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Representations of the Lorentz-Screw Transformation

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Abstract A Lorentz 4-screw is a particular transformation, defined and studied by J.L.Synge. We advance his thwry, by presenting two representations of this transformation, the spinorial, and the tridimensional (on the surface of the unit sphere).

1. INTRODUCTION

J.L.Singe[1] is responsible for the name and study of the Lorentz 4-screw transformation. A Lorentz 4-screw is a Lorentz transformation consisting of a rotation in a timelike 2-flat π, followed by (or preceded by) a rotation in a spacelike 2-flat π^*, the two 2-flat π and π^* being orthogonal to one another.

If we call $(\vec{I}, \vec{J}, \vec{K}, \vec{L})$ a unit orthogonal tetrad, $\vec{L}$ timelike and pointing into the future, so that $\vec{I}$ and $\vec{J}$ lie in π^* while $(\vec{K},\vec{L})$ lie in π, we can write for the overall transformation:

$$\begin{aligned}
\vec{I}'' &= \vec{I}\cos\theta + \vec{J}\sin 0,\\
\vec{J}'' &= -\vec{I}\sin\theta + \vec{J}\cos\theta,\\
\vec{K}'' &= \vec{K}\cosh\chi + \vec{L}\sinh\chi \quad ,\\
\vec{L}'' &= \vec{K}\sinh\chi + \vec{L}\cosh\chi
\end{aligned} \tag{1.1}$$

We shall choose

$$\begin{aligned}
\cosh\chi &= \sec a \quad ,\\
\text{senh}\,\chi &= \tan a \quad .
\end{aligned} \tag{1.2}$$

In our notation, the two angles of rotation are 0 and a. We shall study two types of representations:

a - Spinorial representation;

b - Representation on the three-dimensional surface, or, said in other terms;
Representation by transformation of triads of null rays.

2. SPINORIAL REPRESENTATION OF 4-SCREW

We may represent a spinor by two points ξ and μ in the Argand plane (Synge[1]). If ξ and μ represent the initiat spinor, let ξ'' and μ'' represent the Lorentz 4-screw transformation on them. Synge showed that the θ transfarmation is given by

$$\xi' = \xi e^{i\frac{\theta}{2}} , \qquad \mu = \mu e^{-i\frac{\theta}{2}} \tag{2.1}$$

On the other hand, the χ rotation is given by

$$\xi'' = \xi' e^{-\frac{\chi}{2}} , \qquad \mu'' = \mu' e^{\frac{\chi}{2}} . \tag{2.2}$$

By combining both results, we obtain

$$\xi'' = \xi e^{-\frac{\chi}{2} + \frac{i\theta}{2}} , \qquad \mu'' = \mu e^{\frac{\chi}{2} - \frac{i\theta}{2}} . \tag{2.3}$$

Relations (2.3) give the 4-screw spinor representation.

3. REPRESENTATION OF 4-SCREW ON THE TRIDIMENSIONAL SURFACE

Any two triads of null rays (M,N,P) and (M', N', P') transform in a manner that can be associated to the transformation of the tetrad $(\vec{I},\vec{J},\vec{K},\vec{L})$ into $(\vec{I}', \vec{J}', \vec{K}', \vec{L}')$, as they were introduced earlier. By cutting spacetime across by the hyperplane $X_4/i = 1$, the section of the null cone $X_r X_r = 0$ is the sphere:

$$X_1^2 + X_2^2 + X_3^2 = 1 .$$

To each null ray there corresponds a point on its surface, so that (M, N, P) are points, from now on, on this tridimensional surface of the sphere. Let us choose the following points

$$\begin{aligned} \vec{M} &\equiv (0,0,1,i) \ , \\ \vec{N} &\equiv (0,0,-1,i) \ , \\ \vec{P} &\equiv (1,0,0,i) \ . \end{aligned} \tag{3.1}$$

The tetrad $(\vec{I},\vec{J},\vec{K},\vec{L})$ corresponding to (M,N,P) is

$$\begin{aligned} \vec{I} &= (0,1,0,0) \ , \\ \vec{J} &= (-1,0,0,0) \ , \\ \vec{K} &= (0,0,1,0) \ , \\ \vec{L} &= (0,0,0,\ ) \ . \end{aligned} \tag{3.2}$$

The χ (or a) rotation is then given by

$$\begin{aligned} \vec{M}' &\equiv (0,0,1,i) \ , \\ \vec{N}' &\equiv (0,0,-1,i) \ , \\ \vec{P}' &\equiv (\cos\alpha,\ 0,\ \sin\alpha\ ,\ i) . \end{aligned} \tag{3.3}$$

This corresponds to M and N fixed, while P runs around the meridian great circle throuph the angle o. This corresponds to the following ($\vec{I}$, $\vec{J}$, $\vec{K}$,$\vec{L}$) tetrad transformations

$$\begin{aligned} \vec{I}' &= \vec{I} \ , \\ \vec{J}' &= 3 \ , \\ \vec{K}' &= \sec\alpha\vec{K} + \tan\alpha\vec{L} \ , \\ \vec{L}' &= \tan\alpha\vec{K} + \sec\alpha\vec{L} \ . \end{aligned}$$

So this is a timelike rotation through an angle α.

