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On the Existence of Singularities in the Geometrization of Lagrangian Dynamics

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Abstract It is shown that the standard geometric picture of an irnportant class of nonrelativistic Lagrangian motions has the origin of the generalized velocity space as a singular point. This occurs when the motion's generating force has a less than quadratic dependence on the generalized velocities. The important cases of a gradient force-field and that of a Rayleigh force-field are considered as examples. The corresponding dynamical connections are constructed and present polos of order two and one, respectively, at the origin of velocity space. This implies that well-benaved Lagrangian dynamics may originate ill-behaved gauge-fields in configuration space.

1. INTRODUCTION

Geometry enters the realm of mechanics in connection with the inertial quality of rnass. The principles of analytical mechanics have shown that the really fundamental quantity which characterizes the inertia of mass is not momentum, but kinetic energy. Numerous attempts have been made to link nonrelativistic Lagrangian motion to geometry. What is common to all of them is that under the concept of geometrization of the motion when a field of forces is given, the idea is to find such a mathematical structure to the configuration space that the path defined by the time development of the system turns out to be a geodesic line. In this sense, the geometrization of conservative motion was solved by Douglas[1] and Eisenhart[2] who showed that, for this case, the geometrical structure of a Riemannian space is sufficient. After these classical papers, there were many attempts to describe nonconservative motion. It was shown, first by Lichnerowicz[3] and later by others[4,5], that in this case one needs a more complex mathematical structure than the

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Riemannian one more recently, Trumper[6], in his construction of a geometrized version of rheonomic motion, exploited the fact that the Lagrangian mechanics of nonrelativistic rheonomic systems is contained in the theory of a class of linear symmetric connections in configuration spacetime. But Trumper's geometrization, in spite of being defined in configuration spacetime, excludes all relativistic mechanics and also all types of forces which depend on powers higher than quadratic in the generalized velocities.

The main purpose of this paper is to show the existence of singularities in the geometric picture of the Lagrangian motion when generated by forces with a less than quadratic dependence on the generalized velocities. The particular cases of a gradient force-field and that of a viscous force, linear in the generalized velocities (Rayleigh's force), are considered. The coefficients of the corresponding dynamical connections are constructed with the metric kinetic energy tensor and the generalized force. These connections present poles of order two and order one, respectively, at the origin of velocity space. Meanwhile, contrarily to Trumper's configuration spacetime geometric model, the motion's geometric picture in configuration space is valid and without singularities for forces which depend on powershigher than, or equal to, quadratic in the generalized velocities. It is conjectured that the singular behaviour of the geometric description will disappear, if the configuration space of the Lagrangian description, is considered a stochastic space[13], that is, a space in which the point-like events are not ascribed definite coordinates X.

2. THE DYNAMICAL LINEAR CONNECTION

Let us consider a holonomic scleronomic system with n degrees of freedom whose configuration-velocity description is represented by the generalized coordinates $\{q^\nu\}$ and the generalized velocities $\{\dot{q}^\nu = dq^\nu/dt$; $\nu = 1,\ldots,n$. Here t, the time, is the affine parameter. The kinetic energy

$$T(q,\dot{q}) = \frac{1}{2}\, a_{\mu\nu}(q)\dot{q}^\mu\dot{q}^\nu \geq 0 \ , \tag{2.1}$$

defines the scalar field $T(q,\dot{q})$ in terms of the covariant, symmetric, metric tensor $a_{\mu\nu}(q)$ and the contravariant generalized velocity $\dot{q}^{\mu}$. From the geometric meaning of $a_{\mu\nu}(q)$, we may write eq. (1.2) in the scalar product form

$$\frac{1}{2T}\,(\dot{q}|\dot{q}) = 1 = (U|U) \quad . \tag{1'.2}$$

Then,

$$U^{\beta}(q,q) = \frac{\dot{q}^{\beta}}{\sqrt{2T}} \quad , \tag{1''.2}$$

is a contravariant vector of unit length.

