Revista Brasileira de Física, Vol. 18, nº 1, 1988

On Riemann Spaces with Proper Affine Collineations

B. LESCHE and M.L. BEDRAN
Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, Rio de Janeiro, 21944, RJ, Brasil

Recebido em 14 de outubro de 1987

Abstract It is shown that a Riemannian manifold (with positive definite metric) which carries an affine collineation can locally be written as a Cartesian product of manifolds in each one of which the affine collineation acts as a homothetic motion.

1. INTRODUCTION

Affine collineations are symmetries of affine spaces defined by the vanishing Lie derivative of the affine connections[1]

$$\underset{\xi}{L}\, \Gamma^{\alpha}_{\gamma\beta} = \xi^{\alpha}{}_{;\beta;\gamma} + R^{\alpha}{}_{\beta\gamma\sigma}\, \xi^{\sigma} = 0 \qquad (1.1)$$

where ; denotes covariant derivative and $R^{\alpha}{}_{\beta\gamma\sigma}$ is the Riemann tensor. In Riemannian and pseudo Riemannian spaces eq. (1.1) is equivalent to

$$\xi_{(\alpha;\beta);\gamma} = 0 \qquad (1.2)$$

where the ( ) indicate symmetrization in the indices α,β. Special cases of affine collineations in Riemannian spaces with metric $g_{\alpha\beta}$ are homothetic motions

$$\xi_{(\alpha;\beta)} = \text{const.}\ g_{\alpha\beta} \qquad (1.3)$$

and Killing vectors

$$\xi_{(\alpha;\beta)} = 0 \ . \qquad (1.4)$$

Affine collineations are transformations that keep the set of geodesics unchanged, although they may change the metric. Hojman et al[2] showed that affine collineations are non-Noetherian symmetries and constructed new constants of motion associated to them. An example of affine collineation was given by Katzin and Levine[3] in a two dimensional affine space without metric. Bedran and Lesche[4] gave an example of affine collineation in the Robertson-Walker metric (a pseudo Riemannian space) which turned out to be a homothetic motion.

In this paper we shall derive conditions on the geometry of a Riemannian space in order to admit a proper affine collineation, i.e. an affine collineation that is not a homothetic motion. The study of affine collineations in Riemannian spaces is of interest for dynamic systems. The orbits of a system of particles under scleronomic constraints without forces are given by the geodesics in configuration space, where the metric is given by the kinetic energy. Affine collineations define symmetries in the set of orbits.

2. RESTRICTIONS ON THE GEOMETRY IMPOSED BY THE EXISTENCE OF PROPER AFFINE COLLINEATIONS

Let M be a finite dimensional Riemannian manifold with positive definite metric $g_{\alpha\beta}$. If there exists an affine collineation on M, then according to eq. (1.2) there must exist a symmetric tensor $S_{\alpha\beta}$ such that

$$S_{\alpha\beta;\gamma} = 0 \tag{2.1}$$

If the affine collineation is a proper one we have with eq. (1.3)

$$S_{\alpha\beta} \neq \text{const.} \times g_{\alpha\beta} \tag{2.2}$$

As $S_{\alpha\beta}$ is symmetric, $S^{\alpha}{}_{\beta}$ defines a linear mapping of the tangent spaces into the tangent spaces which is self-adjoint with respect to the scalar product defined by the metric tensor. Thus, applying the spectral theorem, we can write

$$S^{\alpha}{}_{\beta} = \sum_{\ell=1}^{k} s_{\ell}\, P_{\ell}{}^{\alpha}{}_{\beta} \tag{2.3}$$

with

$$P_{\ell}{}^{\alpha}{}_{\beta}\, P_{n}{}^{\beta}{}_{\mu} = \delta_{\ell n}\, P^{\alpha}{}_{\mu} \tag{2.4}$$

and

$$P_{\ell\alpha\beta} = P_{\ell\beta\alpha} \tag{2.5}$$

and

$$\sum_{\ell=1}^{k} P_{\ell\ \beta}^{\ \alpha} = \delta^{\alpha}_{\ \beta} \qquad (2.6)$$

and $S_\ell \in R$.

