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Separable Coordinates and Particle Creation I: the Klein-Gordon Equation

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Abstract We present a very simple derivation of the 10 orthogonal coordinate systems where the Klein-Gordon equation separates. It is based on the conformal structure of the two dimensional Minkowski space. Horizons, proper time and acceleration of observes that follow the time coordinate line, as well as other physical properties of the systems, are obtained. The relevance of these coordinates is discussed, specially in the context of quantum field theory in curved space.

1. INTRODUCTION

We present here a geometrical derivation of the orthogonal coordinates in two-dimensional Minkowski space with which the Klein-Gordon equation separates. They are the Minkowski equivalents of polar, elliptic and parabolic coordinates of non-relativistic physics, and have a very important role as well. They include the known Rindler coordinates, that are adapted to accelerated observers and are much used in quantum field theory in curved space[1] in connection with the Hawking effect of particle creation. These coordinates are also adapted to very interesting physical situations like a global boost - the coordinate lines are the world lines of asymptotically inertial particles that experience a boost in their velocities; this could eventually lead to a link between global quantum field theory and local relativity. Another system describes an observer that is inertial in the past and uniformly accelerated in the future, providing the opportunity of examining particle creation in the realm of only one system of observers, thus eliminating one of the major episthermlogical dificultíes of the Hawking effect. They are also useful to study compact regions of the Minkowski plane. And above all they are

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such that the Klein-Gordon equation (and most interesting quantities) are exactly solvable, distillating the physical understanding from the mathematical difficulties of the problem of particle creation even in flat space-time. A following paper will handle this last case.

Separable coordinates are well known in two- and three- dimensional Euclidean space since the work of Eisenhardt[2]. In Minkowski space they have also been classified by Urbantke[3] with eleyant methods of projective geometry. Kalnins and Miller[4] related orthogonal separable systems of coordinates for a given equation with symmetry operators of this equation. They obtained all such systems for the Klein-Gordon equation in two, three and four dimensional Minkowski space-time. They are in number of 10, 53 and 262, respectively. These authors solved the more general problem of nonorthogonal R-separable coordinates in complex riemannian spaces and also acliieved a general theory of R-separation, in a coordinate-free manner[5]. Recently some work has been done on the more abstract question of separable manifolds[6].

Here we present a simple derivation of orthogonal separable coordinates in two dimensional Minkowski space for the Klein-Gordon equation following the reasoning of Morse, Feshbach[7] and rnaking use of conformal transformations. We obtain the coordinate transformation, the metric, the coordinate lines and the acceleration along them in the region where they are time-like. Other geometrical properties of the systems and analytical properties of the separated equations are derived elsewhere[8]

2. DERIVATIONS

We begin with a Minkowski space described by cartesian coordinates (t,x) and the metric

$$\eta = \eta_{ab}\, dx^a\, dx^b \qquad\qquad a,b,\ldots = 0 \text{ or } 1 \qquad\qquad (1$$

η has signature $(+,-)$. We also define normalized light coordinates

$$\begin{cases} \ell^- = 2^{-1/2}(t+x) \\ \ell^+ = 2^{-1/2}(t-x) \end{cases} \qquad\qquad \begin{cases} t = 2^{-1/2} \quad - \\ x = 2^{-1/2} \end{cases} \qquad (2)$$

where the metric has only one independent component

$$\eta = 2\, d\ell^- \, d\ell^+ \; . \qquad (3)$$

Now consider another orthogonal coordinate system with the metric

$$g = 2\, dL^- \, dL^+ \; . \qquad (4)$$

Since a conformal transformation keeps angles invariant, there is such a transformation between these two systems

$$\eta = \Omega^2 g \qquad \text{or} \qquad (5)$$

$$d\ell^- \, d\ell^+ = \Omega^2 (L^-, L^+) dL^- \, dL^+ \quad , \qquad (6)$$

but, as we remain in the same space, there must be also a coordinate transformation

$$\ell^a = \ell^a (L^-, L^+) \qquad (7)$$

between the two systems so that

$$d\ell^- \, d\ell^+ = \frac{\partial \ell^-}{\partial L^a} \, \frac{\partial \ell^+}{\partial L^b} \, dL^a \, dL^b \; . \qquad (8)$$