Now, it is well known that the Lagrange equations of first kind may be put in the covariant form

$$\ddot{q}^{\lambda} + \{{}^{\lambda}_{\mu\nu}\}\,\dot{q}^{\mu}\dot{q}^{\nu} = Q^{\lambda}(q,\dot{q}) \quad , \tag{2.2}$$

where $Q^{\lambda}(q,\dot{q})$ is the generalized force and the $\{{}^{\lambda}_{\mu\nu}\}$ are the Christofell symbols of second kind, constructed with the metric tensor $a_{\mu\nu}(q)$. However, eq. (2.2) is not qualified as an acceptable way of geometrization since the force $Q^{\lambda}(q,\dot{q})$ is not included as part of the geometric structure. Then, instead of eq. (2.2) we consider the geodesic equation

$$\ddot{q}^{\lambda} + L^{\lambda}(q,\dot{q})_{\mu\nu}\,\dot{q}^{\mu}\dot{q}^{\nu} = 0 \tag{2.3}$$

The quantity $L^{\lambda}(q,\dot{q})_{\mu\nu}$ is a generalized connection, which may be velocity dependent and contains the Christoffel symbol as a velocity independent term

$$L^{\mu}(q,\dot{q})_{\nu\rho} = \{{}^{\mu}_{\nu\rho}\} + \gamma^{\mu}(q,\dot{q})_{\nu\rho} \quad . \tag{2.4}$$

Then, eq. (2.3) may be written in the form

$$\ddot{q}^{\lambda} + [\{{}^{\lambda}_{\mu\nu}\} + \gamma^{\lambda}(q,\dot{q})_{\mu\nu}]\,\dot{q}^{\mu}\dot{q}^{\nu} = 0 \quad , \tag{2.5}$$

Now, comparing eqs. (2.2) e (2.3) we have

$$\gamma^{\lambda}(q,\dot{q})_{\mu\nu}\,\dot{q}^{\mu}\dot{q}^{\nu} = -\,Q^{\lambda}(q,\dot{q}) \quad . \tag{2.6}$$

To obtain the explicit form of the third order tensor $\gamma^{\lambda}(q,\dot{q})_{\mu\nu}$, we must recall that the relevant dynamical quantities at hand are the force $Q^{\lambda}(q,\dot{q})$ and the metric $a_{\mu\nu}(q)$. Motivated by the general form of the relativistic linear connection proposed in 1971 by Oliveira[8] (in his geometrization of nondissipative long range interactions), we were able to construct the general solution of the nonrelativistic eq. (2.6)

$$\gamma^{\lambda}(q,\dot{q})_{\mu\nu} = -\frac{1}{2T}\left[Q^{\lambda} a_{\mu\nu} + Q_{\mu} a^{\lambda}{}_{\nu} - Q_{\nu} a^{\lambda}{}_{\mu}\right] \qquad (2.7)$$

with $a^{\lambda}{}_{\nu} = \delta^{h}{}_{\nu}$.

The form of connection (2.7) is coincident with that used by Vujanovic[9] in 1967, in his geometrization of classical nonconservative dynamics. Only the symmetric part of *(2.7)* contributes to eq. (2.7). Now it is convenient to define

$$\frac{Q^{\lambda}(q,\dot{q})}{2T} = \Phi^{\lambda}(q,\dot{q}) \qquad (2.8)$$

and to split $\gamma^{h}{}_{\mu\nu}$ into the form

$$\gamma^{\lambda}{}_{\mu\nu} = (-\Phi^{\lambda} a_{\mu\nu}) + (-\Phi_{\mu}\delta^{\lambda}{}_{\nu} + \Phi_{\nu}\gamma^{\lambda}{}_{\mu}) = \gamma^{\lambda}{}_{(\mu\nu)} + \gamma^{\lambda}{}_{|\mu\nu|} \qquad (2.9)$$

which will be useful later.

3. THE GEOMETRY GEMERATED BY $\gamma^{\lambda}{}_{(\mu\nu)}$

In this case, the general dynamical connection (2.4) is restricted to having the symmetric form

$$L^{\mu}{}_{\nu\rho} = L^{\mu}{}_{(\nu\rho)} = \{{}^{\mu}_{\nu\rho}\} + \gamma^{\mu}{}_{(\nu\rho)} = \{{}^{\mu}_{\nu\rho}\} - \Phi^{\mu} a_{\nu\rho} \qquad (3.1)$$

The symmetric conaition (3.1) defines a torsionless geometry. It is well known that in this case it is possible to construct, at an arbitrarily chosen point, $(q^{\nu}{}_{0})$, of the configuration manifold, a local coordinate system in which $L^{\mu}{}_{(\nu\rho)} = 0$. More generally, given an arbitrary curve in such space, we can always introduce (geodesic) coordinates such that $L^{\mu}{}_{(\nu\rho)}$ vanishes at all points of the curve[11]. Then, at

(q^{ν}_{0}) we have

$$\{{}^{\mu}_{\nu\rho}\}_0 \, a^{\nu\rho}(q_0) = \Phi^{\mu}(q_0,\dot{q}) = \frac{Q^{\mu}(q_0,\dot{q})}{2T_0} \qquad (3.2)$$

Eq.(3.2) is a restriction on the velocity $(\dot{q}^{\nu})$ at (q_0).

