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The Soliton Sector of the Quantum Field Theory Associated with the Two-Dimensional Ising Model

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Abstract We define an anti-periodic finite lattice Hamiltonian through the transfer matrix of the two-dimensional Ising model with anti-periodic boundary conditions in the space direction and periodic boundary conditions in the imaginary time direction. An infinite lattice quantum field theory is obtained by taking limits of vector states on the algebra of observables generated by finite products of spin operators. Explicit representations of spin and energy-momentum operators are obtained in terms of free Fermions acting in a Fermionic Fock space. The infinite lattice energy-momentum spectrum is analogous to the odd spectrum of a free, scalar, massive field theory: in particular, the vacuum and two-particle states are absent. The algebra of observables admits a decomposition into two subalgebras corresponding to the soliton, anti-soliton decomposition, Particular vector states show soliton behavior. The scaling limit is also obtained. An algebraic construction of the soliton sector is given and soliton field operators are defined. It is shown that the imaginary time soliton two-point function is the two-point function of the disorder operator introduced by Kadanoff and Ceva,

INTRODUCTION

In ref.1 we defined a quantum field theory associated with the ising model, the associated Schwinger functions being equal to the infinite lattice limit of the correlation functions with periodic boundary conditions in the horizontal (space) and vertical (imaginary time) directions of the lattice. The scaling limit was obtained in ref.2. Explicit representations were obtained for the energy, H, momentum, P, and space time zero field or spin operator σ_1^x. The infinite lattice construction is the natural limit of the finite lattice Feynmann-Kac (F-K) formula where the diagonalization of the finite lattice Hamiltonian as-particle state, the rest of the spectrum begins with three-particle states, In this limiting process the limit of the spectrum of the additively adjusted finite lattice a-p Hamiltonian is the same as that of the infinite lattice Hamiltonian.

Even though the finite periodic and a-p lattice Ising Hamiltonians are invarianf under spin flip the $M\to\infty$ limiting spectrum are different; they are the same for the quantum theory obtained from the correlation functions. We expect that similar phenomena will occur for other lattice Harniltonians. For example, if ϕ_i is the time-zero field operator at the point i of the $[-M,M]$ lattice one space dimensional sine-Gordon Hamiltonian then by imposing the boundary condition $\phi_{M+1}=\phi_{-M}+2\pi$ the $M\to\infty$ limiting spectrum should be that of the n-soliton sector. The vacuum sector of the lattice sine-Gordon model has been shown to be exactly soluble in ref.11. For more complicated non-scalar lattice field theories in 1,2 and 3 space dimensions different boundary conditions on the Harniltonian are expected to give different topological charged sectors.

Similar to the periodic case the energy-momentum and field operators are expressed in terms of two sets of Fermi operators obeying CAR; the two sets being related by a plct. The representation space is a Fermion Fock space and the state can be any vector is this Fock space. Similar to the periodic case tha algebra of observables admits a decomposition into two commuting subalgebras, By specializing to specific state vectors we obtain localized soliton and anti-soliton type states.

We establish the connection between the dual algebra approach to the soliton sector (see ref.8) and our anti-periodic constructions functions, infinite lattice representations for the spin and disorder operator in the pure phase of the $M\to\infty$ limit of the $[-M,M]$ periodic lattice model.

Furthermore we give an algebraic construction of the soliton sector which is an imaginary time version of the algebraic automorphism construction of soliton sectors in two-dimensional continuum field theory models as given in refs.6,7. Imaginary time is technically simpler as we exploit a finite propagation speed property present in this case, In our construction we introduce soliton field operators and show that the sociated with the transfer matrix was achieved by LMS[3] in terms of two sets of Fermions related by a linear canonical transformation.

For $T > T_c$, T_c the critical temperatures, the energy-momentum spectrum is the lattice analog of a free scalar massive field theory. For $T < T_c$ the energy-momentum spectrum shows a doubly-degenerate space-time translationally-invariant ground state and a continuum starting strictly above the ground state. The analog of the single particle hyperbola is missing. This is not a phenomena special to the Ising model model expected for $a\phi^4 + b\phi^2$ interacting two-dimensional scalar relativistic field theory models with $a > 0$ and $b << 0$. Numerical studies by[4] of the energy spectrum of a discrete approximation to the Hamiltonian on the line indicate similar behavior for periodic boundary conditions. However, with Diriclet boundary conditions a spectrum anaiogous to a free particle is obtained but with an almost degenerate ground state. This is clearly seen in by taking only a 16 point approximation to the line. This spectrum is expected to be the approximate spectrum of the vacuum and the soliton sector.

