)}{6!}\theta^6 - \dots\right] \cos^{2m}\theta \, d\theta \qquad (31)$$

$$F(0) = (\mu + \tfrac{1}{2}) \log(1 - k^2) \; .$$

Let

$$(\tfrac{\lambda}{\pi})^2 = \frac{F^{(2)}(0)}{2!}$$

then

$$K_\mu(k,m) = (1-k^2)^{-(\mu+\frac{1}{2})} \int_0^\pi \cos^{2m}\theta . e^{-(\frac{\lambda}{\pi})^2\theta^2} \left[1 - \frac{F^{(4)}(0)}{4!}\theta^4 - \frac{F^{(6)}(0)}{6!}\theta^6 \dots\right] d\theta \qquad (32)$$

If we write $t = \lambda\theta/\pi$ and let

$$L_{(p,m)}(\lambda) = \int_0^\lambda t^{2p} e^{-t^2} \cos^{2m} (\tfrac{\pi t}{\lambda}) \, dt \qquad (33)$$

then

$$K_\mu(k,m) = \pi(\lambda)^{-1} (1-k^2)^{-(\mu+\frac{1}{2})} \int_0^\lambda \mathrm{cor}^{2m} (\tfrac{\pi t}{\lambda}) e^{-t^2} \left[1 - \frac{F^{(4)}(0)}{4!} (\tfrac{\pi t}{\lambda})^4 - \frac{F^{(6)}(0)}{6!} (\tfrac{\pi t}{\lambda})^6 - \dots\right] dt$$

$$= \pi(\lambda)^{-1} (1-k^2)^{-(\mu+\frac{1}{2})} \left[L_{(0,m)}(\lambda) - \frac{F^{(4)}(0)}{4!} (\tfrac{\pi}{\lambda})^4 L_{(2,m)}(\lambda) - \frac{F^{(6)}(0)}{6!} (\tfrac{\pi}{\lambda})^6 L_{(3,m)}(\lambda) - \dots\right] \; . \qquad (34)$$

Using the result

$$\cos^{2m} A = 2^{-2m} \binom{2m}{m} + 2^{1-2m} \sum_{r=1}^{m} \binom{2m}{m-r} \cos 2rA \qquad (35)$$

we can write

$$L_{(p,m)}(\lambda) = \int_0^{\lambda} t^{2p} e^{-t^2} \cos^{2m}\left(\frac{\pi t}{\lambda}\right) dt$$

$$= 2^{-2m} \binom{2m}{m} \int_0^{\lambda} t^{2p} e^{-t^2} dt + 2^{1-2m} \sum_{r=1}^{m} \binom{2m}{m-r} \sum_{j=0}^{\infty} \left(\frac{2\pi r}{\lambda}\right)^{2j} ((2j)!)^{-1} \cdot \int_0^{\lambda} t^{2p+2j} e^{-t^2} dt$$

$$= 2^{-2m-1} \binom{2m}{m} \gamma(p+\tfrac{1}{2}, \lambda^2) + 2^{-2m} \sum_{r=1}^{m} \binom{2m}{m-r} \sum_{j=0}^{\infty} \left(\frac{2\pi r}{\lambda}\right)^{2j} ((2j)!)^{-1} \gamma(p+j+\tfrac{1}{2}, \lambda^2) \qquad (36)$$

where $\gamma(a,\lambda)$ are the incomplete gamma functions, defined as

$$\gamma(a,\lambda) = \int_0^{\lambda} e^{-t} t^{a-1} dt, \quad \mathrm{Re}(a) > 0 . \qquad (37)$$

This process is also known as the method of steepest descent. Another alternative form to deal with $L_{(p,m)}(\lambda)$ is to obtain a suitable recurrence formula. For example, integrating eq. (33) by parts we obtain

$$(2p+1)L_{(p,m)}(\lambda) = e^{-\lambda^2} \lambda^{2p+1} + 2L_{(p+1,m)}(\lambda) + \frac{2m\pi}{\lambda} \int_0^{\lambda} e^{-t^2} t^{2p+1} \cos^{2m-1}\left(\frac{\pi t}{\lambda}\right) \sin\left(\frac{\pi t}{\lambda}\right) dt . \qquad (38)$$