To satisfy eq. (6) we need also

$$\frac{\partial \ell^-}{\partial L^+} = \frac{\partial \ell^+}{\partial L^-} = 0 \quad \text{or} \quad \frac{\partial \ell^-}{\partial L^-} = \frac{\partial \ell^+}{\partial L^+} = 0 \; . \qquad (9)$$

There is no restriction in taking the first condition, so that we have

$$\Omega^2 (L^a) = \frac{d\ell^- (L^-)}{dL^-} \; \frac{d\ell^+ (L^+)}{dL^+} \; . \qquad (10)$$

The d'Alembert operator is defined as

$$\Box_\eta = \partial_t^2 - \partial_x^2 = 2 \partial_{\ell^-} \partial_{\ell^+} \; , \qquad (11)$$

so that

$$\Box_g = \Omega^2 (L^a) \; \Box_\eta \; . \qquad (12)$$

The Klein-Gordon equation, defined as

$$\Box_n \; U(t,x) = - m^2 \; U(t,x) \; , \tag{13}$$

written in the conformal coordinates (T,X), which are related to the L's as their low case relatives in eq.2, takes the following form

$$(\partial_T^2 - \partial_X^2) U(T,X) = -m^2 \; \Omega^2(T,X) U(T,X) \; . \tag{14}$$

Eq.14 holds then in any system of coordinates that satisfies orthogonality in Minkowski space. We now ask for separability and write

$$\Omega^2(T,X) = Y^2(T) - Z^2(X) \; . \tag{15}$$

That means

$$\partial \;\; \partial \;\; \Omega^2 = 0 \tag{16}$$

and from

$$2 \; \partial_T \; \partial_X = \partial^2_{L^-} - \partial^2_{L^+} \tag{17}$$

we have

$$\frac{dL^-}{d\ell^-} \frac{d^3\ell^-}{dL^{-3}} = \frac{dL^+}{d\ell^+} \frac{d^3\ell^+}{dL^{+3}} = \nu \; . \tag{18}$$

These differential equations define the orthogonal separable coordinates. They are hyperbolic for $\nu > 0$, parabolic for $\nu = 0$ and elliptic for $\nu < 0$.

Hyperbolic coordinates are given by

$$\begin{cases} t+x = \frac{1}{w} \left[d_1 \; e^{-w(T+X)} + d_2 \; e^{w(T+X)} \right] \\ t-x = \frac{1}{w} \left[d_3 \; e^{-w(T-X)} + d_4 \; e^{w(T-X)} \right] \end{cases} \tag{19}$$

with

$$d_i = \mathrm{sgn} \; d_i \quad \text{and} \quad \nu = 2w^2 \; . \tag{20}$$

The conformal factor is

$$\Omega^2 = d_1 d_3 \; e^{-2wT} + d_2 d_4 \; e^{2wT} - d_1 d_4 \; e^{-2wX} - d_2 d_3 \; e^{2wX} . \tag{21}$$

Parabolic coordinates are given by

$$\begin{cases} \ell^- = d_1 L^- + \frac{d_2}{2} (L^-)^2 \\ \ell^+ = d_3 L^+ + \frac{d_4}{2} (L^+)^2 \end{cases} \tag{22}$$

with conformal factor

$$\Omega^2 = d_1 d_3 + \frac{d_1 d_4 + d_2 d_3}{\sqrt{2}} T - \frac{d_1 d_4 - d_2 d_3}{\sqrt{2}} X + \frac{d_2 d_4}{2} (T^2 - X^2) \ . \tag{23}$$

