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The Behaviour of the Gross-Neveu Model Generating Functional under Chiral Rotations

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Abstract The Thirring model is invariant under global chiral rotations. However, using the method developed by Gamboa-Saraví, Muschietti, Schaposnik and Solomin to calculate the Jacobian associated with this kind of transformation, a non-trivial result has been obtained. This fact may indicate that their regularization does not respect the symmetries of the theory. We thus decided to investigate how the Gross-Neveu model Green functions behave under global chiral rotations. (The Gross-Neveu model with a single fermionic field is equivalent to the Thirring model). We show that the contribution which comes from the Jacobian can be eliminated by exploiting the invariance of the theory under finite renormalization and that the Green functions do not depend on the chiral rotation parameter.

1. INTRODUCTION

In quantum field theory currents which are conserved at the classical levei may no longer be conserved after quantization. This breaking of a classical symmetry by the quantization procedure is usually called anomaly. The first example of this phenomenon, the chiral anomaly, was studied by Adler, Bell and Jackiw[1] in quantum electrodynamics. Until recently divergences of anomalous currents were calculated laboriously in perturbation theory using Feynman diagrams. In 1979, Fujikawa[2] showed in the context of path integrals that the anomaly comes from the non-invariance of the fermionic measure under chiral rotations. In other words, he showed that a non trivial Jacobian results from a chiral rotation. Since then, there has been a renewed interest in the evaluation of functional Jacob'ians associated with these transformations. In 1983, Gamboa-Saraví, Muschietti, Schaposnik and Solomin (G-SMSS) proposed[3] an elegant method based on the ζ-function regularization[4] and Seeley's expansion coefficients[5], which allows in a natural way the evaluation of chiral Jacobians for theories which contain non-hermitian operators. They have shown[6] that their method is equivalent to Fujikawa's for the case of hermitian operators but yields different results when non-hermitian operators are present. The question as to which method is correct has not been settled yet and further investigation is necessary.

Recently, Carneiro, Mignaco and Thomaz[7] (CMT) used the G-SMSS method to study how different parametrizations of the Thirring model[8] behave under chiral rotations. Some parametrizations lead to hermitian while others lead to non-hermitian Dirac operators. The G-SMSS method, which treats both types of operators on an equal footing, is specially adequate for their problem. CMT have shown that, although the Thirring model is invariant under global chiral rotations, one obtains a non trivial Jacobian even in this case. This may indicate that the ζ-function regularization breaks the global chiral invariance of the theory. We then decided to investigate this point further by studying in this paper the behaviour of the Green functions under chiral rotations. We show that, at least in perturbation theory, the G-SMSS method is consistent and the Green functions do not depend on the chiral rotation parameter.

We study the Gross-Neveu model[9] which contains N fermionic fields. Since it is equivalent to the Thirring model when $N=1$, we can easily recover the CMT case at any stage of the calculation. The Gross-Neveu model is not invariant under chiral rotations except for $N = 1$. However, performing a chiral rotation on the generating function is analogous to performing a change of variables in an ordinary integral and the final result does not depend on the integration variables. This paper is organized as follows. In section 2 we show how to calculate the chiral Jacobian when sources are present. In section 3 we show that the Green functions do not depend on the chiral rotation parameter. The results are discussed in section 4, and the calculation of the chiral Jacobian is sketched in an appendix.

2. THE BEHAVIOUR OF THE GROSS-NEVEU MODEL WITH SOURCES UNDER CHIRAL ROTATIONS

The Gross-Neveu model is described in two dimensional Euclidean space by the Lagrangian

$$L = i\,\bar{\psi}^a \not\partial \psi^a + \frac{g^2}{2} (\bar{\psi}^a \psi^a)^2 \ , \qquad (2.1)$$

where $a = 1,2,\ldots,N$; $\not\partial = \gamma_\mu \partial_\mu$; $\mu = 1,2$ and we choose a representation in which the γ_μ matrices are hermitian, namely $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\gamma_5 = \sigma_3$ and $\varepsilon_{12} = 1$.

