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Loop Variables in Godel and Friedmann Universes

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Abstract The ordered integrals for several paths in Gödel and Friedmann gravitations are computed in a compact form. When the path is closed we calculate the corresponding Wilson Loop,

The loop variables, which in the theory of gravity are matrices representing parallel transport along contours in a space with a given affine connection, were first considered in gravitation by Mandelstam[1], who established some equations involving these variables.

Recently, Bollini, Giambiagi and Tiomno[2] computed the loop variables (path ordered integral) and the Wilson loop (for the case whenthe path is closed) for the configuration of the gravitational field corresponding to the Kerr metric[3].

In this paper, we perform the calculations of the loop variables (path ordered integrals) and the corresponding Wilson loop for the configurations of the gravitational fields corresponding to: (a) the Godel metric[4], which presents an unusual property of existence of closed time-like curves; and (b) the Friedmann (or Robertson-Walker) metric[5].

As the loop variables are related to the parallel displacement of a vector along a path, which in the gravitational case has a space-time geometrical meaning, we believe that it is important to perform these calculations, because they could also help in the understanding of a possible theory of gravity based on loop variables.

The loop variables, as we have pointed out, are the matrices representing parallel transport along contours which we shall denote by

$$U(C) = Pe^{\int_C \Gamma_\mu dx^\mu} \tag{1}$$

where P means ordered product along a curve C and Γ_μ is the tetradic connection. Taking the trace of eq. (1) for a closed curve, we obtain the well known Wilson loop.

Firstly we consider a gravitational field corresponding to the Godel solution described by the metric[4]

$$ds^2 = dt^2 - dr^2 - dz^2 + 2f(r)dtd\phi + g(r)d\phi^2 \tag{2}$$

where

$$g(r) = \sinh^4 r - \sinh^2 r \text{ and } f(r) = \sqrt{2}\sinh^2 r . \tag{3}$$

To compute the tetradic connection corresponding to Godel's solutions, we start by defining the one-forms θ^A $(A = 1,2,3,4)$

$$\begin{aligned} \theta^1 &= dr \\ \theta^2 &= h(r)d\phi \\ \theta^3 &= dz \\ \theta^4 &= dt + f(r)d\phi \end{aligned} \tag{4}$$

in which we have set $h(r) \equiv (f - g)^{1/2}$. The geometry given by eq. (2) is obtained from eq. (4) by the expression

$$ds^2 = \theta^A\theta^B\eta_{AB} = -(\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 + (\theta^4)^2 \tag{5}$$

where η_{AB} is the Minkowski tensor diag $(-,-,-,+)$.

The tetrad frame defined by $\theta^A = e_\alpha^{(A)}\, dx^\alpha$ is given by

$$e_1^{(1)} = 1 \;;\quad e_2^{(2)} = h(r) \;;\quad e_3^{(3)} = 1 \;;\quad e_2^{(4)} = f(r) \;;\quad e_4^{(4)} = 1. \tag{6}$$

A straighforward calculation gives the values of the $\Gamma_\mu\, dx^\mu$'s (or $\Gamma^A_{\mu B}\, dx^\mu$ where A and B are tetradic indices). The non-null $\Gamma^A_{\mu B}\, dx^\mu$ are

$$\Gamma^4_{\mu 1} dx^\mu = \frac{1}{2} f' d\phi = \Gamma^1_{\mu 4}\, dx^\mu$$

$$\Gamma^4_{\mu 2} dx^\mu \doteq -\frac{1}{2}\frac{f'}{h}\, dr = \Gamma^2_{\mu 4}\, dx^\mu$$

$$\Gamma^2_{\mu 1} dx^\mu = -\frac{1}{2}\frac{f'}{h}\, dt + \left(h' - \frac{1}{2}\frac{f'f}{h}\right) d\phi = -\Gamma^1_{\mu 2} dx^\mu \tag{7}$$

where the prime indicates derivative with respect to r.