Elliptic coordinates are given by

$$\begin{cases} t + x = \frac{1}{w} \left[d_1 \cos w(T+X) + d_2 \sin w(T+X)\right] \\ t - x = \frac{1}{w} \left[d_3 \cos w(T-X) + d_4 \sin w(T-X)\right] \end{cases} \tag{24}$$

with

$$\nu = - w^2 \ . \tag{25}$$

The conformal factor is

$$\Omega^2 = \frac{1}{2} \left[(d_1 d_3 + d_2 d_4) \cos 2wT - (d_1 d_4 + d_2 d_3) \sin 2wT \right.$$

$$\left. + (d_2 d_4 - d_1 d_3) \cos 2wX + (d_1 d_4 - d_2 d_3) \sin 2wX\right] \ . \tag{26}$$

3. COMMENTS AND CONCLUSION

In this way we reproduce the non-cartesian coordinate systems of Kalnins[4]. His treatment of non-orthogonal coordinate is similar to ours. but the requirement of flatness is implement here by introducing a cartesian system of coordinates which is then transformed. Instead, Kalnins made explicit use of the Riemann tensor. It has also been proved to be more economical to demand separability after flatness.

So let us look at the coordinates. Figures 1 to 13 show the orthogonal separable systems of coordinates, with $w=1$. It is interesting to note a few properties of these coordinates.

Systems A is the only one that covers the whole Minkowski-space, It is cartesianlike in a neighbourhood of the origin. The asymptotic

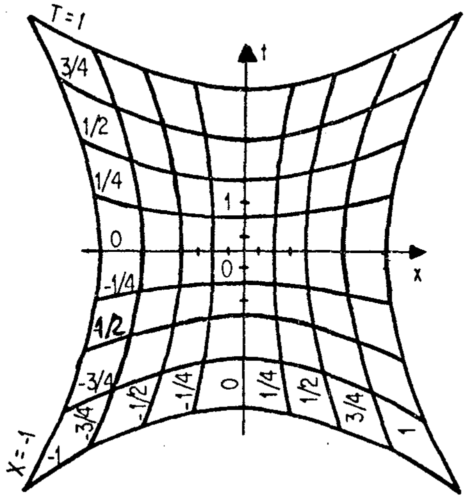

Fig.1 - The system of coordinates A.

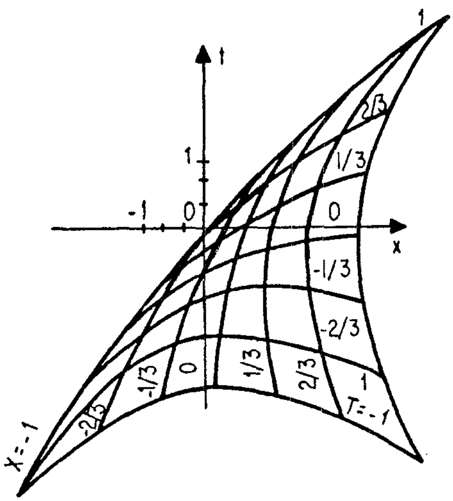

Fig.2 - The system of coordinates B.

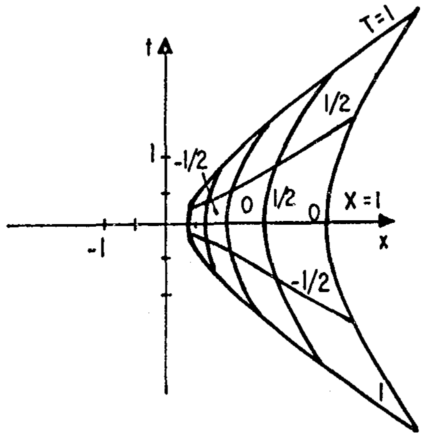

Fig.3 - The system of coordinates C_R.

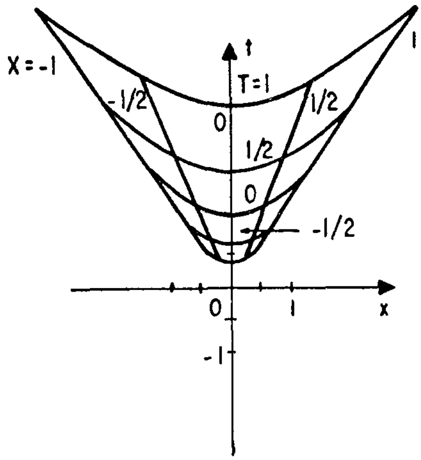

Fig.4 - The system of coordinates C_M.

