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Note on Quantum Singularities

NIVALDO A. LEMOS
Instituto de Física, Universidade Federal Fluminense, 24210, Niterói, RJ, Brasil

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Abstract An explicit counterexample is presented which shows that DeWitt's boundary condition on the wave function is not a valid criterion for quantum singularity avoidance. The example further proves that probability-conserving "slow-time" quantum dynamics may become singular.

In quantum cosmology the usual criterion for the nonexistence of singularity is DeWitt's boundary condition that the universal wave function vanishes at classically singular metrics[1,2]. In the case of Friedmann-Robertson-Walker models, whose sole degree of freedom is the scale factor R, $\psi(R=0,t)\neq 0$ for some t would indicate a quantum cosmological singularity, whereas $\psi(R=0,t)=0$ for all t would bean indication of absence of singularity. The idea behind the previous boundary condition is that it "...makes the probability amplitude for catastrophic three-geometrics vanish, and hence gets the physicist out of his classical collapse predicament."[1] Notwithstanding its intuitive appeal, this condition has been a source of controversy and criticism in the physical literature[3], but it keeps being applied to classifying certain quanturn cosrnological models as nonsingular[4].

Although originally conceived in the realmof quantum cosmology, one may consider DeWitt's proposal as a general criterion for existence of singular states in any problem of quantum mechanics. Espousing this point of view, our aim in this note is to show by means of a counter-exarnple that DeWitt's condition $\psi(x=0,t)=0$ for all t is not a valid criterion for quantum singularity avoidance. To our knowledge, however much criticized, no concrete disproof has ever been given of the adequacy of DeWitt's criterion.

The example we shall be considering is a harmonic oscillator in a Newtoniam expanding universe[5]. The Hamiltonian operator is

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$$\hat{H}(t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{m}{2}\left(\omega_0^2 + \frac{2}{9t^2}\right)x^2 \tag{1}$$

where t denotes cosmic time. A normalized solution to Schrödinger's equation is[5]

$$\psi_1(x,t) = \left(\frac{2\beta}{\sqrt{\pi}}\right)^{1/2} \beta x \exp\left[-3i\gamma/2 + \frac{im}{2\hbar}(i\dot{\gamma} + \dot{s}/s)x^2\right] \tag{2}$$

with

$$\beta = (m\dot{\gamma}/\hbar)^{1/2} , \tag{3}$$

$$s(t) = (\pi t/2)^{1/2}\left[J^2_{1/6}(\omega_0 t) + Y^2_{1/6}(\omega_0 t)\right]^{1/2} \tag{4}$$

and

$$\dot{\gamma}(t) = 1/s^2(t) . \tag{5}$$

In eq. (4) $J_{1/6}$ and $Y_{1/6}$ are Bessel functions of the first and second kinds, respectively. A straightforward calculation yields

$$<x>_t = \int_{-\infty}^{\infty} x|\psi_1(x,t)|^2\, dx = 0 , \tag{6}$$

while

$$<x^2>_t = \frac{3}{2\beta^2(t)} \xrightarrow[t\to 0]{} 0 , \tag{7}$$

the last result following from eqs. (3)-(5) and from the asymptotic behaviour of Bessel functions. Thus eqs. (6) and (7) lead us to conclude that

$$\lim_{t\to 0} |\psi_1(x,t)|^2 = \delta(x) , \tag{8}$$

which may also be verified directly from eq.(2). The quantum state is unequivocally singular at $t = 0$ since the probability density is sharply concentrated at $x = 0$. A further evidence of the singular nature of the quantum state at $t = 0$ is given by the expectation value of the Hamiltonian. Let us first notice that

$$\langle p\rangle_t = -i\hbar \int_{-\infty}^{\infty} \psi_1^*(x,t) \frac{\partial\psi_1(x,t)}{\partial x} dx = 0 \tag{9}$$

because, as one readily checks, the integrand is an odd function of X. As a consequence,

$$(\Delta p)_t^2 = \langle p^2\rangle_t - \langle p\rangle_t^2 = \langle p^2\rangle_t \ . \tag{10}$$

Since from eqs. (6) and (7) it follows that $(\Delta x)_t \to 0$ as $t \to 0$, the uncertainty principle requires that

$$\langle p^2\rangle_t = (\Delta p)_t^2 \xrightarrow[t\to 0]{} \infty \ . \tag{11}$$

Therefore

$$\langle \hat{H}(t)\rangle_t = \frac{\langle p^2\rangle_t}{2m} + \frac{m}{2}\left(\omega_0^2 + \frac{2}{9t^2}\right)\langle x^2\rangle_t \geq \frac{\langle p^2\rangle_t}{2m} \xrightarrow[t\to 0]{} \infty \ . \tag{12}$$

The expectation values of both the kinetic energy and the total energy are infinite at $t = 0$, and one concludes that these fundamental observables become undeniably singular at that instant. Nevertheless, one easily sees from eq. (2) that $\psi_1(0,t) = 0$ for all t, thus establishing that DeWitt's condition is inadequate to avoid quantum singularities.

At first sight our example might be considered as very pathological because $\hat{H}(t)$ is singular at $t = 0$. However, Hamiltonians of this sort do indeed occur in quantum cosmology, and an interesting example can be found in the work of Blyth and Isham (see eq. (3.10) in ref. 3), who quantized a Friedmann universe filled with a scalar field. Besides, a singular Hamiltonian operator should not be regarded as unnatural whenever the system it describes is itself endowed with a singularity at the classical level.

As our last remark, we call the reader's attention to the fact that our example also shows that, contrary to a strong indication[6] given in the context of quantum cosmology, in the case of "slow-time" dynamics conservation of probability is not powerful enough to rule out singular states. In fact, $||\psi_1(t)|| = 1$ for all t, even in the limit $t \to 0$, so that conservation of probability is rigorously satisfied. In spite of this

$$\lim_{t\to 0} X_t = \lim_{t\to 0} \int^{\infty} |x||\psi_1(x,t)|^2 \, dx = 0 \ . \tag{13}$$

This last result, that a strictly positive operator has a vanishing expectation value at a finite time t, is the criterion for the existence of a singularity introduced by Lund[7] and adopted in ref.6. Therefore, probability-conserving "slow-time" quantum dynamics may become singular.

The above results obtained within the frameworkofquantum mechanics in a curved spacetime have incited us to undertake a search for similar situations in quantum cosmology proper. The outcome of such an investigation will be reported in a separate publication[8].

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Resumo

Apresenta-se um contra-exemplo explícito que mostra que a condição de contorno de DeWitt sobre a funçao de onda não é um critério válido para evitar singularidades quânticas. O exemplo prova, ainda, que a dinâmica quântica com conservação da probabilidade e um "tempo lento" pode tornar-se singular.'