It is important to remark that owing to the presence of the kinetic energy in the denominator of Q^{μ} in eq. (3.2), and because of the general nature of the forces Q^{μ}, the connection (2.4) is defined only along real trajectories. This means that the geometry is defined on a space of paths. Then, on the trajectory in configuration space, we can define the covariant derivative of a tensor of any valence. In particular, the covariant derivative of the metric tensor $a_{\mu\nu}$, belonging to the symmetric connection (3.1) is

$$\nabla_{\mu}(a_{\lambda\tau}) = 2a_{\lambda(\mu}\Phi_{\tau)} = a_{\lambda\mu}\Phi_{\tau} + a_{\lambda\tau}\Phi_{\mu} \quad . \qquad (3.3)$$

If the more general, non-symnetric connection (2.7) is used in the construction of the covariant derivative, we obtain instead of eq. (3.3)

$$\nabla_{\mu}(a_{\lambda\tau}) = 2\Phi_{\mu}a_{\lambda\tau} \quad . \qquad (3.3')$$

Equations of type (3.3') define a semi-metric geometry[12], which is not a Weyl geometry because the connection (2.4) is not symmetric[13].

4. THE SPACE OF PATHS AND ITS CURVATURE

Quantities such as $a_{\mu\nu}$, Q^A, T, need not be globally defined as functions of position but only along trajectories. Nevertheless, by considering a continuous family of trajectories simply covering a neighborhood of the original path $q^A(t)$, we can extend the domain of definition of such quantities in order to make their partial derivatives meaningful. The continuous famity of trajectories may be constructed, for example, with trajectories starting from a fixed configuration point, towards different directions with the same kinetic energy, but we could also take the trajectories starting with equal kinetic energies from the points of a configuration hypersurface towards the directions normal to that surface. Now, possessing a manifold of trajectories, we may define its curvature tensor by the standard form[10]

$$R_{\nu\mu\lambda}{}^{\tau} = \partial_{\nu} L^{\tau}{}_{\mu\lambda} - \partial_{\mu} L^{\tau}{}_{\nu\lambda} + L^{\tau}{}_{\nu\sigma} L^{\sigma}{}_{\mu\lambda} - L^{\tau}{}_{\mu\sigma} L^{\sigma}{}_{\nu\lambda} \qquad (4.1)$$

From the symmetric connection eq.(3.1) and with eq.(4.1) we obtain

$$R_{\nu\mu\lambda}{}^{\tau} = \overset{0}{R}_{\nu\mu\lambda}{}^{\tau} + a_{\lambda\mu}\nabla_{\nu}\Phi^{\tau} - a_{\lambda\nu}\nabla_{\mu}\Phi^{\tau} \quad , \qquad (4.2)$$

where $\overset{0}{R}_{\nu\mu\lambda}{}^{\tau}$ is the curvature tensor, belonging to the Christoffel connection, constructed with $a_{\mu\nu}$. The tensor $\nabla_{\nu}\Phi^{\tau}$ is the covariant derivative constructed with the symmetric connection (3.1).

$$\nabla_{\nu}\Phi^{\tau} = \partial_{\nu}\Phi^{\tau} + \{{}^{\tau}_{\nu\sigma}\}\,\Phi^{\sigma} - \Phi_{\nu}\Phi^{\tau} \ . \qquad (4.3)$$

Now, two second-order curvature tensors may be obtained from eq.(4.1). By contracting $R_{\nu\mu\tau}{}^{\tau}$ with respect to the indices ν, τ we obtain the Ricci tensor

$$R_{\tau\mu\lambda}{}^{\tau} = R_{\mu\lambda} = \overset{0}{R}_{\mu\lambda} + a_{\lambda\mu}\nabla_{\tau}\Phi^{\tau} - a_{\lambda\tau}\nabla_{\mu}\Phi^{\tau} \qquad (4.4)$$