For $m = 0$, eq. (38) reduces to a known result:

$$L_{(p+1,0)}(\lambda) = \frac{1}{2} \left[(2p+1)L_{(p,0)}(\lambda) - \lambda^{2p+1} e^{-\lambda^2}\right]. \qquad (39)$$

Further, we note that

$$L_{(0,0)}(\lambda) = \int_0^{A} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\,\mathrm{Erf}.\lambda \tag{40}$$

where Erf.λ is the error function, defined as [7, p.297]

$$\mathrm{Erf}.\lambda = \frac{2}{\sqrt{\pi}} \int_0^{\lambda} e^{-t^2}\, dt \quad . \tag{41}$$

An alternative recurrence relation for $L_{(p,m)}(\lambda)$ can be derived in the following form:

$$\begin{aligned} L_{(p,m)}(\lambda) = {} & \left[2m^2 + \left(\frac{\lambda}{\pi}\right)^2 (4p+1)\right]^{-1} \left\{\left(\frac{\lambda}{\pi}\right)^2 e^{-\lambda^2} \left(\lambda^{2p+1} - p\lambda^{2p-1}\right)\right. \\ & + \left(\frac{\lambda}{\pi}\right)^2 \left[p(2p-1)L_{(p-1,m)}(\lambda) + 2L_{(p+1,m)}(\lambda)\right] \\ & \left. + m(2m-1)L_{(p,m-1)}(\lambda)\right\} \quad . \end{aligned} \tag{42}$$

For $m = 0$, eq. (42) reduces to

$$\begin{aligned} (4p+1)L_{(p,0)}(\lambda) = {} & \left(\lambda^{2p+1} - p\lambda^{2p-1}\right)e^{-\lambda^2} + p(2p-1)L_{(p-1,0)}(\lambda) \\ & + 2L_{(p+1,0)}(\lambda) \end{aligned} \tag{43}$$

4. SOME RESULTS FOR $S_\mu(k,\nu)$

1. The differencing technique for $S_\mu(k,\nu)$

We shall use the differencing technique mentioned in section 2 for the function *f(8)* given, for this case, by

$$f(\theta) = \frac{\sin^{2\nu}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} \tag{44}$$

where $f^*(\theta)$ is given by

$$f^*(\theta) = \sin^{2\nu}\theta\left[e^{(\mu+\frac{1}{2})(k^2\cos\theta)} + \frac{1}{2}(\mu+\frac{1}{2})(k^2\cos\theta)^2 e^{(\mu+\frac{7}{6})(k^2\cos\theta)}\right] \quad (45)$$

Using the transformation cos θ = 1-2w and binomial expansion, we obtain after simplification

$$\int_0^\pi f^*(\theta)d\theta = \int_0^\pi \sin^{2\nu}\theta\; e^{(\mu+\frac{1}{2})(k^2\cos\theta)} + \frac{1}{2}(\mu+\frac{1}{2})k^4\cos^2\theta\; e^{(\mu+\frac{7}{6})(k^2\cos\theta)}\, d\theta \quad (46)$$

$$= 2^\nu e^{(\mu+\frac{1}{2})k^2}\int_0^1 \omega^{\nu-\frac{1}{2}}(1-w)^{\nu-\frac{1}{2}} e^{-2(\mu+\frac{1}{2})k^2 w}\,dw$$

$$+ (\mu+\frac{1}{2})2^{2\nu-1} e^{k^2(\mu+\frac{7}{6})}k^4 \left\{\int_0^1 w^{\nu-\frac{1}{2}}(1-w)^{\nu-\frac{1}{2}} e^{-2k^2(\mu+\frac{7}{6})w}\,dw\right.$$

$$- 4\int_0^1 w^{\nu+\frac{1}{2}}(1-w)^{\nu-\frac{1}{2}} e^{-2k^2(\mu+\frac{7}{6})w}\,dw$$

$$\left.+ 4\int_0^1 w^{\nu+\frac{3}{2}}(1-w)^{\nu-\frac{1}{2}} e^{-2k^2(\mu+\frac{7}{6})w}\,dw\right\} \quad (47)$$