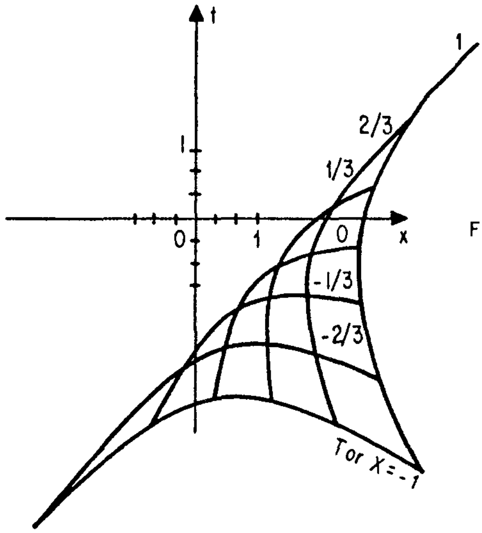

Fig.5 - The system of coordinates D.

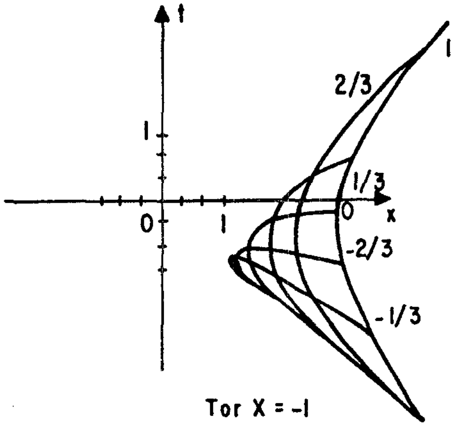

Fig.6 - The system of coordinates E_R.

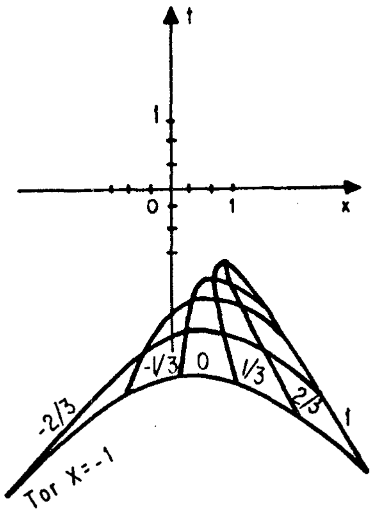

Fig.7 - The system of coordinates E_M.

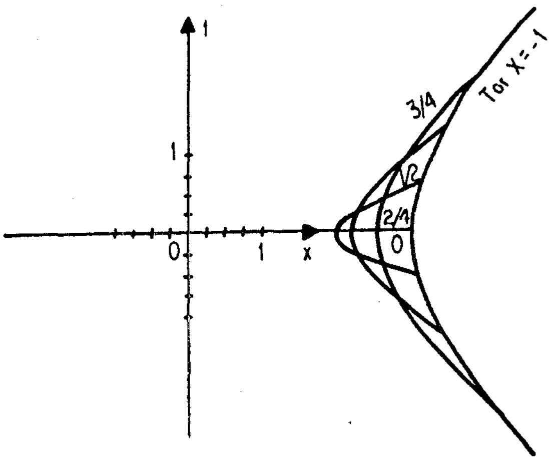

Fig. 8 - The system of coordinates F_R.

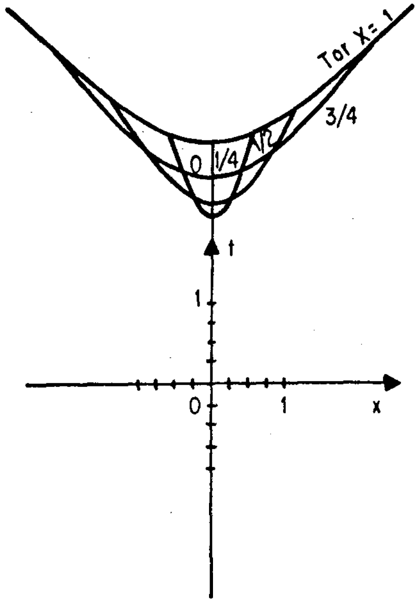

Fig.9 - The system of coordinates F_M.