The other curvature tensor may be constructed by contracting the indices λ, τ and we obtain

$$R_{\mu\nu\tau}{}^{\tau} = a_{\tau\mu}\nabla_{\nu}\Phi^{\tau} - a_{\tau\nu}\nabla_{\mu}\Phi^{\tau} = \partial_{\nu}\Phi_{\mu} - \partial_{\mu}\Phi_{\nu} \quad . \qquad (4.5)$$

5. THE DYNAMICAL CONNECTION GENERATED BY A GRADIENT FORCE-FIELD

In this simple case, the manifold of paths is constituted by trajectories with the same total energy. Here, owing to the presence of the potential field, the curvature tensor, eq.(4.5), vanishes identically. As is well known, the potential and kinetic energies are restricted by the total energy conservation relation $E = T + V(q)$. Then, from eq. (2.8), we have

$$\Phi_\mu = \frac{Q_\mu}{2T} = \frac{1}{2T}\,\partial_\mu V(q) = -\,\partial_\mu V(q)\,\frac{1}{(\dot{q}\,\dot{q})} = -\,\partial_\mu V(q)\,\frac{1}{\dot{q}} \qquad (5.1)$$

But, from eq. (3.1) and eq. (5.1) we obtain

$$L^\mu(q,\dot{q})_{\nu\rho} = \{^{\ \mu}_{\nu\rho}\} + a_{\nu\rho}(q)\,\partial^\mu V(q)\,\frac{1}{|\dot{q}|^2} \quad . \qquad (5.2)$$

The connection eq. (5.2) has a simple pole of second degree located at the origin of velocity space. This means that the geometric picture of dynamics remains highly singular in some neighborhood of that point. It is interesting to remark that from the restriction $E=$ cte, we may write eq. (5.2) in the convenient form

$$L(^\mu_q)_{\nu\rho} = \{^{\ \mu}_{\nu\rho}\} + a_{\nu\rho}(q)\,\frac{1}{2\left[E-V(q)\right]}\,\partial^\mu V(q) =$$

$$= \{^{\ \mu}_{\nu\rho}\} = \frac{1}{2}\,a_{\nu\rho}\,\partial^\mu\left[\log(1-V/E)\right] =$$

$$= \{^{\ \mu}_{\nu\rho}\} = -\,a_{\nu\rho}\,\partial^\mu\left[\log\sqrt{1-V/E}\,\right] \quad . \qquad (5.2')$$

Now it is easy to see that the singular behaviour has a logarithmic derivative form and occurs exactly when $V = E$, (at a turning point, for example) in configuration space. From eqs. (5.1) and (4.5), we have in the gradient force-field case

$$R_{\nu\mu\tau}{}^{\tau} = \frac{1}{2T}\,(\partial_\nu V\partial_\mu V - \partial_\mu V\partial_\nu V) = 0 \quad . \qquad (5.3)$$

Then, only one curvature tensor is left, namely that one defined by eq. (4.4). The scalar curvature constructed with Ricci's tensor, (4.4) in this case is

$$R = \overset{0}{R} + 2\,\nabla_\mu\Phi^\mu \quad , \qquad (5.4)$$

where Φ^μ is defined by eq. (5.1') and $\overset{0}{R}$ is the scalar curvature belonging to the Christoffel connections constructed with $a_{\mu\nu}$. If we represeni eq. (5.3) in the form

$$\frac{1}{2}\,(R-\overset{0}{R}) = \tag{5.5}$$

we see that the Φ^{μ} field has $(R-R)$ as a source field.

For the sake of completeness and after some tedious calculations, it is possible to obtain the following relevant equations

$$\begin{aligned} &R_{(\nu\mu)\lambda}{}^{\tau} = 0 \\ &R_{|\nu\mu\lambda|}{}^{\tau} = 0 \\ &R_{\nu\mu(\lambda\tau)} = -\,2\nabla_{[\nu}{}^{a}{}_{\mu]}\,(\lambda^{\Phi}\tau) \\ &\nabla_{\tau}(R_{\nu\mu\lambda}{}^{\tau}) = 2\nabla_{[\nu}{}^{R}{}_{\mu]\lambda} \;. \end{aligned} \tag{5.6}$$

6. THE DYNAMICAL CONNECTION GENERATED BY FORCES LINEARLY DEPENDENT ON THE GENERALIZED VELOCITY

In this case, we will consider the very interesting and well known[7] Rayleigh dissipative force-field, $Q_{\lambda}(q,\dot{q})$ which is defined by

$$Q_{\lambda}(q,\dot{q}) = -\,\eta_{\lambda\theta}(q)\,\dot{q}^{\theta} \;. \tag{6.1}$$

Of course, the dissipative character of any dissipative force is contained in the restriction

$$Q_{\lambda}\,\dot{q}^{\lambda} \leq 0 \;, \tag{6.2}$$

for arbitrary $\dot{q}^{\lambda}$.