In Euclidean space we have

$$\{\psi^a_\alpha, \psi^b_\beta\} = \{\bar{\psi}^a_\alpha, \bar{\psi}^b_\beta\} = \{\psi^a_\alpha, \bar{\psi}^b_\beta\} = 0 \qquad \alpha,\beta = 1,\ 2 \qquad (2.2)$$

If there is only one fermion field $(N=1)$ then

$$(\bar{\psi} M \psi)^2 = \det M (\bar{\psi}\psi)^2 \ , \qquad (2.3)$$

where M is any 2x2 matrix and det M stands for the determinant of M. Using this identity it is easy to show that $2g^2(\bar{\psi}\psi)^2 = -g^2(\bar{\psi}\gamma_\mu\psi)^2$ and we obtain the well-known equivalence between the Thirring model and the $N=1$ Gross-Neveu model.

In order to study the Green functions it is convenient to use the generating functional

$$Z = \int D\bar{\psi}\, D\psi\, \exp\left\{- \int d^2x \left[i\,\bar{\psi}^a \not\partial \psi^a + \frac{g^2}{2} (\bar{\psi}^a\psi^a)^2 - \bar{\eta}^a\psi^a - \bar{\psi}^a \eta^a \right]\right\}, \qquad (2.4)$$

where $\bar{\eta}^a$ and η^a are sources for the fermionic fields ψ^a and $\bar{\psi}^a$ respectively.

Before applying a chiral rotation to the fermionic fields we introduce an auxiliary field σ, to reduce the quartic term to a quadratic one in $\bar{\psi}$, ψ, and shift $\bar{\psi}$, ψ to eliminate the linear terms. These steps are necessary because the G-SMSS method requires a quadratic Lagrangian in the fermionic fields which is regularized from the beginning using the ζ-function[10]

$$\int D\bar{\psi}\, D\psi\, \exp\left[- \int \bar{\psi} D \psi\right] = \det D \equiv \exp\left\{ -\frac{d\zeta(s,D)}{ds} \right\}_{s=0} \qquad (2.5)$$

A transformation over the fermionic fields is defined as

$$\psi = \Omega\psi' \quad , \quad \bar{\psi} = \bar{\psi}'\tilde{\Omega} \tag{2.6}$$

After this transformation the fermionic path integral becomes

$$\int D\bar{\psi}\, D\psi\, J \exp\left\{-\int \bar{\psi}\tilde{\Omega}D\Omega\psi\right\} \equiv \exp\left\{-\zeta'(0,\tilde{\Omega}D\Omega)\right\} \tag{2.7}$$

and the Jacobian J can be expressed as the ratio of two regularized determinants

$$\log J = \zeta'(0,\tilde{\Omega}D\Omega) - \zeta'(0,D) \quad . \tag{2.8}$$

The reduction of the quartic term is easily accomplished using the identity

$$\exp\left\{-\frac{1}{2}A^2\right\} = \int D\sigma \exp\left\{-\int d^2x \left[\frac{\sigma^2}{2} + i\sigma A\right]\right\} \ . \tag{2.9}$$

The generating functional becomes

$$Z = \int D\sigma \exp\left\{-\int d^2x \frac{\sigma^2}{2}\right\} \times \\ \times \prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \exp\left\{-\int d^2x \left[\bar{\psi}^a(i\not{\partial} + ig\sigma)\psi^a - \bar{\eta}^a\psi^a - \bar{\psi}^a\eta^a\right]\right\} , \tag{2.10}$$

which is the product of N single fermion Gross-Neveu models coupled through the σ field.

If we define

$$S^{-1}(x,y) = (-i\not{\partial} + i\sigma)\delta(x-y) \tag{2.11}$$

and perform the change of variables

$$\psi^a(x) \to \psi^a(x) + \int dy\ S(x,y)\eta^a(y) \ , \tag{2.12a}$$

$$\bar{\psi}^a(x) \to \bar{\psi}^a(x) + \int dy\ \bar{\eta}^a(y)S(y,x) \ , \tag{2.12b}$$

where $S(x,y)$ is the inverse of $S^{-1}(x,y)$, we obtain

$$Z = \int D\sigma \exp\left\{ -\int d^2x \left[\frac{\sigma^2}{2} - \bar{\eta}S\eta\right]\right\}$$