Consider the Ising model defined on'a $[-M, M]$ $[1, N]$ lattice (where the lattice extends in the space direction from $-M$ to M) with anti-ferromagnetic coupling between $-M$ and M and ferromagnetic between 1 and N. We call this the anti-periodic Ising model. We define a finite lattice anti-periodic (a.p) Hamiltonian through the transfer matrix of this model. This Hamiltonian is diagonalized by the LMS[3] procedure and in the $N \to \infty$ limit a finite lattice F-K formula can be obtained for the anti-periodic correlation functions (c.f) in terms of the a.p. Hamiltonian and its unique ground state eigenvector. We expect that the limit followed by the $M \to \infty$ limit of the c.f. will yield the same c.f. of the periodic Ising model. Thus the resulting quantim field theory obtained from a GNS construction or lattice O-S construction[5] will be the periodic one.

In this article we define, for $T < T_c$, an infinite lattice quantum theory by taking the limit of finite lattice vector states. The observables are taken as finite products of Heisenberg lattice spin operators. The energy-momentum spectrum of the limiting theory has no space-time translationally invariant state and shows an isolated one soliton two-point function is the two-point function of the disorder operator introduced by Kadanoff and Ceva[10].

We describe the organization of this paper. Section 1 develops the finite lattice theory; sections 2, 3 and 4, the infinite lattice theory including the decomposition of the algebra of observables. Also specific state vectors are introduced which exhibit soliton behavior.

In section 5 we obtain the correlation functions and representations for the order and disorder operators of the finite $[-M, M]$ lattice periodic model, their infinite lattice limits and in section 6 define the scaling limit theory. In section 7 we give an albebraic construction of the soliton sector.

In an appendix we give a pedagogical discussion of the 'zero temperature' ground state spectrum of the periodic and a.p. Hamiltonians.

1. THE ANTI-PERIODIC ISING MODEL

We will use the same definitions and notations of ref.1 when no confusion can arise, Consider a two-dimensional lattice which extends from $-M$ to M in the horizontal (space) direction and from 1 to N in the vertical (imaginary time) direction. We define the anti-periodic Ising model in a manner similar to the usual periodic model but which antiferromagnetic coupling between the spins $\sigma(n,-M)$ and $\sigma(n,M)$ for $n =$ $= 1,2,\ldots N$. Thus the partition function is

$$Z_{N,M} = \sum_{\{\sigma\}} \exp\left[\sum_{m=-M}^{M}\left(\sum_{n=1}^{N-1} K\,\sigma(n,m)\sigma(n+1,m)+K\sigma(N,n)\sigma(1,n)\right)\right.$$

$$\left.+\sum_{n=1}^{N}\left(\sum_{m=-M}^{M-1} K\sigma(n,m)\sigma(n,m+1) - K\sigma(n,M)\sigma(n,-M)\right)\right] .$$

Spin operators, $\tau_m^{\pm}$, τ_m^{x}, τ_m^{y}, τ_m^{z} are introduced analogous to and act on the 2^{2M+1} dimensional Hilbert space

$$H_M = \prod_{m=-M}^{M} \otimes H_m$$

In terms of these operators the partition function can be written as the trace of an operator which we state as

Thm.1.1. $Z_{N,M} = \mathrm{Tr}(V_2^{1/2}\ V_1 V_2^{1/2}) = \mathrm{Tr}\ V^N$,

where $V = V_1 V_2$,

$$V_1 = (2\sinh 2K)^{(2M+1)/2} \exp\left\{-2K^* \sum_{m=-M}^{M} (\tau_m^+ \tau_m^- - \tfrac{1}{2})\right\}$$

and

$$V_2 = \exp\left\{K \sum_{m=-M}^{M} (\tau_m^+ + \tau_m^-)(\tau_{m+1}^+ + \tau_{m+1}^-)\right\} ,$$

with $\tau_{M+1}^{\pm} = -\tau_{-M}^{\pm}$ and $\tanh K^* = e^{-2K}$,

Let $B_M = V_2^{1/2} V_1 V_2^{1/2}$, which is self adjoint. By suitable transformations, B_M can be written as $B_M = (2\sin 2K)^{(2M+1)/2} \cdot e^{-H_M}$; H_M is identified as the finite volume a-p Hamiltonian. The diagonalization of H_M is achieved by the LMS procedure which involves making the Jordan-Wignar transformation

$$c_m = \exp(\pi i \sum_{j=-M}^{m-1} \tau_j^+ \tau_j^-)\ \tau_m^-,\ -M < m \leqq M;\ c_{-M} = \tau_{-M}^-$$