Bu using eq. (16) we have

$$\int_0^\pi f^*(\theta)d\theta = 2^\nu e^{(\mu+\frac{1}{2})k^2}\frac{\Gamma(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\Phi(\nu+\frac{1}{2}, 2\nu+1, -2k^2(\mu+\frac{1}{2}))$$

$$+ (\mu+\frac{1}{2})e^{k^2(\mu+\frac{7}{6})} 2^{2\nu-1}k^4\left\{\frac{\Gamma(\nu+\frac{1}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\Phi(\nu+\frac{1}{2}, 2\nu+1, -2k^2(\mu+\frac{7}{6})\right.$$

$$- 4\frac{\Gamma(\nu+\frac{3}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+2)}\Phi(\nu+\frac{3}{2}, 2\nu+2, -2k^2(\mu+\frac{7}{6}))$$

$$\left.+ 4\frac{\Gamma(\nu+\frac{5}{2})\Gamma(\nu+\frac{1}{2})}{\Gamma(2\nu+3)}\Phi(\nu+\frac{5}{2}, 2\nu+3, -2k^2(\mu+\frac{7}{6}))\right\}$$

$$\equiv B(\nu,k,\mu) \quad (48)$$

Now cons'der

$$\int_0^{\pi} (f(\theta)-f^*(\theta))d\theta = \int_0^{\pi} \frac{\sin^{2\nu}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} - \int_0^{\pi} \sin^{2\nu}\theta\, e^{(\mu+\frac{1}{2})(k^2\cos\theta)} d\theta$$

$$- \frac{1}{2}(\mu+\frac{1}{2})k^4 \int_0^{71} \sin^{2\nu}\theta \cos^2\theta\, e^{(\mu+\frac{7}{6})(k^2\cos\theta)} d\theta \,.$$

Let

$$\int_0^{\pi} (f(\theta) - f^*(\theta))d\theta = M_1 - M_2 - M_3$$

where

$$M_1 = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})_{2n}\,\Gamma(\nu+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2n)!\,\Gamma(\nu+n+1)} k^{4n} \,, \tag{49}$$

A result given by Kalla[4].

$$M_2 = \int_0^{\pi} \sin^{2\nu}\theta\, e^{(\mu+\frac{1}{2})(k^2\cos\theta)} d\theta$$

$$= \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})^n}{n!} k^{2n} \int_0^{\pi} \sin^{2\nu}\theta \cos^n\theta\, d\theta$$

since

$$\int_0^{\pi} \sin^{2\nu} \cos^r\theta d\theta = \frac{\Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{2\nu+r}{2}+1\right)} \qquad \text{for r even}$$

$$= 0 \qquad \text{for r odd}$$

Then

$$M_2 = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})^{2n}}{(2n)!} k^{4n} \frac{\Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma\left(\frac{2\nu+2n}{2}+1\right)} \tag{50}$$

Similarly we have .

$$M_3 = \frac{1}{2}(\mu+\frac{1}{2})k^4 \sum_{n=0}^{\infty} \frac{(\mu+\frac{7}{6})^{2n}k^{4n}}{(2n)!} \frac{\Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{2n+2+1}{2}\right)}{\Gamma\left(\frac{2\nu+2n+2}{2}+1\right)} \tag{51}$$

Then, we have

$$\int_0^{\pi} (f(\theta)-f^*(\theta))d\theta = \sum_{n=0}^{\infty} \frac{(\mu+\frac{1}{2})_{2n}\Gamma(\nu+\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2n)!\ \Gamma(\nu+n+1)} k^{4n}$$

$$- \sum_{n=0}^{\infty} \frac{(\nu+\frac{1}{2})^{2n}\ \Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{(2n)!\ \Gamma(\nu+n+1)} k^{4n}$$

$$- \frac{1}{2}(\mu+\frac{1}{2})k^4 \sum_{n=0}^{\infty} \frac{(\mu+\frac{7}{6})^{2n}\ \Gamma\left(\frac{2\nu+1}{2}\right)\Gamma\left(\frac{2n+3}{2}\right)}{\Gamma(\nu+n+2)} k^{4n}$$

$$\equiv N(\nu,k,\mu) \tag{52}$$

Therefore

$$\int_0^{\pi} \frac{\sin^{2\nu}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}} = B(\nu,k,\mu) + N(\nu,k,\mu) \tag{53}$$

Particular cases:

For $\nu = 0$ eq. (48) reduces to

$$\int_0^{\pi} f^*(\theta)d\theta = \int_0^{\pi} \left[e^{(\mu+\frac{1}{2})(k^2\cos\theta)} + \frac{1}{2}(\mu+\frac{1}{2})(k^2\cos\theta)^2 e^{(\mu+\frac{7}{6})(k^2\cos\theta)}\right]d\theta$$

$$= \pi\left[I_0((\mu+\frac{1}{2})k^2) + \frac{k^4}{4}(2\mu+1)I_0((\mu+\frac{7}{6})k^2)\right.$$

$$\left. - \frac{k^2}{4}\frac{(2\mu+1)}{(\mu+\frac{7}{6})} I_1((\mu+\frac{7}{6})k^2)\right] \tag{54}$$

We notice that eq. (54) is equal to eq.(24) and obtained in a similar way. For $\nu = 0$ and $\mu = j$, an integer, eq.(52) reduces to

$$\int_0^{\pi} (f(\theta) - f^*(\theta))d\theta = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{(2n)!\Gamma(n+1)} k^{4n} \Big[(j+\tfrac{1}{2})_{2n} - (j+\tfrac{1}{2})^{2n}$$

$$- \frac{k^4}{2} \frac{(j+\frac{1}{2})(j+\frac{7}{6})^{2n}(n+\frac{1}{2})}{(n+1)}\Big] \quad (55)$$

As

$$\Gamma(n+\tfrac{1}{2}) = \frac{(2n)!\Gamma(\frac{1}{2})}{n!\,2^{2n}} ,$$

therefore eq. (55) becomes

$$\frac{1}{\pi}\int_0^{\pi} [f(\theta) - f^*(\theta)]d\theta = \sum_{n=0}^{\infty} \frac{k^{4n}}{(n!)^2\,2^{2n}} \Big[\frac{(2j+4n)!\;j!}{(j+2n)!\,2^{4n}(2j)!} - \frac{(2j+1)^{2n}}{2^{2n}}$$

$$- \frac{k^4}{2}\frac{(2j+1)}{2}\frac{(6j+7)^{2n}}{6^{2n}}\frac{(2n+1)}{2(n+1)}\Big]$$

$$= \sum_{n=0}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2}\left\{\frac{(2j+4n)!\,j!}{(j+2n)!\,(2j)!} - 2^{2n}(2j+1)^{2n}\right\}$$

$$- \sum_{n=1}^{\infty} (2j+1)18\,(\tfrac{2}{3})^{2n}\,(6j+7)^{2n}\,\frac{n(2n-1)k^{4n}}{6^{2n}(n!)^2}$$

$$= \sum_{n=2}^{\infty} \frac{k^{4n}}{2^{6n}(n!)^2}\Big\{\frac{(2j+4n)!\,j!}{(j+2n)!\,(2j)!} - 2^{2n}(2j+1)^{2n}$$

$$-18(\tfrac{2}{3})^{2n}(2j+1)(6j+7)^{2n-2}n(2n-1)\Big\} \quad (56)$$

We notice that eq.(56) is equal to eq.(27), which is given by Epstein and Hubbell[2].

II. $S_\mu(k,\nu)$ in terms of the Incomplete Gamma Function

$$S_\mu(k,\nu) = \int_0^\pi \frac{\sin^{2\nu}\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}$$

$0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$, $\mathrm{Re}(\nu) > -\frac{1}{2}$.

Let

$$S_\mu(k,\nu) = \int_0^\pi \sin^{2\nu}\theta \; e^{-F(\theta)} d\theta \tag{57}$$

where

$$F(\theta) = (\mu+\tfrac{1}{2}) \log(1-k^2\cos\theta) \tag{58}$$

Expanding $F(0)$ in Maclaurin series, we observe that the derivatives of odd order vanish and hence

$$S_\mu(k,\nu) = \int_0^\pi \sin^{2\nu}\theta \; e^{-\left[F(0) + \frac{\theta^2}{2!} F^{(2)}(0) + \frac{\theta^4}{4!} F^{(4)}(0) + \ldots\right]} d\theta \tag{59}$$