$$\prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \exp\left\{ -\int d^2x\, \psi^a(i\partial\!\!\!/ + ig\sigma)\psi^a\right\}, \qquad (2.13)$$

where

$$\bar{\eta}S\eta \equiv \int d^2y\, \bar{\eta}(x)S(x,y)\eta(y) \quad .$$

A local chiral rotation is defined as in eq.(2.6) with $\tilde{\Omega} = \Omega = e^{\alpha\gamma_5}$. Since only the fermionic fields $\bar{\psi}$, ψ are transformed, the calculation of the Jacobian is the same as in the sourceless case. Notice however that if S^{-1} has zero eigenvalues, one has to be more careful to define S. However, since we are restricted to the trivial topological sector where perturbation theory holds, we may safely neglect this possibility.

The chiral Jacobian for the N=l Gross-Neveu model, calculated in the appendix, is given by

$$\ell n \quad J = -\int d^2x \; \frac{\sigma^2(x)}{2}\frac{g^2}{2\pi}\left[\cosh 4\alpha(x) - 1\right] + \frac{1}{2\pi}\int d^2x\,\alpha(x)\,\partial^2\alpha(x) \qquad (2.14)$$

From now on we shall restrict ourselves to global chiral rotations (α = const.). Examining the expression (2.13) we see that the total Jacobian is the product of N such Jacobians. Hence

$$Z = \int D\sigma \exp\left\{- \int d^2x \left[\frac{\sigma^2}{2}\left(1 + \frac{g^2N}{2\pi}(\cosh 4\alpha - 1)\right) - \bar{\eta}S\eta\right]\right\} \times$$

$$\times \prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \exp\left\{- \int d^2x\, \bar{\psi}^a e^{\alpha\gamma_5} (i\partial\!\!\!/ - ig\sigma)e^{\alpha\gamma_5}\psi^a\right\}. \qquad (2.15)$$

If we shift the fermionic fields as in eq. (2.12) but substitute $S(x,y)$ for $e^{-\alpha\gamma_5} S(x,y)\, e^{-\alpha\gamma_5}$ we obtain

$$Z = \int D\sigma \exp\left\{-\int d^2x \left[\frac{\sigma^2}{2}\left(1 + \frac{g^2N}{2\pi}(\cosh 4\alpha - 1)\right)\right]\right\}$$
$$\times \prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \exp\left\{-\int d^2x\left[\bar{\psi}^a e^{\alpha\gamma_5}(i\partial\!\!\!/ + ig\sigma)e^{\alpha\gamma_5}\psi^a - \right.\right.$$
$$\left.\left. - \bar{\eta}^a e^{\alpha\gamma_5}\psi^a - \bar{\psi}^a e^{\alpha\gamma_5}\eta^a\right]\right\} . \qquad (2.16)$$

This result can also be obtained if we perform the chiral rotation in the generating function (2.10) and use the Jacobian obtained from the sourceless case. In other words, the sources do not affect the calculation of the Jacobian.

3. INDEPENDENCE OF THE GREEN FUNCTIONS ON THE CHIRAL PARAMETER (α)

Since all Green functions can be obtained from the generating functional by calculating derivatives with respect to the sources, the demonstration that eq.(2.16) does not depend on a implies that the Green functions are α-independent.

We can eliminate the factor which multiplies the $\sigma^2/2$ term if we perform a renormalization of the σ-field and the coupling constant. In fact, recall that we can perforn finite renormalization withoutchanging the physical content of the theory. Thus, if we make

$$\sigma \to Z^{1/2}\sigma \quad , \qquad (3.1a)$$

where

$$Z = \left(1 + \frac{g^2N}{2\pi}(\cosh 4\alpha - 1)\right)^{-1} , \qquad (3.1b)$$

the factor which'multiplies $u^2/2$ disappears, but

$$\bar{\psi}^a e^{2\alpha\gamma_5} ig\sigma\,\psi^a \to \bar{\psi}^a e^{2\alpha\gamma_5} ig\, Z^{1/2}\sigma\psi^a \quad . \qquad (3.2)$$