Thus

$$\tau_m^x = (\exp(\pi i \sum_{j=-M}^{m-1} c_j^* c_j))c_m^x = c_m^x(\exp(i\pi \sum_{j=-M}^{m-1} c_j^* c_j)) .$$

where $c_m^x = c_m + c_m^*$. The $\{c_m\}$ obey canonical anti-commutation relations. The vector

$$\Omega = \begin{vmatrix} 0 \\ 1 \end{vmatrix} x \ldots x \begin{vmatrix} 0 \\ 1 \end{vmatrix} \in H_M$$

with $2M+1$ factors has the property $c_m \Omega = 0$ for all $-M \leqq m \leqq M$, $H_e(0)$ denotes the subspace generated by vectors with an even (odd) number of c_m's applied to R.

We define a linear canonical transformation by

$$c_m = (2M+1)^{-1/2}\ \alpha \sum_q e^{iqm}\ \eta_q,\quad \alpha = e^{-i\pi/4}\ ,\ -M = m = M,\ q \in A_e \text{ or } q \in A_0$$

but not both, where $A_{e(0)}$ are the $2M+1$ element sets is $2(2M+1)$ - fold degenerate. The first excited state lies $2g$ above or at E_0+3g. For T finite, $T<T_c$ and $M\to\infty$ the $2(2M+1)$ degenerate ground state becomes rhe analog of the one-particle hyperbola,

The sets $\{\xi_\ell\}$ and $\{\xi_k\}$ are not independent, they are related by the linear canonical transformation

$$\xi_\ell = (2M+1)^{-1} \sum_{k\in A_0} 2\, e^{-i(k-\ell)M}\,(1-e^{i(k-\ell)})^{-1}\,.$$
$$[\cos(\phi_\ell-\phi_k)\xi_k + \sin(\phi_\ell-\phi_k)\xi^*_{-k}]\,,\quad \ell\in A_e\quad .$$

The reverse transformation is

$$\xi_k = (2M+1)^{-1} \sum_{\ell\in A_e} 2e^{-i(\ell-k)M}\,(1-e^{i(\ell-k)})^{-1}$$
$$[\cos(\phi_k-\phi_\ell)\xi_\ell + (1-2\delta_{0\ell})\sin(\phi_k-\phi_\ell)\xi^*_{-\ell}]\,,\quad k\in A_0\,.$$

The vacuum vectors

$$\psi_{A_e} = \eta_0^* \prod_{\ell\in A_0^+} (\cos\phi_\ell + \sin\phi_\ell\,\eta^*_{-\ell}\,\eta^*_\ell)\,\Omega\in H_0$$

$$\psi_{A_0} = \prod_{K\in A_0^+} (\cos\phi_k + \sin\phi_k\,\eta^*_{-k}\,\eta^*_k)\,\Omega\in H_e$$

satisfy $\varepsilon_\ell\,\psi_{A_e}=0$ for all $\ell\in A_e$ and $\xi_k\,\psi_{A_0}=0$ for all $k\in A_0$. $A^+_{e(0)}$ are the positive elements (>0 and $<\pi$) of $A_{e(0)}$. We emphasize ihat in contrast to the periodic case neither of the vacuum vectors are eigenvectors of the Hamiltonian H_M. By the Perron-Frobenius theorem the eigenspace associated with the lowest eigenvalue of H_M is one-dimensional and it is found that the associated eigenvector is $\xi_0^*\,\psi_{A_e}\in H_e$.