The fact that $S_{\alpha\beta}$ is covariantly constant means that $S_{\alpha\beta}(x)$ can be obtained from $S_{\alpha\beta}(y)$ by applying a parallel transport operator which corresponds to a path leading from x to y

$$S_{\alpha\beta}(y) = T_{\alpha}^{\ \nu}\, T_{\beta}^{\ \mu}\, S_{\nu\mu}(x) \qquad (2.7)$$

Inserting eq. (2.3) into eq. (2.7) gives

$$\sum_{\ell} S_\ell(y) P_{\ell\alpha\beta}(y) = \sum_{R} S_\ell(x) T_{\alpha}^{\ \nu} T_{\nu}^{\ \mu} P_{\ell\nu\mu}(x) \qquad (2.8)$$

$$\bar{P}_{\ell\alpha\beta}(y) \equiv T_{\alpha}^{\ \nu} T_{\beta}^{\ \mu} P_{\ell\nu\mu}(x)$$

fulfills eqs. (2.4) -(2.6). Then the uniqueness of the spectral decomposition implies

$$S_\ell(y) = S_R(x) \equiv S_R \qquad (2.9)$$

and

$$P_{\ell\alpha\beta}(y) = T_{\alpha}^{\ \nu}\, T_{\beta}^{\ \mu}\, P_{\ell\nu\mu}(x) \qquad (2.10)$$

By definition of parallel transport eq. (2.10) gives

$$P_{\ell\alpha\beta;\gamma} = 0 \qquad (2.11)$$

This result means that the tangent spaces T_x can be written as a direct sum of orthogonal subspaces

$$T_x = \bigoplus_{\ell=1}^{k} V_{\ell x} \qquad (2.12)$$

in such a way that a parallel transport τ_γ along any curve γ leading from x to y maps $V_{\ell x}$ into $V_{\ell y}$

$$\tau_\gamma V_{\ell x} = V_{\ell y} \qquad (2.13)$$

Consequently the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ defines a map

$$R^{\alpha}{}_{\beta\mu\nu}\,\partial_{\alpha}\otimes dx^{\beta}:\; T_x \to T_x$$

such that it maps $V_{\ell x}$ into $V_{\ell x}$

$$(R^{\alpha}{}_{\beta\mu\nu}\,\partial_{\alpha}\otimes dx^{\beta}\,V_{\ell x})\subset V_{\ell x} \tag{2.14}$$

and because of $R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}$ we also have

$$(R_{\alpha\beta}{}^{\mu}{}_{\nu}\,\partial_{\mu}\otimes dx^{\nu}\,V_{\ell x})\subset V_{\ell x}. \tag{2.15}$$

From eq. (2.15) and

$$R_{\mu\alpha\beta\gamma}+R_{\mu\gamma\alpha\beta}+R_{\mu\beta\gamma\alpha}=0$$

and from the orthogonality of the spaces $V_{\ell x}$ one further concludes for $a_m\in V_{mx}$, $b_\ell\in V_{\ell x}$, $c_n\in V_{nx}$:

$$R^{\mu}{}_{\alpha\beta\gamma}\,a^{\alpha}_{m}\,b^{\beta}_{\ell}\,c^{\gamma}_{n}\neq 0 \;\Rightarrow\; m=\ell=n \tag{2.16}$$

In the following we will denote covariant derivatives by D.

Lemma: Let $x(\xi,\lambda)$ be a one-parameter family of geodesics, where ξ is the curve parameter (proportional to the path length s) and λ is the family parameter. Denote $a^{\alpha}(\xi,\lambda)=\partial x^{\alpha}/\partial\lambda$ and $t^{\alpha}(\xi,\lambda)=\partial x^{\alpha}/\partial\xi$ and let λ_0, ξ_0 be some fixed values of λ and ξ. If the vectors $a(\xi,\lambda_0)$ and $Da/D\xi|_{\xi_0\lambda_0}$ are in $V_{\ell x(\xi_0,\lambda_0)}$ then $a(\xi,\lambda_0)$ is in $V_{\ell x(\xi_0\lambda_0)}$ for all ξ.