In eq. (6.1), the real, symmetric, tensor field, $\eta_{\lambda\theta}(q)$, is the assumed mathematical entity which represents the viscosity of the configuration manifold and it must be such that

$$\eta_{\lambda\theta}(q)\,\dot{q}^{\lambda}\dot{q}^{\theta} \geq 0 \tag{6.3}$$

A trivial consequence of eq.(6.1) and eq.(6.3) is

$$Q_\lambda(q)\dot{q}^\lambda = - \eta_{\lambda\theta}\, \dot{q}^\lambda \dot{q}^\theta \leqslant 0 \ . \tag{6.4}$$

Now, with eq. (6.4) and eq. (3.1), we have for the Rayleigh case

$$L^\mu(q,\dot{q})_{\nu\rho} = \{^{\,\mu}_{\nu\rho}\} + a_{\nu\rho}(q)\,\frac{1}{2T}\,\eta^\mu{}_\theta \dot{q}^\theta = \{^{\,\mu}_{\nu\rho}\} + a_{\nu\rho}(q)\,\frac{\eta^\mu{}_\theta \dot{q}^\theta}{|\dot{\bar{q}}|^2} =$$

$$= \{^{\,\mu}_{\nu\rho}\} + a_{\nu\rho}(q)\,\eta^\mu{}_\theta U^\theta(q,\dot{q})\,\frac{1}{|\dot{\bar{q}}|} \ , \tag{6.5}$$

where $U^\theta(q,\dot{q})$ is the unit generalized velocity defined by eq. (2.1").

The formal structure of the dynamical connection (6.5), shows that in the regime of high velocities, the geometric picture of motion is basically Riemannian, but in the low velocity regime, the geometry presents a singular behavior due to the presence of the first order pole, located at the origin of velocity space.

7. CONCLUSIONS AND CONJECTURES

The geometry of configuration space does not in itself exist, since it is determined by the interaction of dynamical systems, Each form of interaction creates its own geometry. In this paper we have shown that well-behaved force-fields may create ill-behaved geometries. The geometrization results as a consequence of the transformation of the Lagrangian equation of motion into geodesic equations. In order to do this, as we have shown, it is necessary to construct a dynamical connection field in terms of the generalized force field. Now, this dynamical connection, in Utiyama's[13] sense, is a gauge field. But if this gauge field proceeds from a force-field with less than quadratic dependence on the velocities, it is singular at the origin of velocity space. This implies that well-behaved Lagrangian dynamics may originate an ill-behaved gauge field. For the class of force-fields considered in this paper, the geometric ill behavior is a direct consequence in the a priori hypothesis of sharp localizability in the manifold. Naturally, if a small neighborhood of the velocity space's origin is subtracted from the manifold, the ill-behavior will disappear. Then, if the concept of well defined localization is relaxed, the singular behavior is overcome and the geometry stays well behaved. This suggests that if we intend to construct a well-behaved geometry picture of the Lagrangian motion in configuration space we must employ a stochastic geometry[14], because in a stochastic configuration space, the particle coordinates q^v are determined only up to a certain accuracy Δq^v, while the average value of the coordinate is sharply defined. Then, stochastic geometries generalize conventional geometries in the sense that these arerecovered if the space consists exclusively of sharp values. In particular, Blokhintsev's[15] stochastic geometric model appears, to us, very interesting for the geometrization of the type of force-fields considered in this paper. This is a task for another paper.

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Resumo

Demonstra-se que a geometrização usual de uma importante classe de movimentos não relativistas, Lagrangianos, é singular na origem do espaço das velocidades generalizadas. Exemplifica-se com os importantes particulares casos de uma força dependente de um potencial e de uma força dissipativa do tipo Rayleigh. As correspondentes afinidades dinâmicas são construidas e elas têm polos de segunda e primeira, ordens, respectivamente. Isto é uma evidência de que dinâmicas bem comportadas podem gerar campos de gauge, mal comportados, no espaço das configurações.