Analogous to the periodic momentum operator we define e^{iP_M}, P_M self-adjoint, by the properties

$$e^{iP_M}\,\Omega=\Omega\,,$$

$$e^{iP_M}\,\tau^x_m\,e^{-iP_M} = \tau^x_{m-1}\,,\quad -M<m\leq M\,,$$
$$e^{iP_M}\,\tau^x_{-M}\,e^{-iP_M} = -\,\tau^x_M\quad .$$

$$A_e = \{q \equiv \ell \,|\, 0,\ \pm 2\pi/(2M+1),\ \ldots\ \pm 2M\pi/(2M+1)\}$$

$$A_0 = \{q \equiv k \,|\, \pm \pi/(2M+1),\ \pm 3\pi/(2M+1),\ldots,\ \pm(2M-1)\pi/(2M+1),\pi\}\ .$$

Throughout this section the letter $\ell(k)$ will refer to elements of A_e (A_0).

From now on we consider $T < T_c$ only. A Bogoluibov-Valatin transformation is defined by

$$\varepsilon_q = \cos\phi_q\, \eta_q + \sin\phi_q\, \eta^*_{-q}\ ,$$

$q \neq 0$, $\xi_0 = \eta^*_0$, and the same convention of ref.1 for ϕ_q, $\Theta(z)$ and ε_q is maintained. We find that

$$c_m + c^*_m = (2M+1)^{-1/2}\ (\alpha \sum_q e^{iqm}\, e^{i\phi q}\, \xi_q + \bar{\alpha} \sum_q e^{-iqm}\, e^{-i\phi q}\, \xi^*_q)$$

where $q \in A_e$ or A_0, but not both.

In terms of the ξ_q operators we have

Thm.1.3. $H_M = H^+_M \oplus H^-_M$, $H^{-(+)}_M = H_M \restriction H_{e(0)}$

$$H^-_M \restriction H_e = \sum_{\ell \in A_e} \varepsilon(\ell)\,(\xi^*_\ell\, \xi_\ell - \tfrac{1}{2})$$

$$H^+_M \restriction H_0 = \sum_{k \in A_0} \varepsilon(k)\,(\xi^*_k\, \xi_k - \tfrac{1}{2})\ .$$

We define a renormalized H_M by $\hat{H}_M = H_M + \frac{1}{2} \sum_{\ell \in A_e} \varepsilon(\ell)$ so that inf sp $\hat{H}_M = \varepsilon(0)$. For large M the structure of the energy spectrum of $\hat{H}_M$ is quite different than the periodic spectrum. This difference can be understood qualitatively by considering the 'zero temperature' Hamiltonian

$$V_0 = -g \sum_{m=-M}^{M} \tau^x_m\, \tau^x_{m+1}$$

(see the appendix). With periodic b.c. ($\tau^x_{M+1} = \tau^x_{-M}$) the ground state E_0 is doubly degenerate (all up, all down) and the first excited state lies $2g$ above E_0. With a-p b.c ($\tau^x_{M+1} = -\tau^x_{-M}$) the ground state is at E_0+g and

Then

$$\tau_m^x = e^{-iP_M(M+m)}\ \tau_{-M}^x\ e^{iP_M(M+m)}$$

and by an analysis similar to that of[9] we have

Thm.1.4 $P_M = P_M^+ \oplus P_M^-$, $P_M^{-(+)} = P_M \upharpoonright H_{e(0)}$

$$P_M^- = \sum_{\ell \epsilon A_e} \ell\ \xi_\ell^*\ \xi_\ell\ ,$$

$$P_M^+ = \sum_{k \epsilon A_0} k\ \xi_k^*\ \xi_k\ .$$

The finite volume correlation functions are given by

Thm.1.5. If $1 \leqq n_1 \leqq n_2 = \ldots n_k \leqq N$, then

$$<\sigma(n_1,m_1)\ldots\ \sigma(n_k,m_k)>_{N,M} =$$

$$= \frac{\mathrm{Tr}(e^{H_M n_1}\ \tau_m^x\ e^{-H_M(n_2-n_1)} \ldots\ \tau_{m_k}^x\ e^{-H_M(N-n_k)})}{e^{-H_M N}}$$

Letting N $\to \infty$ we obtain the finite volume F-K formula Corollary 1.5.1. If $1 \leqq n_1 \leqq ,\ldots n_k$ then

$$<\sigma(n_1,m_1)\ \ldots\ \sigma(n_k,m_k)>_M\ =$$

$$= (\xi_0^*\ \psi_{A_e},\ \prod_{j=1}^{k-1} \tau_{mj}^x\ e^{-\hat{H}_M(n_{j+1}\ -\ n_j)}\ \tau_{m_k}^x\ \xi_0^*\ \ \psi_{A_e})$$

$$= (\xi_0^*\ \ \psi_{A_e},\ \prod_{j=1}^{k-1}\ \tau_{-M}^x\ \ e^{-\hat{H}_M(n_{j+1}-j_n)}\ e^{-iP_M(m_{j+1}-m_j)}\ \tau_{-M}^x\ \xi_0^*\ \psi_{A_e})\ .$$