Proof: $a(\xi)\equiv a(\xi,\lambda_0)$ is determined by the equation of geodesic deviation:

$$\frac{D^2a^{\alpha}}{D\xi^2}=t^{\beta}\,t^{\mu}\,a^{\nu}\,R^{\alpha}{}_{\mu\beta\nu} \tag{2.17}$$

The tangent vector t has a unique decomposition into vectors in the spaces V_{rx},

$$t = \sum_{r=1}^{k} t_r \; , \; t_r \in V \quad . \tag{2.18}$$

With this eq.(2.17) takes the form

$$\frac{D^2 a^\alpha}{D\xi^2} = \sum_{r,n} t_r^\beta \, t_n^\mu \, a^\nu \, R^\alpha{}_{\mu\beta\nu} . \tag{2.19}$$

Suppose that $a(\xi)$ is in $V_{\ell x(\xi,\lambda_0)}$ for some value of ξ. Then eq. (2.16) tells us that the only non-zero contribution in eq.(2.19) comes from $r=n=\ell$

$$\frac{D^2 a^\alpha}{D\xi^2} = t_\ell^\beta \, t_\ell^\mu \, a^\nu \, R^\alpha{}_{\mu\beta\nu} \tag{2.20}$$

Then eq.(2.14) tells os that $D^2a(\xi)/D\xi^2$ is also a vector in $V_{\ell x(\xi,\lambda_0)}$. This means that the entire differential eq.(2.17) can be formulated in V_ℓ provided that the initial conditions $a(\xi_0)$, $Da/D\xi_{|\xi_0}$ also lie in V_ℓ.

Let $p \in M$ be some point. Our aim is to construct a coordinate system in some open neighbourhood N of p which is aligned with the spaces V_{rx}. This means that for these coordinates u^a the vectors $\partial/\partial u$ are all elements of the spaces V_ℓ; $\partial/\partial u^a \in \bigcup_\ell V$ for all $x \in N$. We may then use a double index $a \equiv \langle \ell,i \rangle$ such that $\partial/\partial u^{\langle \ell,i \rangle} \in V_{\ell x}$.

Let us choose basis vectors $e_{\langle \ell,i \rangle} \in V_{\ell p}$. This choice of basis together with the exponential map at p defines coordinates $u^{\langle \ell,i \rangle}$ in some open neighbourhood N of p. We shall show that these coordinates are aligned with the spaces $V_{\ell x}$ in N.

Let $y \in N$ be given; we have to show that $\partial/\partial u^{\langle \ell,i \rangle}(y) \in V_{\ell y}$. The vector $\partial/\partial y^{\langle \ell,i \rangle}(y)$ is the tangent vector of the curve c with parameter X at y whose coordinate description in the u-coordinates is

$$u^{\langle k,r \rangle}(c(\lambda)) = u^{\langle k,r \rangle}(y) + \lambda \, \delta^{k\ell} \, \delta^{ri} \tag{2.21}$$

According to the definition of the exponential map, for each of the points $c(\lambda)$ we have exactly one geodesic $x(\xi,\lambda)$ such that $x(0,\lambda) = p$, $x(1,\lambda) = c(\lambda)$ and that the components of the tangent vector of $x(\xi,\lambda)$ at $\xi = 0$ in the basis $e_{\langle k,r \rangle}$ are given by $u^{\langle k,r \rangle}(c(\lambda))$. We can now apply the above Lemma to this family of geodesics. With the nomenclature of the Lemma we can write the vector $\partial/\partial u^{<\ell,i>}(y)$ as $a(\xi=1,\ \lambda=0)$. Choosing $\lambda_0=0$, $\xi_0=0$ we have $a(\xi_0,\lambda_0) = 0 \in V_{\ell p}$ because $x(0,\lambda) = g$. The tangent vector t and the geodesic deviation vector a fulfill