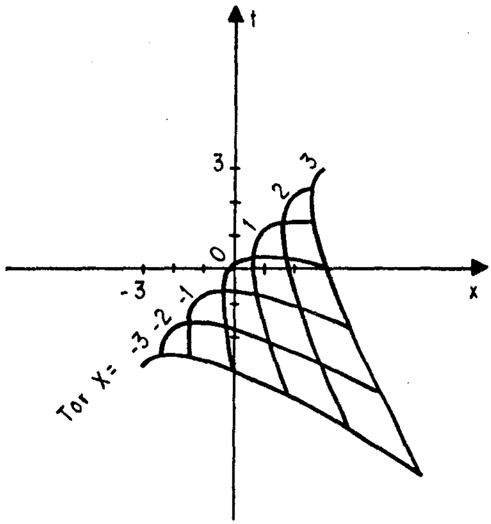

Fig.10 - The system of coordinates G.

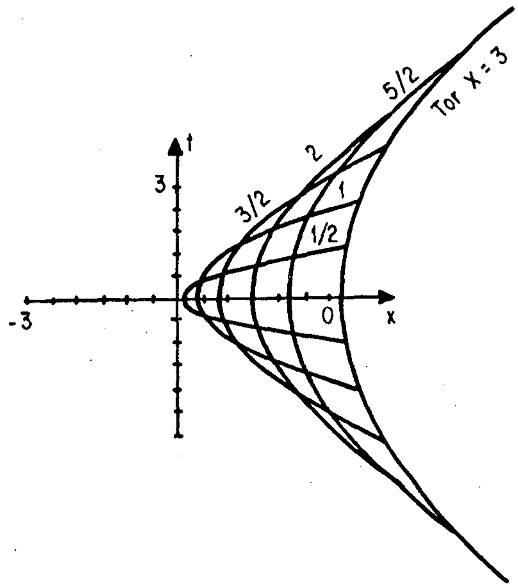

Fig.11 - The system of coordinates H_R.

