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Divergence Law in Riemann-Cartan Space-Time: Cosmological Applications

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Abstract A vector current in Riemann-Cartan Space-Time is proposed. We prove that for homogeneous and irrotational cosmological models this current is conserved. Parallel propagation of spin seems to be a stronger condition for conservation.

1. INTRODUCTION

Recently I have proposed[1] a Killing vector current in Riemann--Cartan Space-Time. Nevertheiess, this current appears to be conserved only for a narrow class of cosmological models. In this paper I propose a new non-Killing vector current which seems to be more natural, since it depends only on the fluid parameters as shear, rotation, expansion, and acceleration. The new current reduces to the proper General Relativistic limit of the Einstein-Cartan-Sciama-Kibble (ECSK) theory[2]. The spinning fluid current is given by

$$J^a \equiv t^a_b \, v^b$$

where t^a_b is the energy-momentum (asymmetric) tensor, and $S^e_{ab} = v^e S_{ab}$ is the Weyssenhoff definition ot the spin-density tensor, S_{ab} is the angular-momentum tensor. The asymmetric energy-momentum tensor t^a_b is related to the matter energy-momentum tensor $T^a{}_b$ by $t^a{}_b = T^a{}_b + \frac{1}{2} \nabla_c S^{ca}{}_b$.

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2. VBCTOR CURRENT IN ECSK COSMOLOGY

Application of the Riemann-Cartan connexion ∇_a on both sides of eq. (1) yields

$$\nabla_a J^a = (\nabla_a t^a_b) v^b + t^a_b \nabla_a v b \tag{2}$$

From ECSK theory we have the following expressions

$$\nabla_a t^a{}_b = S^d{}_{bf} R^f{}_d + \frac{1}{2} R^{ac}{}_{db} S^d{}_{ac} \tag{3}$$

$$\nabla_a v_b = w_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab} \theta + \partial_a v_b - S_{ab} . \tag{4}$$

Where w_{ab}, σ_{ab}, θ and h_{ab} are respectively the rotation, shear and projective tensor

$$h_{ab} = g_{ab} - v_a v_b \tag{5}$$

Using eqs. (3) and (4) into (2) and expanding yields

$$\nabla_a J^a = S^d{}_{bf} R^f{}_d v^b + \frac{1}{2} S^d{}_{ac} R^{ac}{}_{db} v^b + (T^{ab} + \frac{1}{2} \quad {}_c S'^{ab})$$

$$. (w_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab} \quad + \partial_a v_b) \tag{6}$$

Expression (6) is further simplified

$$S_{bf} v^b R^f{}_d v^d = 0 \tag{7}$$

by noticing that the Weyssenhoff equation implies $S_{fb} v^b = 0$, and

$$S^d{}_{bf} R^f{}_d v^b = v^d S_{bf} R^f{}_d v^b .$$

Also

$$S^d{}_{ac} R^{ac}{}_{db} v^b = v^d S_{ac} R^{ac}{}_{db} v^b$$

$$= S_{ac} R^{ac}{}_{[db]} v^{(b} v^{d)} = 0 \tag{8}$$

due to the symmetries of the Riemann-Christoffel Tensor.

3. DUST-FILLED MODELS

For dust-filled cosmological models in ECSK the expression (6) reduces still further, since the energy-momentum tensor is given by

$$T^{ab} = \rho v^a v^b$$

where ρ is the matter density. Using the well-known expressions

$$w_{ab}\, v^b = 0 \qquad (10a)$$

$$a_{ab}\, v^b = 0 \qquad (10b)$$

$$h_{ab}\, v^b = 0 \qquad (10c)$$

$$\partial_a\, v^a = 0 \qquad (10d)$$

we have

$$\nabla_a J^a = \frac{1}{2} \nabla_c S^{cab} \left[w_{ab} + \sigma_{ab} + \frac{1}{3} h_{ab}\, \theta + \partial_a v_b \right] \qquad (11)$$

However, since the shear tensor σ_{ab} and the projection tensor h_{ab} are symmetric and S^{cab} is skew-symmetric on the same pair of indices the expression (11) becomes

$$\nabla_a J^a = \frac{1}{2} \nabla_c S^{cab} \left[w_{ab} + \partial_a\, v_b \right] \qquad (12)$$

At this stage we could say that our current is conserved for non-rotating homogeneous[4] cosmological models based on ECSK, like the FRW model. Nevertheless, expansion of the divergence of the spin- density tensor yields

$$\begin{aligned} \nabla_a J^a &= \frac{1}{2} \left[(\nabla_c\, v^c) S^{ab} + (v^c\, \nabla_c) S^{ab} \right] \cdot \left[w_{ab} + \partial_a\, v_b \right] \\ &= \frac{1}{2} \left[\theta\, S^{ab} + \frac{DS^{ab}}{ds} \right] \cdot \left(w_{ab} + \partial_a\, v_b \right) \\ &= \frac{1}{2} \left[\theta\, S^{ab} v_b\, \partial a + \frac{DS^{ab}}{ds} \cdot \partial_a\, v^b + \left(\theta\, S^{ab} w_{ab} + \frac{DS^{ab}}{ds} \cdot w_{ab} \right) \right] \\ &= \frac{1}{2} \left[\frac{DS^{ab}}{ds} \cdot \partial_a\, v_b + \left(\theta\, S^{ab} + \frac{DS^{ab}}{ds} \right) w_{ab} \right]. \end{aligned} \qquad (13)$$

The right-hand side of expression (13) vanishes in the following cases:

(i) For irrotational, homogeneous cosmological models:

$$a_b = 0 \quad , \; w_{ab} = 0 \tag{14}$$

(ii) For expansion-free models, where the spin is parallel-propagated:

$$\theta = 0 \quad , \; \frac{DS^{ab}}{ds} = 0 \tag{15}$$

In these two cases we óbtain a current conservation given by the Divergente Law in ECSK

$$\nabla_a J^a = 0 \tag{16}$$

Finally we want to comment that one of the interesting features of the current presented here is the possibility to extend Penrose's quasi-local mass[5] construction of GR to ECSK[6].

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Resumo

Propomos a definição de uma corrente vetorial no espaço-tempo de Riemann-Cartan. Mostramos que no caso de modelos cosmologicoshomogeneos e irrotacionais bem como no caso de modelos onde haja propagação paralela do spin, esta corrente é conservada.