Let

$$\left(\frac{\lambda}{\pi}\right)^2 = \frac{1}{2!} F^{(2)}(0)$$

$$S_\mu(k,\nu) = e^{-F(0)} \int_0^\pi \sin^{2\nu}\theta \; e^{-\left(\frac{\lambda}{\pi}\right)^2\theta^2} \left[1 - \frac{F^{(4)}(0)}{4!}\theta^4 - \frac{F^{(6)}(0)}{6!}\theta^6 \ldots\right] d\theta \tag{60}$$

Now

$$F(0) = (\mu + \tfrac{1}{2}) \log(1-k^2)$$

$$t = \frac{\lambda}{\pi}\theta \; ; \; dt = \frac{\lambda}{\pi} d\theta$$

$$S_\mu(k,\nu) = (1-k^2)^{-\mu-\frac{1}{2}} \left(\frac{\lambda}{\pi}\right) \int_0^\lambda \sin^{2\nu}\left(\frac{\pi t}{\lambda}\right) e^{-t^2} \left[1 - \frac{F^{(4)}(0)}{4!}\left(\frac{\pi}{\lambda}\right)^4 t^4 - \ldots\right] dt \tag{61}$$

Let

$$L_{(p,\nu)}(\lambda) = \int_0^\lambda \sin^{2\nu}\left(\frac{\pi t}{\lambda}\right) t^{2p}\, e^{-t^2}\, dt \tag{62}$$

then

$$S_\mu(k,\nu) = (1-k^2)^{-\mu-\frac{1}{2}}\ \frac{\pi}{\lambda}\left[L_{(0,\nu)}(\lambda) - \frac{F^{(4)}(0)}{4!}\left(\frac{\pi}{\lambda}\right)^4 L_{(2,\nu)}(\lambda) - \frac{F^{(6)}(0)}{6!}\left(\frac{\pi}{\lambda}\right)^6 L_{(3,\nu)}(\lambda)\ \ldots\right] \tag{63}$$

for ν *is an integer*, and using

$$\sin^{2\nu}A = \frac{1}{2^{2\nu}}\binom{2\nu}{\nu} + \frac{1}{2^{2\nu-1}}\sum_{r=1}^{\nu}(-1)^r\binom{2\nu}{\nu-r}\cos 2rA \tag{64}$$

we have

$$\begin{aligned} L_{(p,\nu)}(\lambda) &= \int_0^\lambda t^{2p}\, e^{-t^2}\left[\frac{1}{2^{2\nu}}\binom{2\nu}{\nu} + \frac{1}{2^{2\nu-1}}\sum_{r=1}^{\nu}(-1)^r\binom{2\nu}{\nu-r}\cos\left(\frac{2r\pi t}{\lambda}\right)\right] dt \\ &= \frac{1}{2^{2\nu}}\binom{2\nu}{\nu}\int_0^\lambda t^{2p}\, e^{-t^2} dt + \frac{1}{2^{2\nu-1}}\sum_{r=1}^{\infty}(-1)^r\binom{2\nu}{\nu-r}\int_0^\lambda t^{2p}\, e^{-t^2}\cos\left(\frac{2r\pi t}{\lambda}\right) dt \\ &= \frac{1}{2^{2\nu+1}}\binom{2\nu}{\nu}\gamma\left(p+\frac{1}{2};\lambda^2\right) + \frac{1}{2^{2\nu}}\sum_{r=1}^{\nu}(-1)^r\binom{2\nu}{\nu-r}\sum_{j=0}^{\infty}\left(\frac{2\pi r}{\lambda}\right)^{2j}\frac{1}{(2j)!}\,\gamma\left(p+j+\frac{1}{2};\ \lambda^2\right) \end{aligned} \tag{65}$$

Particular cases:

$$\begin{aligned} L_{(p,0)}(\lambda) &= \frac{1}{2}\gamma\left(p+\frac{1}{2};\ \lambda^2\right) \\ L_{(p,1)}(\lambda) &= \frac{1}{2^2}\gamma\left(p+\frac{1}{2};\lambda^2\right) - \frac{1}{2^2}\sum_{j=0}^{\infty}\left(\frac{2\pi}{\lambda}\right)^{2j}\frac{1}{(2j)!}\,\gamma\left(p+j+\frac{1}{2};\lambda^2\right) \end{aligned} \tag{66}$$