We can absorb the factor $Z^{1/2}$ by redefining the coupling constant,

$$g\, Z^{1/2} = g' \quad . \qquad (3.3)$$

We are thus left with the generating functional

$$Z = \int D\sigma \exp\left\{-\int d^2x \frac{\sigma^2}{2}\right\} \prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \times$$
$$\times \exp\left\{-\int d^2x \left[\bar{\psi}^a (i\partial\!\!\!/ + ig' e^{\gamma_5\alpha} \sigma)\psi^a - \bar{\eta}^a e^{\gamma_5\alpha} \psi^a - \bar{\psi}^a e^{\gamma_5\alpha} \eta^a\right]\right\} . \quad (3.4)$$

The perturbative expansion is obtained by expanding the interaction term in powers of g'. Since derivatives with respect to $\bar{\eta}^a(\eta^a)$ bring down factors $e^{\alpha\gamma_5}\psi^a(\bar{\psi}^a e^{\alpha\gamma_5})$, it is easy to see that we can rewrite eq. (3.4) as

$$Z = \int D\sigma \exp\left\{-\int d^2x \left[\frac{\sigma^2}{2} - ig'\sigma \frac{\delta}{\delta\eta^a}\frac{\delta}{\delta\bar{\eta}^a}\right]\right\}$$
$$\times \prod_{a=1}^{N} \int D\bar{\psi}^a D\psi^a \exp\left\{-\int\left[d^2x\, \bar{\psi}^a\, i\partial\!\!\!/\, \psi^a - \bar{\eta}^a e^{\alpha\gamma_5}\psi^a - \bar{\psi}^a e^{\alpha\gamma_5}\eta^a\right]\right\} \quad (3.5)$$

Finally, a transformation in the fermionic fields as in eq.(2.12) with $S^{-1} = -i\partial\!\!\!/\, \delta(x-y)$ leads to

$$Z = \int D\sigma \exp\left\{-\int d^2x \left[\frac{\sigma^2}{2} - ig'\sigma \frac{\delta}{\delta\eta^a}\frac{\delta}{\delta\bar{\eta}^a}\right]\right\}$$
$$\times \exp\left\{\int d^2x\, \bar{\eta}^a e^{\alpha\gamma_5}(i\partial\!\!\!/)^{-1} e^{\alpha\gamma_5}\eta^a\right\} \prod_a \int D\bar{\psi}^a D\psi^a \exp\left\{-\int d^2x\, \bar{\psi}^a i\partial\!\!\!/\psi^a\right\} , \quad (3.6)$$

where the fermionic integrals are constants which can be neglected. Since

$$\bar{\eta}^a e^{\alpha\gamma_5}(i\partial\!\!\!/)^{-1} e^{\alpha\gamma_5}\eta^a = \bar{\eta}^a(i\partial\!\!\!/)^{-1}\eta^a, \quad (3.7)$$

all dependence on α disappears and the Green functions are a-independent.

4. CONCLUSIONS AND DISCUSSION

Finally, for completeness, let us show why the $N=1$ Gross-Neveu model is invariant. For this purpose, it is more convenient to use expression (2.16) which, after integrating over a, becomes

$$Z = \int D\bar{\psi}\, D\psi\, \exp\left\{ - \int d^2x \left[\bar{\psi}^a\, i\partial\!\!\!/\, \psi^a + \frac{g^2}{2}\, Z (\bar{\psi}^a\, e^{2\gamma_5 \alpha}\, \psi^a)^2 \right] \right\} \quad , \qquad (4.1)$$

where we have put the sources equal to zero because we are only interested in the transformation properties of the Gross-Neveu Lagrangian. As in section 3, we can define $g' = gZ^{1/2}$ and all dependence on a is left in $(\bar{\psi}^a\, e^{2\gamma_5\alpha}\, \psi^a)^2$. However, if $N=1$, we can use property eq. (2.3) with $M = e^{2\gamma_5\alpha}$,

$$(\bar{\psi}\, e^{2\gamma_5\alpha}\, \psi)^2 = (\bar{\psi}\psi)^2 \qquad (4.2)$$

and we prove that the-N=l model is invariant.

