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Generation of Kink-like Solutions by Friction

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Abstract Through the example of the ϕ^3 equation it is shown that a dissipative perturbation $\alpha\phi_t$ may generate kink-like solutions even in theories which do not have any such solutions when unperturbed. The generating mechanism is explained by energy considerations. Furthermore, a relationship is conjectured between the results of a Painlevé analysis of an ordinary differential equation and a possible representation of its kink-like solutions in a closed form.

1. INTRODUCTION

The behaviour of kink solutions under the influence of a dissipative force $\alpha\phi_t$ and a constant externa1 force F is a problem of current interest, both in applied and mathematical physics. The former since friction and constant forces frequently appear in nature, the latter because of the difficulties to find exact solutions to such problems.

A well known example is the damped and driven sine-Gordon (sG) equation, used as a *mathematical model* of physics in a Josephson junction[1,2]. Damped and driven 2π kinks were proved to exist[3] and studied numerically, but their exact solution is unknown. Only unstable kinks of height $\pi + 2\arcsin F$ and $\pi - 2\arcsin F$ and velocity $v = \pm 1$ were found for $|F| < 1$ and an array of kinks for $|F| > 1$[4]. They all satisfy a relatively simple ordinary differential equation (ode) of first order.

Unstable kinks of height $f_3 - f_2$ and $f_2 - f_1$ (where $f_1 < f_2 < f_3$ are the real and different roots of $f^3 - f - F = 0$, $0 \leq F^2 < 4/27$ and of velocity $v = \pm 1$ were given in implicit form for the damped anddriven ϕ^4 equation[5]. Also stable kinks of height $f_3 - f_1$ (generalizing the kinks known for $\alpha = F = 0$) and unstable kinks of height $f_3 - f_2$ and $f_2 - f_1$, all of velocity $|v| < 1$, were found[5,6]. The terms *stable* and *unstable* are meant with respect to small perturbations in the far field ($X \to \pm\infty$).

In this note the generation of kink-like solutions by a dissipative perturbation $\alpha\phi_t$ is examined. For this purpose we choose a theory which, in unperturbed form, does not have kink solutions at all: ϕ^3 theory. In section 2, kink-like solutions to the dissipative ϕ^3 equation are derived. The mechanism of this generation is explained in section 3 by an energy analysis. In the following section 4, a Painlevé analysis[7] of the damped and driven ϕ^3 equation, reduced to an ordinary differential equation (ode), is carried out, and the results are compared with the sG and ϕ^4 equations. Results and conclusions are summarized in section 5.

2. KINK-LIKE SOLUTIONS

The ϕ^3 field equation with a dissipative perturbation $\alpha\phi_t$, $\alpha > 0$, and an external driving force F = const reads

$$\phi_{xx} - \phi_{tt} = \phi^2 - \phi + \alpha\phi_t - F \; . \tag{1}$$

Throughout the following we consider $F > -1/4$ (cf.eq.(4b)) and look for travelling wave solutions. With

$$\phi(x,t) = w(z), \quad z = \gamma(x-vt), \quad \gamma = (1-v^2)^{-1/2} \tag{2a}$$

eq. (1) is reduced to

$$d^2w/dz^2 = w^2 - w - B dw/dz - F, \; B = \alpha\gamma v \; . \tag{2b}$$

The ansatz

$$w(z) = f + g(1 + \exp(D(z - z_0)))^{-2} \tag{3}$$

solves eq. (2b). The exponent -2 follows from a leading order analysis; z_0 is a constant of integration, and f, g, B and D are constant parameters to be determined. For kink-líke solutions the asymptotic (**) behavior of eqs. (2b) and (3) leads one to guess that f is a solution of the quadratic equation

$$f^2 - f - F = 0 \; . \tag{4a}$$

For $F > -1/4$ the solutions of eq. (4a)

$$f_{1,2} = 1/2 \pm \sqrt{F + 1/4} \tag{4b}$$

are real and different. Without loss of generality we consider

$$f = f_1 < 1/2 < f_2 \quad .$$

Inserting the ansatz (3) into eq. (2b) one obtains three equations for g B and D:

$$g(2f + g - 1) = 0 \quad , \tag{5a}$$

$$g(-4D^2 + 2DB + 2f - 1) = 0 \; , \tag{5b}$$

$$g(-2D^2 - 2DB - 4f + 2) = 0 \; . \tag{5c}$$

The nontrivial solutions $(g \neq 0)$ are

$$g = 1 - 2f = f_2 - f_1 = + \sqrt{4F + 1} \quad , \tag{6a}$$

$$D = \pm\sqrt{(1 - 2f)/6} = \pm((4F + 1)/36)^{1/4} \; , \tag{6b}$$

$$B = 5D = \pm 5((4F + 1)/36)^{1/4} \; . \tag{6c}$$

D and B have the same sign and are real since $1 - 2f = 1 - 2f_1$ is not only real $(F > -1/4)$, but positive. The solution (3) describes a kink from f_1 to f_2 if $D < 0$, and an antikink from f_2 to f_1 if $D > 0$. (For $F = -1/4$ eqs. (6a,b,c) yield g = D = B = 0, and the ansatz (3) - by $\exp(D(z - z_0)) \to -\exp(D(z - z_0))$ i.e., $z_0 \ni \hat{z}_0 = z_0 - (\ln(-1)/D)$ - leads to a rational solution $w = 1/2 + 6(z - z_0)^{-2}$, $v = 0$, which is not kink-like and will not be discussed further.) From $B = \alpha\gamma v$ and $\text{sign}(B) = \text{sign}(D)$ it follows that the kink and antikink move with opposite velocities

$$v(\alpha,F) = \pm \sqrt{B^2/(\alpha^2+B^2)}, \; B^2 = 25\sqrt{4F + 1}/6 \quad . \tag{6d}$$

Here and in all following equations with two signs the upper sign is valid for an antikink and the lower one for a kink.

It should be emphasized that the product $B = \alpha\gamma v$ depends only on F, but not on a, and that $B \neq 0$ for F = 0. Therefore the solution (3) in the case $\alpha \to 0$ and $F = 0$ has nothing to do with travellingwave solutions to eq. (1) for a = F = 0. In this context it is instructive to rewrite solution (3) in the form

$$w(z) = (1 - g)/2 + g(\tanh(D(z - z_0)/2) - 1)^2/4 . \qquad (7a)$$

On the other hand, for a = F = 0 eq. (2b) has the particular solution

$$w_0(z) = (-1 + 3\tanh^2((z - z_0)/2))/2 \qquad (7b)$$

which satisfies $w(-\infty) = w_0(\infty) = 1$ and has arbitrary velocity $-1 < v < 1$.

Another solution (kink-like for $F > -1/4$) is obtained when eq. (1) is reduced with $y = x - vt$, $v = \pm 1$, to the first order ode

$$w^2 - w - \alpha v dw/dy - F = 0 . \qquad (8a)$$

Eq. (8a) has the solution (with the constant of integration y_0)

$$w(y) = (f_2 + f_1 Y)/(1 + Y), \quad Y = \exp(\mp(f_1 - f_2)(y - y_0)/\alpha) . \qquad (8b)$$

3. ENERGY ANALYSIS OF KINK GENERATION

The velocities of the solutions (3) and (8b) can be obtained and explained by the following energy analysis. Eq. (1) can (up to the factor $\exp(\alpha t)$) be derived from a lagrangian density

$$L = (\phi_t^2/2 - \phi_x^2/2 - V(\phi))\exp(\alpha t) \qquad (9a)$$

where

$$V(\phi) = \phi^3/3 - \phi^2/2 - F\phi + V_0, \quad V_0 = \text{const.} \qquad (9b)$$

is the potential energy density (including the part due to the external force F). V has a maximum for $\phi = f_1 < 1/2$ and a minimum for $\phi = f_2 > 1/2$ $(F > -1/4)$. The total potential energy is given by*

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* The integral (9c) and therefore the total energy E are infinite or of form $(\infty-\infty)$ for the solutions (3) and (8b). The differences ΔE_p and $dE = E(t+dt) - E(t)$ in eqs. (9d,e) and (10a), however, are finite.