$F^{(2)}(0)$, $F^{(4)}(0)$ and $F^{(6)}(0)$ given above are

$$\begin{aligned} F^{(2)}(0) &= (\mu + \tfrac{1}{2})k^2(1-k^2)^{-1} \\ F^{(4)}(0) &= -(\mu + \tfrac{1}{2})k^2(1-k^2)^{-2}(2k^2+1) \\ F^{(6)}(0) &= (\mu + \tfrac{1}{2})k^2(1-k^2)^{-3}(16k^4+13k^2+1) \end{aligned} \tag{67}$$

and

$$\lambda = \frac{\pi k}{2}\sqrt{\frac{2\mu+1}{1-k^2}} \ .$$

5. COMPUTATIONS

In table 1, we have computed the series expansions of $K_\mu(k,m)$ which are given by eq.(11) and eq.(12) of ref.1, these equations are

$$K_\mu(k,m) = \sum_{r=0}^{\infty} W_r(\mu,m)\ k^{4r} \tag{68}$$

and

$$W_r(\mu,m) = \frac{(\mu+\frac{1}{2})_{2r}\sqrt{\pi}\,\Gamma(m+r+\frac{1}{2})}{(2r)!\ \Gamma(m+r+1)} \tag{69}$$

In fact, eq.(68) and eq.(69) were tabulated in table 1 of ref.1, but we have computed it again due to a small error in the computations of some values of $K_\nu(k,m)$. The numerical integration of eq.(1), by using the trapezoidal rule, is represented by KI.

In table 2, we have computed the series expansion of $S_\mu(k,\nu)$ which is given by eq.(4) of ref.4:

$$S_\mu(k,\nu) = \sum_{r=0}^{\infty} \frac{(\mu+\frac{1}{2})_{2r}\Gamma(\nu+\frac{1}{2})\Gamma(r+\frac{1}{2})}{(2r)!\ \Gamma(\nu+r+1)}\ k^{4r} \tag{70}$$

Equation (4) of ref.4 should be corrected to eq.(70), because the factor $[\Gamma(\nu+\frac{1}{2})]^2$ was typed instead of $\Gamma(\nu+\frac{1}{2})$.

The approximate asymptotic formula of K (k,m) in the neighbourhood of $k=1$ is obtained in simpler form and is given by

Table 1 - Values of $K_\mu(k,m)$ be series expansion eqr. (68), (69) and numerical integration of eq. (1) denoted by (KI).

k	m	μ	$K_\mu(k,m)$	KI
0.00	0	0.0	3.141593	3.1415926
0.05	0	0.0	3.1415963	3.1415963
0.1	0	0.0	3.1416515	3.1416515
0.1	0	1.0	3.1418871	3.1418872
0.3	0	2.5	3.2191074	3.2191075
0.4	0	5.0	3.939221	3.9392216
0.5	0	5.0	5.4212342	5.4212351
0.05	1	0.5	1.5709141	1.5708036
0.05	1	7.0	1.5710309	1.57103103
0.15	1	1.0	1.5719152	1.5719152
0.15	1	1.5	1.5725867	1.5725868
0.25	1	4.0	1.6285054	1.6285054
0.2	1	7.5	1.6394901	1.6394901
0.5	1	0.5	1.6484845	1.6484846
0.5	1	7.5	5.9382731	5.9382746
0.05	2	0.0	1.1780995	1.1780995
0.1	2	0.0	1.1781340	1.1781341
0.1	2	7.5	1.181634	1.18163437
0.15	2	1.0	1.1790296	1.1790297
0.15	2	3.5	1.183075	1.1830751
0.2	2	7.5	1.235377	1.23537708
0.5	2	0.0	1.202069	1.202069
0.5	2	6.0	3.3738182	3.3738188
0.1	3	0.0	0.9817799	0.9817799
0.1	3	7.0	0.9844879	0.9844879
0.2	3	7.0	1.0260863	1.0260863
0.2	5	4.5	0.7902535	0.7902535
0.1	5	1.0	0.773259	0.773259
0.15	6	2.0	0.7101581	0.7101581
0.30	6	7.5	0.914499	0.914499
0.50	6	7.5	3.294679	3.2946807

$$K_\mu(k,m) \sum_{r=0}^{2m} \frac{(-1)^r 2^r \binom{2m}{r} \left(r+\frac{1}{2}\right)\Gamma\left(\mu - r - \frac{1}{2\lambda'}\right)}{(1-k^2)^{\mu+\frac{1}{2}} \lambda'^{r+\frac{1}{2}} \Gamma\left(\mu+\frac{1}{2}-\frac{1}{2\lambda'}\right)} \quad (71)$$

where $\lambda' = 2k^2/1-k^2$.