To summarize, we have applied the G-SMSS method to the Gross--Neveu model with sources, extending in this way the work by Carneiro, Mignaco and Thomaz. The inclusion of sources allowed us to prove that the Green functions do not depend on the parameter of the chiral rotation. The result holds for arbitrary N. However, that does not mean that the Gross-Neveu model is invariant under chiral rotation. Only the $N=1$ model is invariant. As explained in the introduction, the chiral rotation is just a change of integration variables, and the final result must not depend on them. Since the G-SMSS yields a non trivial Jacobian even for global chiral rotations, there was the suspicion at the beginning that there might be some inconsistency in the ζ-function regularization. We have shown that this is not the case.

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APPENDIX

In this appendix we sketch the calculation of the chiral Jacobian. According to G-SMSS[3], if we define a_0 such that

$$e^{r\gamma_5\alpha}\, D\, e^{r\gamma_5\alpha} \equiv i\partial\!\!\!/ + a_0 \ , \qquad (A.1a)$$

where

$$D = i\partial\!\!\!/ + ig\sigma \qquad (A.1b)$$

and the following Seeley's coefficients

$$b_{-1}(x,\xi,\lambda) = -(\xi+\lambda)^{-1}\quad , \quad b_{-2}(x,\xi,\lambda) = -b_{-1}\, a_0\, b_{-1}\ ,$$

$$b_{-3}(x,\xi,\lambda) = -b_{-2}\, a_0\, b_{-1} - i\,\frac{\partial b_{-1}}{\partial \xi_\mu}\,\frac{\partial a_0}{\partial x_\mu}\, b_{-1}\ , \qquad (A.2)$$

the two dimensional Jacobian associated with the chiral rotation is then given by

$$\ln J = -\frac{2i}{(2\pi)^2}\int dx \int_0^1 dr \int_{|\xi|=1} d\xi \int_0^\infty du\ \mathrm{Tr}\left[b_{-3}(x,\xi,iu)\gamma_5\right]\alpha(x)\,, \qquad (A.3)$$

where the parameter r is used to produce a finite chiral rotation from infinitesimal ones by iteration.

In our case

$$a_0 = -\,i r\gamma_5 \not\partial\alpha + i g\sigma \cosh 2r\alpha + i g\sigma\gamma_5 \sinh 2r\alpha\ . \qquad (A.4)$$

Substituting eq.(A.4) into eq.(A.2) we determine b_{-3}, which is used to calculate

$$\mathrm{Tr}\left[b_{-3}(x,\xi,\lambda)\gamma_5\right] = \frac{2\lambda\, g^2\sigma^2(x)\,\sinh 4r\,\alpha(x)}{(\lambda^2-\xi^2)^2} - \frac{2\lambda\, r\,\partial^2\alpha(x)}{(\lambda^2-\xi^2)^2} \qquad (A.5)$$

After substituting this result into eq. (A.3) we arrive at the final result

$$\ln J = -\int dx\,\frac{1}{2}\,\sigma^2(x)\,\frac{g^2}{2\pi}\left[\cosh 4\alpha(x) - 1\right] + \frac{1}{2\pi}\int dx\ \alpha(x)\,\partial^2\alpha(x)\,. \qquad (A.6)$$

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Resumo

O modelo de Thirring é invariante sob transformações quirais globais. Entretanto, utilizando-se o método desenvolvido por Gamboa-Saraví, Muschietti, Schaposnik e Solomin para calcular o Jacobiano associadocom este tipo de transformação, obtém-se um resultado não trivial. Este fato poderia estar indicando que a regularização por eles usada estaria desrespeitando as simetrias da teoria. Decidimos então investigar como as funções de Green do modelo de Gross-Neveu se comportam sob transformações quirais globais (o modelo de Gross-Neveu com um campo fermiônico é equivalente ao modelo de Thirring). Mostramos que a contribuição que vem do Jacobiano pode ser eliminada explorando-se a invariância da teoria sob renormalização finitas e que as funções de Green não dependem do parâmetro da rotação quiral.