Now, calculating a spacelike rotation θ, where the triad $(\vec{M}, \vec{N}, \vec{P})$ is transformed into

$$\vec{M}' \equiv (0,0,1,i) \ ,$$

$$\vec{N}' \equiv (0,0,-1,i) \ ,$$

$$\vec{P}' \equiv (\cos\theta, \ \text{sen}\,\theta, \ 0, \ i) \ .$$

This corresponds to a rotation of P through the angle θ on the plane X_1X_2. The tetrad $(\vec{I},\vec{J},\vec{K},\vec{L})$ transforms into

$$\vec{I}' = \vec{I}\cos\theta + \vec{J}\sin\theta \ ,$$

$$\vec{J}' \quad -\vec{I}\sin\theta + \vec{J}\cos\theta,$$

$$\vec{K}' = \vec{K} \ ,$$

$$\vec{L}' = \vec{L} \ .$$

The obvious result (we shall demonstrate it ir! next section) is that a Lorentz 4-screw is represented by the points (M,N,P) on the following transformation:

a) M and N do not move

b) moves on the (X_1X_3) plane along an angle α, and then moves along the (X_1X_2) plane an angle θ (the reverse order is also possible). The movement is always on the tridimensional surface of the unit sphere.

4. DEMONSTRATION OF THE RESULT STATED IN LAST SECTION

Let us begin with the θ rotation (spacelike). The normalized null vectors $\vec{M}$, $\vec{N}$, and $\vec{P}$ are

$$\vec{M} = \frac{1}{\sqrt{2}}(0,0,1,i)$$

$$\vec{N} = \frac{1}{\sqrt{2}}(0,0,-1,i) \ , \tag{4.1}$$

$$\vec{P} = \sqrt{2}\,(1,0,0,i) \ .$$

The normalized $\vec{M}'$, $\vec{N}'$, $\vec{P}'$ null vectors are

$$\vec{M}' = \frac{\sqrt{2}}{2}\,(0,0,1,i) \quad ,$$

$$\vec{N}' = \frac{\sqrt{2}}{2}\,(0,0,-1,i) \quad , \qquad (4.2)$$

$$\vec{P}' = \sqrt{2}\,(\cos\theta,\ \text{sen}\,\theta,\ 0,\ i) \quad .$$

Now, let us obtain the a rotation (timelike), The normalized $\vec{M}''$, $\vec{N}''$, $\vec{P}''$ null vectors are

$$\vec{M}'' = \frac{1}{\sqrt{2}}\left[\frac{1+\sin\alpha}{1-\sin\alpha}\right]^{1/2}(0,0,1,i) \quad ,$$

$$\vec{N}'' = \frac{1}{\sqrt{2}}\left[\frac{1-\sin\alpha}{1+\sin\alpha}\right]^{1/2}(0,0,-1,i) \quad , \qquad (4.3)$$

$$\vec{P}'' = \frac{\sqrt{2}}{\cos\alpha}\,(\text{cos a cos 8, cos a, sin a, i}) .$$

The $(\vec{I}'', \vec{J}'', \vec{K}'', \vec{L}'')$ tetrad can be obtained by the following relations due to Synge[1]

$$\vec{I}''_r = -\,i\varepsilon_{rsmn}J''_s K''_m L''_n \quad ,$$

$$\vec{J}'' = \frac{1}{\sqrt{2}}\,(\vec{M}'' + \vec{N}'' - \vec{P}'') \quad ,$$

$$\vec{K}'' = \frac{1}{\sqrt{2}}\,(\vec{M}'' - \vec{N}'') \quad , \qquad (4.4)$$

$$\vec{L}'' = \frac{1}{\sqrt{2}}\,(\vec{M}'' + \vec{N}'') \quad ,$$

where ε_{rsmn} is the well-known permutation symbol. Taking eq. (4.3) into (4.4) we get at last

$$\vec{I}'' = (-\sin\theta,\ \cos\theta,\ 0,\ 0) \quad ,$$

$$\vec{J}'' = (-\cos\theta,\ -\sin\theta,\ 0,\ 0), \qquad (4.5)$$

$$\vec{K}'' = (0,\ 0,\ \sec\alpha,\ i\tan\alpha) \quad ,$$

$$\vec{L}'' = (0,\ 0,\ \tan\alpha,\ i\sec\alpha) \quad ,$$

If we compare relations (1.1) and (3.2) with (4.5) we shall find that they are equivalent. So, what we constructed was a 4-screw Lorentz transformation indeed!

5. CONCLUDING REMARKS

As Synge[1] remarked, the six parameter group of projective transformations of Euclidean 3-space into itself is equivalent to the general Lorentz transformation of spacetime. By considering the three points (M, N, P) on the tridimensional surface of the unit sphere, we were able to generate the Lorentz 4-screw, which is then easily represented. The spinor representation also gives us another insight of a Lorentz 4-screw.

REFERENCE

1. J.L. Synge, *Relativity: The Special Theory*, 2nd. ed., North Holland, 1964, pages. 69-110.

Resumo

Um parafuso-4 de Lorentz é una tracsformação de Lorentz particular que foi definida e estudada por J.L.Synge. Desenvolvemos sua teoria exibindo duas representações dessa transformação, a espinorial e a tridimensional (na superfície da esfera unitária).