We shall consider circles centered at the origin with fixed values of r, z and t. So, in this case

$$\Gamma^A_{\mu B} dx^\mu = \Gamma_\phi d\phi \tag{8}$$

where

$$\Gamma_\phi = \begin{bmatrix} 0 & -h' + \frac{1}{2}\frac{ff'}{h} & 0 & \frac{1}{2}f' \\ h' - \frac{1}{2}\frac{ff'}{h} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}f' & 0 & 0 & 0 \end{bmatrix} \tag{9}$$

As Γ_ϕ is independent of ϕ

$$U = P\, e^{\int_{\phi_1}^{\phi_2} \Gamma_\phi d\phi} = e^{\Gamma_\phi(\phi_2 - \phi_1)} \tag{10}$$

From eqs. (3) and (9) and the definition of $h(r)$ it is easy to see that

$$\Gamma^3_\phi = -(1 - 2\sinh^2 r - 2\sinh^4 r)\Gamma_\phi \equiv -L_r^2 \Gamma_\phi \tag{11}$$

(definition of L_r).

This relation implies that for a complete circle, eq. (10) becomes

$$= 1 + \frac{\Gamma_\phi}{L_r}\sin(2\pi L_r) + \frac{\Gamma^2_\phi}{L^2_r}\left[1 - \cos(2\pi L_r)\right] \tag{12}$$

Taking the trace of eq. (12), we get the corresponding Wilson Loop which is given by

$$\Gamma_r = -\sqrt{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} . \tag{16}$$

So, in this case

$$U = e^{\int_{r_1}^{r_2} \Gamma_r dr} = 1 - \sinh\left[\sqrt{2}(r_2-r_1)\right]\Gamma' - \{1 - \cosh\left[\sqrt{2}(r_2-r_1)\right]\}\Gamma'^2 \tag{17}$$

where

$$\Gamma' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} .$$

It is easy to see that a segment in the z-direction contributes with a unit factor.

Now, consider a translation in time

$$\Gamma_\mu dx^\mu = \Gamma_t dt \tag{18}$$

with Γ_t (from eq. (7)) given by

$$\Gamma, = \sqrt{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{19}$$

and

$$U(C) = 1 + \sin\left[\sqrt{2}(t_2 - t_1)\right]\Gamma'' + \{1 - \cos\left[\sqrt{2}(t_2 - t_1)\right]\}\Gamma''^2 \tag{20}$$

where

$$\Gamma'' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

$$W(C) = 2\left[1 + \cos(2\pi L_r)\right] \tag{13}$$

If we define a critical radius R_C such that $\sinh R_C = 1$ (or $R_C = \ell n(1+\sqrt{2})$), then for $R > R_C$ we have $g(r) = \sinh^4 r - \sinh^2 r > 0$. Thus, the circle defined by r = const. $> R_C$, t = const. and z = const., is a closed time-like curve. The existence of such curves makes possible the violation of causality. If we observe the definition of L_r^2 given by eq. (11), we conclude that for $r > \ell n(1+\sqrt{2}) = R_C$, $L_r^2 < 0$, and therefore, for such values of r the U matrix and the Wilson Loop contain hyperbolic functions rather than natural trigonometric functions. Then, eqs. (12) and (13) become

$$U = 1 + \frac{\Gamma_\phi}{L_r} \sinh(2\pi L_r) + \frac{\Gamma_\phi^2}{L_r^2}\left[1 - \cosh(2\pi L_r)\right] \tag{14}$$

and

$$W = 2\left[1 + \cosh(2\pi L_r)\right] \tag{15}$$

for $r > \ell n(1+\sqrt{2})$.

The region $r < \ell n(1+\sqrt{2})$ can be divided into two parts:

(a) $\left(\frac{-1+\sqrt{3}}{2}\right)^{1/2} < r < \ell n(1+\sqrt{2})$ and (b) $0 < r < \left(\frac{-1+\sqrt{3}}{2}\right)^{1/2}$.

In the first part, the U matrix and the Wilson Loop contain hyperbolic functions and in the second they contain natural trigonometric functions.

Therefore, the U matrix (transport operator) and the Wilson loop each have a unique expression in the region where the existence of closed time-like curves is allowed, and a double expression outside this region. Then, the U matrix and the Wilson Loop distinguish *partially* the two regions. Equations (12) and (14) are the exact expressions for the holonomy transformations for a circle for the Gödel solution.