$$E_p = \int_{-\infty}^{w} V dx \qquad (9c)$$

The kinks ($v < 0$) and the antikinks ($v > 0$) lose, when moving the distance $\Delta x \gtrless 0$ in time Δt, the potential energy

$$\Delta E_p = \mp ((f_2^3 - f_1^3)/3 - (f_2^2 - f_1^2)/2 - F(f_2 - f_1))\Delta x$$

$$= \mp (4F + 1)^{3/2} \Delta x/6 < 0 , \qquad (9d)$$

or per unit time

$$\Delta E_p/\Delta t = \mp (4F + 1)^{3/2} v/6 < 0 . \qquad (9e)$$

Their kinetic energy is conserved since their velocity is constant. Therefore the loss (9e) is equal to the frictional loss of the total energy

$$dE/dt = - \int_{-\infty}^{\infty} \alpha \phi_t^2 \, dx . \qquad (10a)$$

For the solution (3) one gets (replacing t by z and z by $Z = \exp(D(z-z_0))$)

$$dE/dt = -\alpha\gamma v^2 (4F + 1)^{5/4}/(5\sqrt{6}) , \qquad (10b)$$

and for the solution (8b) (admitting an arbitrary velocity v and replacing t by y and y by Y)

$$dE/dt = -v^2 (f_2 - f_1)^3/6 = -v^2 (4F + 1)^{3/2}/6 . \qquad (10c)$$

Equating expressions (9e) and (10b) one obtains the result (6c) with $\alpha\gamma v = B$ for the solution (3) while expressions (9e) and (10c) yield $v=\pm 1$ as the velocity of solution (8b).

This analysis explains particularly why the kinks and antikinks can move, for F = 0, with a constant velocity $v \neq 0$ in spite of the dissipative force. There is an infinite amount of potential energy (9c) available since the potential energy density (9b) takes on a maximum at $\phi = f_1$ (or more generally since $V(\phi=f_1) \neq V(\phi=f_2)$). Such an analysis was already indicated for ϕ^4 theory[8], and now results have been extended to

$F \neq 0$ and to the solution with $v = \pm 1$. Since for v = const $\neq 0$ the kinetic energy is constant, but the potential energy is not, it becomes also clear that the solutions (3) and (8b) cannot exist for $a = 0$. Of course, due to the maximum of V for $\phi = f_1$ the solutions (3) and (8b) are unstable against small perturbations of the far field.

4. PAINLEVÉ ANALYSIS

Now we will examine eq. (2b) for the Painlevé property. Inserting the ansatz

$$w(z) = \sum_{n=p}^{n=\infty} a_n (z - z_0)^n \tag{11a}$$

into eq. (2b) one finds (besides the trivial case $p = 0$ with arbitrary a_0 and a_1) the resonances for $n = -3$ i.e., $p = -2$ corresponding to the arbitrariness of z_0, and for $n = 4$. The coefficients $a_{-2}, a_{-1}, \ldots, a_3$ are

$$a_{-2} = 6, \quad a_{-1} = -6B/5, \; a_0 = 1/2 - B^2/50, \quad a_1 = -B^3/250 \quad ,$$

$$a_2 = -7B^4/5000 + F/10 + 1/40, \quad a_3 = 11Ba_2/15 - B^5/37500 \; . \tag{11b}$$

a_4 is arbitrary provided the resonance condition

$$B^6/6250 - 4B^2 a_2 = 0 \tag{12a}$$

is satisfied. The solutions of eq.(12a) are

$$B = 0 \text{ for arbitrary } F \tag{12b}$$

and

$$B^4 = 625(4F + 1)/36 \; . \tag{12c}$$

In the simple case (12b) a first integral to eq. (2b) is

$$(dw/dz)^2/2 = w^2/3 - w^2/2 - Fw + C$$

and can be integrated once more in terms of elliptic integrals. The solution is free of movable singularities other than poles. In case

(12c) eq. (2b) can be brought into the canonical form[7]

$$d^2u/ds^2 = 6u^2 \tag{13a}$$

by the transformation

$$w = \exp(-2Bz/5)u + 1/2 - 3B^2/25\ , \tag{13b}$$

$$s = -5\exp(-Bz/5)/(\sqrt{6}\,B)\ . \tag{13c}$$

Eq. (13a) has the Painlevé property. Condition (12c) includes the values (6c) of B for which the kink-like solution (3) exists. The latter one is found from eq.(13a) when, in the first integral $(du/ds)^2/2 = 2u^3 + C$, the constant of integrations C is chosen equal to zero. Only the solutions (6c) of eq.(12c) yield real values of v if $F > -1/4$. For $F < -1/4$ no real values for v arise from eq. (12c) at all.