$$\frac{Da}{D\xi} = \frac{Dt}{D\lambda} \tag{2.22}$$

which is simply the consequence of not having torsion in our Riemmanian connection. With eq.(2.21) we have that $Dt/D\lambda$ at $\lambda = 0$, $\xi = 0$ is an elementof $V_{\ell p}$. Eq.(2.22) ensures that $Da/D\xi$ at $\lambda = 0$, $\xi = 0$ is also an element of $V_{\ell p}$. Then the Lemma gives the desired result $\partial/\partial u^{<\ell,i>}(y)$ E $\in V_{\ell y}$.

Let us now write the metric tensor in these aligned coordinates. As the spaces $V_{\ell x}$ are orthogonal it follows that g takes block diagonal form

$$(g_{<\ell,i><n,j>}) = \begin{pmatrix} g^{(1)}_{ij} & & & \\ & g^{(2)}_{ij} & & \\ & & g^{(3)}_{ij} & \\ & & & \ddots \end{pmatrix}$$

Further, from eq. (2.13) for the Christoffel symbols we conclude

$$\Gamma^{<\ell,i>}_{<n,j><m,k>} \neq 0 \longrightarrow \ell = n = m \tag{2.24}$$

with

$$g_{\alpha\beta,\nu} = \Gamma^{\sigma}_{\nu\alpha}\, g_{\sigma\beta} + \Gamma^{\sigma}_{\nu\beta}\, g_{\alpha\sigma}$$

we get from eq. (2.24)

$$g_{<m,i><m,j>,<n,k>} = \Gamma^{<p,s>}_{<n,k><m,i>} g_{<p,s><m,j>} + \Gamma^{<p,s>}_{<n,k><m,j>} g_{<m,i><p,s>}$$

$$= 0 \quad \text{if } n \neq m \tag{2.25}$$

So $\overset{(\ell)}{g}_{ij}$ depends only on the coordinates $u^{<\ell,k>}$. The set $M_\ell \subset M$ with $u^{<m,i>} = 0$ for $m \neq \ell$ is locally a submanifold of M. $\overset{(\ell)}{g}$ can be thought of as a metric tensor for M_ℓ. On M_ℓ the affine collineation ξ acts as a homothetic motion because the tangent spaces of M_ℓ are the V_ℓ and $\xi_{(<\ell,i>;<\ell,j>)} = s_\ell \overset{(\ell)}{g}_{ij}$ with s_ℓ from eq.(2.3). So locally M is a Cartesian product of Riemannian spaces M_ℓ in such a way that the affine collineation ξ acts as a homothetic motion in each M_ℓ.

One may summarize this result by saying that, at least locally, affine collineations in Riemannian spaces are necessarily *almost trivial*. As the argument is based on the spectral theorem, which in turn supposes a positive definite metric, one may expect much more interesting affine collineations in pseudo-Riemannian manifolds.

We thank Dr.S.Hojman for stimulating discussion and criticism. Part of this work was carried out at the Centro de Estudios Nucleares (U.N.A.M. - México). We thank the Centro de Estudios Nucleares and especially Dr. M.P.Ryan Jr. for kind hospitality.

REFERENCES

1. G.H.Katzin, J.Levine, W.R.Davis, J.Math.Phys. *10*, 617 (1969).
2. S.Hojman, L.Nuñez, A.Patiño, H.Rago, J.Math.Phys.*27*, 281 (1986).
3. G.H.Katzin, J.Levine, J.Math.Phys. 9, 8 (1968).
4. M.L.Bedran, B.Lesche, J.Math.Phys. 27, 2360 (1986).

Resumo

Mostramos que uma variedade Riemanniana (com métrica positivamente definida) que admite uma colineação afim pode ser escrita localmente como produto Cartesiano de variedades nas quais a colineação afim atua como movimento homotético.