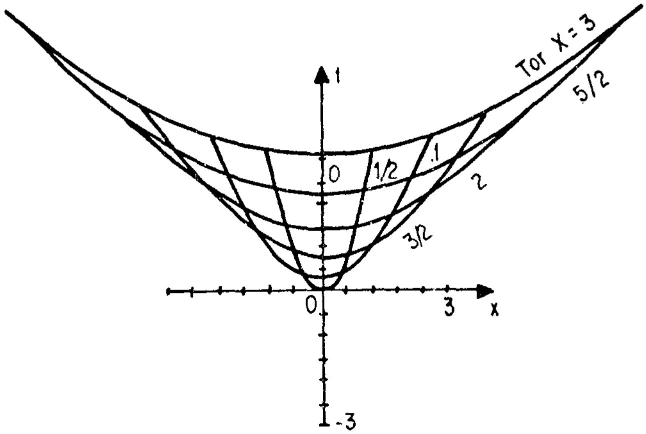

Fig.12 - The system of coordinates $H_{M'}$

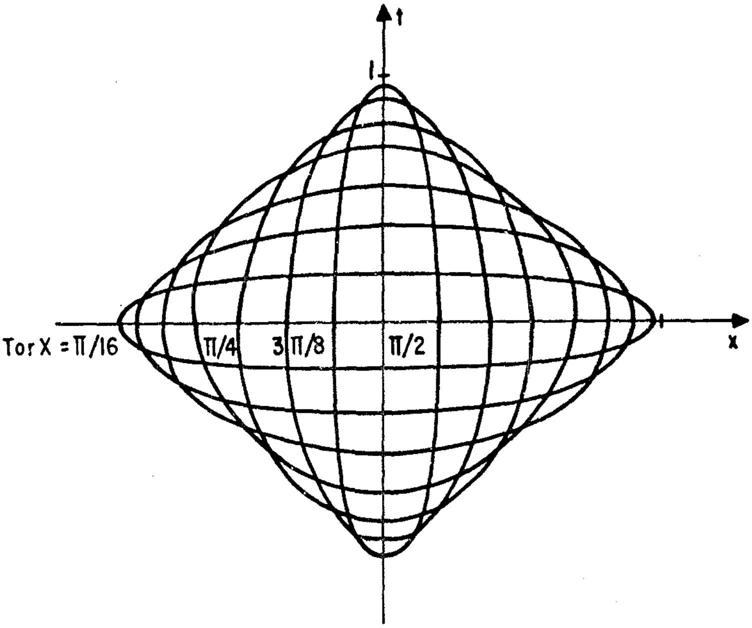

Fig.13 - The system of coordinateç I.

velocity of a point that follows the timelike coordinate line $X=$const. is $\pm$ tanh (wX) at $T=\pm\infty$, much like a boost. The integral of the acceleration of such a point is finite[9].

Systems B, D and G have one coordinate singularity at $\overset{+}{R}$=const.; it is an event-horizon for observers goiny with the timelike coordinate line. For the last two systems the coordinate line touches the horizon needing for that an infinite amount of acceleration in a finite time t. In system B the observers are initially inertial and are smoothly accelerating, so that they become of Rindler type asymptotically. This system is very interesting for investigating the mechanism of particle creation with exact aethods and without comparing two different systems.

Systerns C (the well known Rindler coordinates[10]), E, F and H have two event-horizons at constant ℓ^{+} and ℓ^{-}. In system E the horizons are of two different types.

The last system, I, covers only a compact region of Minkowski--space. It is a very pleasant figure to look at, where one can most easily see one of the important properties of the orthogonal separable coordinates, that is the fact that the coordinate lines are in each case confocal lines in the Minkowski sense[3].

Furthar information on the separable orthogonal system of coordinates can be drawn from tables 1 and 2 (for details on the derivations see reference 8). There the acceleration a at the curve X=const, is given by

$$a = \frac{1}{2\Omega^3}\frac{\partial(\Omega^2)}{\partial X} = \frac{ZZ'}{(Y^2+Z^2)^{3/2}} \quad ; \tag{27}$$

A is the proper-time integration of the acceleration

$$A = ZZ'\int_{H^-}^{H^+}\frac{dT}{Y^2+Z^2} \quad . \tag{28}$$

θ and ϕ are respectively the angle necessary to rotate the hyperbola so that its axes coincides with the R-coordinate axes and the angle between its asymptote and these axes, as indicated in figure 14. They are given by

Table 1 - The hyperbolic systems of coordinates.