Comparing the results of a Painlevé analysis for the damped and driven sG, ϕ^4 and ϕ^3 equations (all reduced to ode's by the transformation (2a)) one observes an hierarchy in the sense that the sG equation (after the transformation $u = \exp(iw)$) does not possess the Painlevé property for any set $(\alpha\gamma v, F) \neq (0,0)$ as follows from the resonance conditions[9] $\pm iF = 2(\alpha\gamma v)^2$. (Note that in order to get both conditions one has to investigate solutions of leading order $(z - z_0)^{-2}$ as well as $(z - z_0)^2$). The ϕ^4 equation is of Painlevé type only if either $F = 0$ and $\alpha\gamma v = \pm 3/\sqrt{2}$ or $\alpha\gamma v = 0$ and F arbitrary[7,10]. Only the ϕ^3 equation (2b) preserves the Painlevé property for simultaneously nonvanishing values of $\alpha\gamma v$ and F, provided that the velocity satisfies condition (12c). This hierarchy can be intuitively explained by the fact that the nonlinearity is less complex in the ϕ^3 equation and most complex in the sG equation.

Although the ϕ^4 equation is not of Painlevé type for any simultaneously nonvanishing $\alpha\gamma v$ and F, it is interesting that for all sets of $\alpha\gamma v$ and F which lead to kink solutions, one of the two resonance conditions can be satisfied[5,10]. In sG theory neither of the two conditions $2(\alpha\gamma v)^2 = \pm iF$ can be solved for real, nontrivial, $\alpha\gamma v$ and F, and kink solutions to the damped and driven sG equation (reduced by eq.(2a)) are unknown in closed form.

Finally, the analysis of the ode (2b), without the need to perform a generalized Painlevé analysis[11] of the partial differential eq. (1), indicates that for a $\neq 0$ eq.(1) is not completely integrable. This result is, of course, expected and can be obtained since, for given a and F, the reduction (2a) leads to ode's which do not have the Painlevé property for all values of v which do not satisfy eqs.(12b) or (12c).

5. DISCUSSION

In unperturbed dissipationless systems a kink $\phi(x,t)$ from $\phi(-\infty,t) = \phi_1$ to $\phi(\infty,t) = \phi_2$ with $\phi_1 \neq \phi_2$ can only exist if the theory has degenerate vacua $V(\phi_1) = V(\phi_2)$. Such a solution would become stationary in the presence of dissipation. In a theory with $V(\phi_1)\neq V(\phi_2)$ and dissipation $\alpha\phi_t$, a kink when moving loses potential energy which is consumed by friction. This compensation determines sign and absolute value of the kink's velocity. (The velocity cannot be arbitrary already because the term $\alpha\phi_t$ destroys the relativistic invariance of the equation). The potential energy eq.(9c) of such solutions is infinite, i.e. the kink can move with constant velocity for an infinite time. Thus in the presence of friction, kink-like solutions can be expected if the potential $V(\phi)$ has at least two extremum values. In particular, the solutions exist also for $F = 0$, and it seems convenient to include the constant external force as $F\phi$ in the potential $V(\phi)$.

Physical realizations of such kinks will be only approximate (as any kink when realized) because of the then finite spatial extension of the system. The energy of the *truncated* kinks is finite, and they can propagate with approximately constant velocity for a finite time. Kink-like solutions with one or both asymptotic values ϕ_1 and ϕ_2 corresponding to a maximum of $V(\phi)$, as in damped ϕ^3 theory, meet with a further problem of physical realization due to their asymptotic instability. However, such solutions may be stable against small perturbations of the near field[5].

Another question is whether and under what conditions kink solutions can be given in a closed form, i.e. in terms of known mathematical functions. We feel that an answer is suggested by the Painlevé analysis and that it is likely to be *yes*, if the Painlevé test is positive at least for one branch (possible leading order behavior) of the solution.

In conclusion, we think that the generating mechanism of kink-like solutions by friction $\alpha\phi_t$ has been clarified. This may throw new light on certain models of physical interest, as e.g. $\phi^4 + \phi^3$ theory, which - without dissipation - do not have any kink solutions.

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Resumo

Mostra-se, através da equação ϕ^3 como exemplo, que uma perturbação dissipativa $\alpha\phi_t$ pode gerar soluções de tipo kink mesmo em teorias que sem perturbação não tem soluções de tal tipo. O mecanismo gerador é explicado através de considerações energéticas. Sugere-se, também, uma conexão entre os resultados de uma análise Painlevé de uma equação diferencia? ordinária e uma possível representação de suas soluçoes de tipo kink numa forma analítica.