This formula is much better than that one given by eq.(24) of ref.1, see table 3 of ref.1.

In table 3, eq. (71) is computed and is represented by $K3$.

We have also computed the asymptotic formula for $K_\mu(k,m)$ in the neighbourhood of k = 1 which is given by eq.(31) of ref.1,

Table 2 - Values of $S_\mu(k,\nu)$ for ν = 0,1 and some selected values of μ and k^2 by using series expansion eq. (70).

ν=0	k^2 \ μ	1	2	3	4	5	6	7	8	9
	0.01	3.141887	3.142280	3.142830	3.143537	3.144402	3.145423	3.146603	3.147940	3.149434
	0.02	3.142771	3.144343	3.146545	3.149387	3.152842	3.156939	3.161670	3.167037	3.173043
	0.03	3.144264	3.14786	3.152748	3.159135	3.166954	3.176208	3.186908	3.199061	3.212675
	0.04	3.146312	3.152616	3.161456	3.172846	3.186805	3.203351	3.222507	3.244300	3.268760
	0.05	3.148974	3.158840	3.172591	3.190562	3.212492	3.238531	3.268736	3.303171	3.341910
	0.10	3.171338	3.211392	3.368106	3.342036	3.433876	3.544462	3.674782	3.825987	3.999397
	0.15	3.209361	3.301768	3.434487	3.610523	3.833699	4.108742	4.441404	4.838579	5.308465
	0.20	3.264233	3.434460	3.683954	4.023088	4.465477	5.028575	5.731418	6.610573	7.691301
	0.25	3.337744	3.616446	4.035910	4.624564	5.421231	6.477918	7.863508	9.668643	12.012170
	0.30	3.432453	3.858021	4.520192	5.486993	6.855988	8.764916	11.406770	15.050960	20.072980
	0.35	3.551918	4.174221	5.182451	6.726625	9.034357	12.447290	17.475920	24.882040	35.802340
	0.40	3.701079	4.586979	6.093474	8.535160	12.421700	18.573190	28.306960	43.739580	68.275140
	0.45	3.886838	5.128978	7.365833	11.239530	17.870950	29.210280	48.650950	82.103690	139.871600
	0.50	4.119012	5.850103	9.184889	15.421710	27.022583	48.673690	89.240820	105.56970	309.567000

ν=1	k^2 \ μ	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
	0.01	1.570831	1.570866	1.570910	1.570964	1.571028	1.571101	1.571185	1.571278	1.571381	1.571494
	0.02	1.570949	1.571087	1.571263	1.571479	1.571734	1.572029	1.572364	1.572737	1.573150	1.573603
	0.03	1.571145	1.571455	1.571852	1.572340	1.572915	1.573579	1.574332	1.575174	1.576105	1.577125
	0.04	1.571420	1.571971	1.572679	1.573545	1.574569	1.575752	1.577093	1.578593	1.580252	1.582071
	0.05	1.571775	1.572636	1.573743	1.575098	1.576701	1.578552	1.580652	1.583003	1.585604	1.588457
	0.10	1.574739	1.578203	1.582671	1.588152	1.594652	1.602184	1.610759	1.620392	1.631097	1.642890
	0.15	1.579728	1.587608	1.597806	1.610366	1.625336	1.642777	1.662749	1.685327	1.710590	1.738629
	0.20	1.586822	1.601040	1.619544	1.642469	1.669932	1.702273	1.739565	1.782102	1.830167	1.884074
	0.25	1.596134	1.618791	1.648479	1.685560	1.730461	1.783686	1.845825	1.917552	1.999643	2.092984
	0.30	1.607820	1.641256	1.685454	1.741210	1.809492	1.891450	1.988452	2.102102	2.234277	2.387164
	0.35	1.622091	1.668974	1.731616	1.811614	1.910944	2.032007	2.177714	2.351557	2.557728	2.387164
	0.40	1.639216	1.702662	1.788532	1.899825	2.040326	2.214754	2.428960	2.690171	3.007283	3.391240
	0.45	1.659554	1.743277	1.858344	2.010127	2.205577	2.453613	2.765614	3.156070	3.643406	4.251043
	0.50	1.683569	1.792108	1.944018	2.148632	2.418390	2.769764	3.224505	3.811293	4.567959	5.544431