COORD.	Conformal factor	$d_1 d_2$ $d_3 d_4$	Transformation of coordinates or curve parameters θ, ϕ	Comments
A	$2(\mathrm{ch}2wT + \mathrm{ch}2wX)$	-1 1 -1 1	$t = 2/w\ \mathrm{sh}\ wT\ \mathrm{ch}\ wX$ $x = 2/w\ \mathrm{ch}\ wT\ \mathrm{sh}\ wX$	$A = 2wT$
B	$e^{-2wT} + e^{2wX}$	-1 1 -1 0	$\tan 2\theta = \tan 2\phi = -e^{-2wT}$ $\tan 2\theta = \tan 2\phi = e^{2wX}$	$a = -\dfrac{w\ e^{2wx}}{(e^{-2wT}+e^{2wX})^{3/2}}$
C_R	e^{2wX}	0 1 -1 0	$t = \frac{1}{w} e^{wX}\ \mathrm{sh}\ wT$ $x = \frac{1}{w} e^{wX}\ \mathrm{ch}\ wT$	$a = we^{-wX}$, $A = wT$
C_M	e^{2wT}	0 1 0 1	$t = \frac{1}{w} e^{wT}\ \mathrm{ch}\ wX$ $x = \frac{1}{w} e^{wT}\ \mathrm{sh}\ wX$	$a = 0$
D	$2(\mathrm{sh}2wX - \mathrm{sh}2wT)$	-1 1 -1 -1	$\mathrm{tg}\theta = -\mathrm{tgh}wT, \tan\phi = -\pm 1$ $\mathrm{tg}\theta = -\mathrm{ctgh}wX, \tan\phi = -\pm 1$	$a = \dfrac{w\ \mathrm{ch}\ 2wX}{[2(\mathrm{sh}2wX - \mathrm{sh}2wT)]^{3/2}}$
E_R	$e^{2wT} - e^{2wX}$	0 1 -1 -1	$\tan 20 = -e^{2wT}$ $\tan 2\phi = \tan 28$	$a = \dfrac{-w\ e^{2wX}}{(e^{2wT} - e^{2wX})^{3/2}}$
E_M	$e^{2wX} - e^{2wT}$	0 -1 -1 -1	$\tan 28 = e^{2wT}$ $\tan 24 = -\tan 20$	$a = \dfrac{w\ e^{2wX}}{(e^{2wX} - e^{2wT})^{3/2}}$
F_R	$2(\mathrm{ch}2wT - \mathrm{ch}2wX)$	1 1 1 1	$t = 2/w\ \mathrm{ch}\ wT\ \mathrm{ch}\ wX$ $x = 2/w\ \mathrm{sh}\ wT\ \mathrm{sh}\ wX$	$a = \dfrac{-w\ \mathrm{sh}\ 2wX}{[2(\mathrm{ch}2wT - \mathrm{ch}2wX)]^{3/2}}$
F_M	$2(\mathrm{ch}2wT + \mathrm{ch}2wX)$	1 1 -1 -1	$t = 2/w\ \mathrm{sh}\ wT\ \mathrm{sh}\ wX$ $x = 2/w\ \mathrm{ch}\ wT\ \mathrm{ch}\ wX$	$a = \dfrac{w\ \mathrm{sh}\ 2wX}{[2(\mathrm{ch}2wT + \mathrm{ch}2wX)]^{3/2}}$

Table 2 - The parabolic and elliptic systems of coordinates.

C R D.	Conformal factor	d_1 d_2 d_3 d_4	Tranformation of coordinates	Acceleration
G	$\frac{2}{d}(T+X)$	$0 \quad \frac{\sqrt{8}}{d}$ $1 \quad 0$	$a^- = (T+X)^2/\sqrt{2}d$ $\ell^+ = (T-X)/\sqrt{2}$	$\sqrt{\frac{d}{8}}\,(T+X)^{-3/2}$
H_R	$\frac{2}{d}(T^2-X^2)$	$0 \quad \frac{\sqrt{8}}{d}$ $0 \quad \frac{\sqrt{8}}{d}$	$t = \frac{1}{d}(T^2+X^2)$ $x = \frac{2}{d}\,T\,X$	$-\frac{d}{2}\,\frac{X}{(T^2-X^2)^{3/2}}$
H_M	$\frac{-2}{d^2}(T^2-X^2)$	$0 \quad \frac{\sqrt{8}}{d}$ $0 \quad -\frac{\sqrt{8}}{d}$	$t = \frac{2}{d}\,T\,X$ $x = \frac{1}{d}(T+X)$	$-\frac{d}{2}\,\frac{X}{(X^2-T^2)^{3/2}}$
I	$\frac{(\cos 2\omega T-\cos 2\omega X)}{2}$	$1 \quad 0$ $1 \quad 0$	$t = -\frac{1}{\omega}\sin\omega T\,\sin\omega X$ $x = \frac{1}{\omega}\cos\omega T\,\cos\omega X$	$\sqrt{2}\omega\,\frac{\sin 2\omega X}{(\cos 2\omega T-\cos 2\omega X)^{3/2}}$

$$\tan\theta = \frac{d_1d_2 - d_3d_4 + \sqrt{(d_1^2\,e^{-2\omega T} + d_4^2\,e^{2\omega T})(d_3^2\,e^{-2\omega T} + d_2^2\,e^{2\omega T})}}{d_1d_3\,e^{-2\omega T} + d_2d_4\,e^{2\omega T}} \tag{29}$$