Let us compute $U(C)$ corresponding to a radial segment. Frorn eq. (7) we obtain that Γ_r is independent of r and is given by

Using formulae (17) and (20) together with the fact that $P\exp(\Gamma_z dz) = 1$, one can compute more general loops. In particular for the *plaquette* $z_1 < z < z_2$ and $t_1 < t < t_2$ the result is

$$U(C) = 1 \tag{21}$$

From eq. (21) we get for the Wilson loop

$$W = 4$$

which is the *vacuum* (or flat) value in gravitation. The same result is obtained for the plaquette $r_1 < r < r_2$ and $z_1 < z < z_2$.

Now, consider the Friedmann metric which can be written, by a convenient choice of coordinates, in the following form

$$ds^2 = dt^2 - a^2(t)\left[d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{22}$$

where $d\chi^2 + \sigma^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)$ is the metric of a three-space of constant curvature and is independent of time. The geometry for this three-space is qualitatively different according to the value of the scalar curvature R, which can assume the values -1, 0 or +1. The function $\sigma(\chi)$ depends on the values of the curvature

$$\sigma(\chi) = \begin{cases} \sin\chi , & \text{if } R = +1 \\ \chi , & \text{if } R = 0 \\ \sinh\chi , & \text{if } R = -1 \end{cases} \tag{23}$$

The coordinate χ runs from 0 to ∞ if R = 0 or -1, but from 0 to 2π if $R = +1$.

Defining the one-forms θ^A (A = 1,2,3,4)

$$\begin{aligned} \theta^1 &= a(t)d\chi \\ \theta^2 &= a(t)\sigma(\chi)d\theta \\ \theta^3 &= a(t)\sigma(\chi)\sin\theta d\phi \\ \theta^4 &= dt \end{aligned} \tag{24}$$

the line element given by eq.(22) may be written as

$$ds^2 = \eta_{AB}\theta^A\theta^B = -(\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 + (\theta^4)^2 \tag{25}$$

where η_{AB} is the Minkowski metric tensor, namely,

$$\eta_{AB} = \mathrm{diag}(-,-,-,+) \ .$$

With definition eq. (24), the tetratic connection corresponding to the Friedmann metric is given explicitly by

$$\begin{aligned}
\Gamma^4_{\ \mu 2} dx^\mu &= \dot{a}(t)\sigma(\chi)d\theta = \Gamma^2_{\ \mu 4} dx^\mu \\
\Gamma^4_{\ \mu 3} dx^\mu &= \dot{a}(t)\sigma(\chi)\sin\theta d\phi = \Gamma^3_{\ \mu 4} dx^\mu \\
\Gamma^2_{\ \mu 1} dx^\mu &= \sigma'(\chi)d\theta = -\Gamma^1_{\ \mu 2} dx^\mu \\
\Gamma^3_{\ \mu 1} dx^\mu &= \sigma'(\chi)\sin\theta d\phi = -\Gamma^1_{\ \mu 3} dx^\mu \\
\Gamma^3_{\ \mu 2} dx^\mu &= \cos\theta d\phi = -\Gamma^2_{\ \mu 3} dx^\mu
\end{aligned} \tag{26}$$

where

$$\dot{a}(t) \equiv \frac{da(t)}{dt} \quad \text{and} \quad \sigma'(\chi) \equiv \frac{d\sigma(\chi)}{d\chi} \ .$$

We shall consider only circles with $\chi = t = $ const. and $\theta = \pi/2$. Then, in this case we have from eq.(26) that

$$\Gamma_\mu dx^\mu = \Gamma_\phi d\phi$$

where

$$\Gamma_\phi = \begin{bmatrix} 0 & 0 & -\sigma' & 0 \\ 0 & 0 & 0 & 0 \\ \sigma' & 0 & 0 & \dot{a}\sigma \\ 0 & 0 & \dot{a}\sigma & 0 \end{bmatrix} \tag{27}$$

From eq. (27) it is easy to see that

$$\Gamma^3_\phi = -(\sigma'^2 - \dot{a}^2\sigma^2)\Gamma_\phi \equiv -L^2_\sigma\Gamma_\phi \tag{28}$$

(definition of L_σ)

Using eq. (28) and the fact that Γ_ϕ is independent of ϕ, we obtain the following result for a complete circle,

$$U = 1 + \frac{\Gamma_\phi}{L_\sigma}\sin(2\pi L_\sigma) + \frac{\Gamma^2_\phi}{L^2_\sigma}\left[1 - \cos(2\pi L_\sigma)\right] \tag{29}$$

Eq.(29) is the exact expression for the holonomy transformation for this case..