Table 3 - Tabulation of $K_\mu(k,m)$ by using eq. (71) denoted by K3 and numerical integration (KI).

k	m	μ	K3	KI
0.99	0	0.4	0.1270754 D 02	0.1216152 D 02
0.99	0	1.9	0.1675509 D 04	0.1677898 D 04
0.90	0	1.9	0.2574906 D 02	0.2617420 D 02
0.90	0	8.6	0.7711231 D 06	0.7714421 D 06
0.99	0	8.6	0.1868443 D 15	0.18685004D 15
0.99	1	7.6	0.3960815 D 13	0.39509723D 13
0.99	2	1.6	0.5272625 D 03	0.5424871 D 03
0.99	2	8.1	0.2702932 D 14	0.2703030 D 14
0.93	3	8.6	0.1319975 D 08	0.1320232 D 08
0.99	3	8.6	0.18536288D 15	0.18536835D 15
0.90	3	8.6	0.7076432 D 06	0.7078339 D 06
0.966	0	1	22.146192	22.48500
0.999	0	1	708.386432	709.35153
0.999	2	3	0.94342702D 08	0.9434021D 08
0.999	3	4	0.40449339D 11	0.40450023D 11
0.999	3	6	0.81860997D 16	0.81861535D 16

$$K_\mu(k,m) = \frac{k^{-2\mu-1}}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r}$$

$$\left[\sum_{n=0}^{\nu_r-1} \frac{\Gamma(r+n+\frac{1}{2})\Gamma(\frac{1}{2}+n)\Gamma(\nu_r-n)}{\Gamma(\frac{1}{2})n!\, 2^{n+\frac{1}{2}}} \left(\frac{1-k^2}{k^2}\right)^{n-\nu_r} \right. \tag{72}$$

$$\left. + \sum_{n=\nu_r}^{\infty} \frac{(-1)^{n-\nu_r+1}\, \Gamma(r+n+\frac{1}{2})\, (\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\; n!\; 2^{n+\frac{1}{2}}\, (n-\nu_r)!} ((1-k^2)/k^2)^{n-\nu_r} \log\left(\frac{1-k^2}{k^2}\right)\right]$$

This formula is valid for ν_r (positive integer) $= \mu - r$.

In table 4 have computed eq. (72) which is represented by $K4$. It is compared with $K3$ and KI.

Table 4 - Tabulation of $K_\mu(k,m)$ by using eq. (72) denoted by K4, compared with K3 and (KI).

k	m	μ	K3	KI	K4
0.999	0	5	0.180131799D 14	0.18013358D 14	0.18013333 D 14
0.999	1	3	0.9943098D 08	0.94434242D 08	0.94432703D 08
0.999	0	2	0.236207D 06	0.236233 D 06	0.236600D 06
0.966	1	5	0.438736D 06	0.438886D 06	0.438883D 06
0.999	1	5	0.180131799D 14	0.18013358D 14	0.18013333D 14
0.999	2	6	0.81893723 D 16	0.81894262D 16	0.81893723D 16
0.999	3	8	0.176551136D 22	0.17655156D 22	0.17660177D 22

Results were computed on a VAX/VMS electronic computing machine.

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Resumo

Este trabalho é uma continuação de nossos artigos anteriores [Kalla e Al-Saqabi: Rev.Bras.Fis.*16*, pp.145-156; Kalla: Mathematical Structures, Computational Mathematics - Math.Modelling 2, Sofia, 1984, pp. 216-219]. Usando a técnica de diferenciamento, expressamos integrais de tipo elíptico generalizadas em termos de funções hipergeométricas confluentes. O método de ponto de sela é empregado para obter relações entre $K_\mu(k,m)$, $S_\mu(k,\nu)$ e funções gama incompletas. Estas integrais do tipo elíptico são tabuladas, através do uso de fórmulas apropriadas. Alguns resultados conhecidos seguem como casos particulares das fórmulas aqui estabelecidas.