$$\tan\phi = \pm\frac{d_1d_2 + d_3d_4 + \sqrt{(d_1^2\,e^{-2\omega T} + d_4^2\,e^{2\omega T})(d_3^2\,e^{-2\omega T} + d_2^2\,e^{2\omega T})}}{d_1d_3\,e^{-2\omega T} - d_2d_4\,e^{2\omega T}} \tag{30}$$

for the curves with T = constant and for the curves with X = constant by

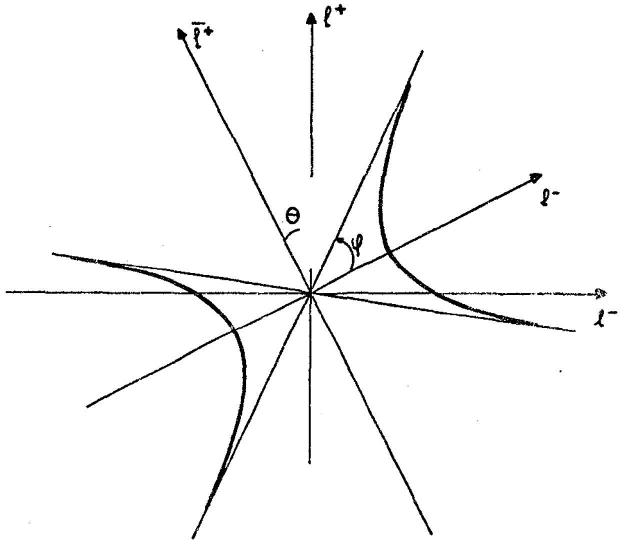

Fig.14 - The parameters θ and ϕ.

$$\tan\theta = \frac{d_1d_3 - d_2d_4 + \sqrt{(d_1^2\, e^{-2wX} + d_4^2\, e^{2wX})(d_3^2\, e^{-2wX} + d_2^2\, e^{2wX})}}{d_1d_4\, e^{-2wX} + d_2d_3\, e^{2wX}} \tag{31}$$

$$\tan\phi = \pm\frac{d_1d_2 + d_3d_4 + \sqrt{(d_1^2\, e^{-2wX} + d_3^2\, e^{2wX})(d_4^2\, e^{-2wX} + d_2^2\, e^{2wX})}}{d_1d_4\, e^{-2wX} - d_2d_3\, e^{2wX}} \tag{32}$$

There are minor differances between Kalnins'[4] results and ours. He does not mention the presence of the event-horizont at $\ell^+ = -2/w$ in system D, either in the text or in his figure 4. In systein A the hyperbolic axes coincide with the (t,x)-axes, as $\tan\theta = \pm 1$; theirasymptotes have the direction $\tan\phi = \pm\tanh wT$, $\tan\phi = \pm\text{ctanh}\, wX$, which is not clear from his figure 7.

We have shown here a very easy derivation of the separable orthogonal coordinates of Minkowski space. These were indeed known after the work of Kalnins and Miller[5], who however used atgebraic method and didn't discuss the physical significance of such coordinate systems. As we know, one of them originated a major evolution in quantum field theory in curved space and the others are adapted to very interesting situations which we intend to study with more detail in later papers.

I want to express my thanks to Professor H.K. Urbantke for his enthusiasm on the geometrical properties of this problem and to the whole Institut fuer Theoretische Physik of the University of Vienna for its hospitality during my stay there. I am also very glad with the atmosphere of work and friendship found here at Centro Brasileiro de Pesquisas Fisicas. This work was supported in part by CAPES-Education Ministery of Brasil, and CNPq-Science Ministery of Brasil.

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Resumo

Apresentamos uma derivação muito simples dos 10 sistemas de coordenadas ortogonais em que a equação de Klein-Gordon se separa. A derivação é .baseada na estrutura conforme do espaço de Minkowski bidimensional. Horizontes, tempo próprio e aceleração de observadores que seguem as linhas de coordenadas de tempo bem como outras propriedades físicas dos sistemas são deduzidos. É discutida a importância destas coordenadas, especialmente no contexto da teoria quântica de campos em espaços curvo.