Taking the trace we have for the Wilson loop,

$$W = 2\left[1 + \cos(2\pi L_\sigma)\right] \tag{30}$$

As we know, the density decreases as the universe expands, and conversely the density was higher in the past, increasing without bound as $a(t) \to 0$.

Consider now the following expressions for $a(t)$[5]:

$$\begin{aligned}
&a(t) = a_0(\cosh\tau - 1) \qquad && t = a_0(\sinh\tau - \tau), \qquad && \text{if} \quad R = -1 \\
&a(t) = \tau^2 && t = a_0^{3/2}\,\tau^3\,, && \text{if} \quad R = 0 \\
&a(t) = a_0(\cos\tau - 1) && t = a_0(\sin\tau - \tau), && \text{if} \quad R = 1
\end{aligned} \tag{31}$$

From the definition of L, (cf. eq. *(28))* and frorn eq. (31) we see that for $t \to 0$, $\dot{a}(t) \to 0$; consequently the U-matrix tends to the unitary matrix in 4-dimensions and the Wilson Loop tends to its vacuum value. Then, near the singularity the Wilson Loop does not distinguish the three different geometries.

If we consider only the part of the metric corresponding tothe three-space which has a constant curvature and is independent of time, we obtain the following expression for the U-matrix

$$U = 1 + \frac{\Gamma_\phi}{(\sigma')^2}\sin(2\pi L_\sigma) + \frac{\Gamma_\phi^2}{(\sigma')^2}\left[1 - \cos(2\pi L_\sigma)\right] \qquad (32)$$

where

$$\Gamma_\phi = \begin{bmatrix} 0 & 0 & -\sigma' \\ 0 & 0 & 0 \\ \sigma' & 0 & 0 \end{bmatrix} \qquad (33)$$

If we transport a vector parallel to the circle around this closed curve, it will, in general, not return to itself but will undergo rotation through a certain angle. Consider, then, that neither the original vector nor the transposed one has a temporal cornponent. In this case we can define the real angular deviations which are given by

$$\cos \alpha_A = U^A{}_A \qquad (34)$$

where A is a tetradic index.

The non-null angular deviations occur in our case for A=l and 3; $\sigma(\chi) = \sin\chi$, and $\sigma(\chi) = \sinh\chi$. For $\sigma(\chi) = \chi$, all the angular deviations are zero, which is consistent with the fact that the scalar curvature is zero in this case. Then, in the three dimensional case the parallel transport operator distinguishes between diferents solutionsas we can see from the distinct angular deviations. For $\sigma(\chi) = \chi$ we obtain the expect result, that is, all angular deviations are null

For the Friedrnann case we can compute $U(C)$ for a radial segment, for a pure temporal segment and for a curve in a meridian plane. In these cases the results are, respectively,

$$U_{\chi_1\chi_2}(C) = e^{\int_{\chi_1}^{\chi_2} \Gamma_\chi d\chi} = 1 \qquad (35)$$

$$U_{t_1 t_2}(C) = e^{\int_{t_1}^{t_2} \Gamma_t dt} = 1 \qquad (36)$$

$$U_{\theta_1\theta_2}(C) = 1 + \frac{\Gamma_\theta}{L_\sigma}\sin[(\theta_2-\theta_1)L_\sigma] + \frac{\Gamma_\theta^2}{L_\sigma^2}\{1-\cos[(\theta_2-\theta_1)L_\sigma]\} \qquad (37)$$

where L_σ is given by eq. (28) and

$$\Gamma_\theta = \begin{bmatrix} 0 & -\sigma' & 0 & 0 \\ \sigma' & 0 & 0 & \dot{a}\sigma \\ 0 & 0 & 0 & 0 \\ 0 & \dot{a}\sigma & 0 & 0 \end{bmatrix} \qquad (38)$$

Eq. (37) is the exact expression for the holonomy transformation for an arc of circle corresponding to the Friedmann's solution. With the use of eqs. (29), (35), (36) and (37), we can compute $U(C)$ for more general loops.

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Resumo

São calculadas integrais de trajetória ordenadas, em forma compacta, para várias trajetárias, para as soluções de Gödel e Friedrnann. Quando a trajetória é fechada, calculamos o "Loop" de Wilson correspondente.