In order to obtain the soliton sector in the M $\to \infty$ limit we will not be interested in the limit of the correlation functions of Thm.1.5 but in a state defined on an algebra of observables. For M finite and $f_n, g_n \epsilon S(R^n)$

an arbitrary vector in H_M can be written

$$\psi_M = \sum_{\substack{\ell_1,\dots,\ell_m \\ m \text{ odd}}} (2M+1)^{-m/2}\, g_m(\ell_1,\dots,\ell_m)\; \xi^*_{\ell_1} \cdots \xi^*_{\ell_m}\, \psi_{A_e}$$

$$+ \sum_{\substack{k_1,\dots,k_n \\ n \text{ odd}}} (2M+1)^{-n/2}\, f_n(k_1,\dots,k_n)\; \xi^*_{k} \cdots \xi^*_{k_n}\, \psi_{A_0}$$

and we define a vector state by the expectations

$$(\psi_M \prod_{i=1}^{k} e^{iH_M n_i}\, \tau_{m_i}\, e^{-iH_M n_i}\, \psi_M) \tag{1.1}$$

The above will be used to define the soliton sector in the $M\to\infty$ limit. The contribution of $\xi_0\, \psi_{A_e}$ to the $M\to\infty$ in (1.1) will be zero.

In addition to the operators $\{\tau^x_m\}$ we also introduce the so-called lattice disorder operator

$$\mu_m = e^{i\pi \sum_{j=-M}^{m-1} c^*_j c_j} = (c_n + c^*_m)\, \tau^x_m = \tau^x_m (c_m + c^*_m) \tag{1.2}$$

for $-M < m \leq M$. These operators satisfy the relations, for $-M < m, n \leq M$, $|\mu_m,\mu_n| = \delta_{nm}$

$$\tau^x_m\, \mu_n = \begin{cases} \mu_n\, \tau^x_m, & n < m \\ -\mu_n\, \tau^x_m, & n > m \end{cases} \tag{1.3}$$

$$\begin{pmatrix} c_n \\ c^*_n \end{pmatrix} \mu_m = \pm\, \mu_m \begin{pmatrix} c_n \\ c^*_n \end{pmatrix}, \quad n \gtrless m\,. \tag{1.4}$$

In addition (1.3) is satisfied with $c_n + c^*_n$ replacing μ_n.

2. INFINITE LATTICE OPERATORS AND THEIR PROPERTIES

Here we introduce operators and vectors which will be used in the infinite lattice theory. Since many of the objects are the same as the periodic case we refer the reader to section 2 of ref.1. In what follows it is helpful to keep in mind the following formal correspondences:

$$\psi_{A_0} \to \hat{\psi} \ , \quad \psi_{A_e} \to \tilde{\psi}$$

$$(2M+1)^{1/2}\ e^{-ikM}\ \xi_k \to \hat{\xi}(q)/\sqrt{2\pi}\ ,$$

$$(2M+1)^{1/2}\ e^{-i\ell M}\ \xi_\ell \to \tilde{\xi}(q)/\sqrt{2\pi}\ ,$$

$$(2M+1)^{-1} \sum_q \cdot \to (2\pi)^{-1} \int_{-\pi}^{\pi} dq.$$

We define in H, the fermion Fock space over $L^2(-\pi,\pi)$, $\hat{\psi} \in \hat{H}_e$, $\tilde{\psi} = U\hat{\psi} \in \hat{H}_0 = \tilde{H}_e$, where $\tilde{\xi}(q) = U\ \hat{\xi}(q)\ U^{-1}$ and U is the unitary which implements the linear canonical transformation $\tilde{\xi}(f) = \hat{\xi}(T_1\ f) + \hat{\xi}^*(T_2 f)$, $f \to L^2$. We define energy-momentum operators by $H(P) = \hat{H} \oplus \tilde{H}\ (\hat{P} \oplus \tilde{P})$ where

$$H(P)\ |\ (\hat{H}_0=\tilde{H}_e) = \int_{-\pi}^{\pi} \varepsilon(q)\quad (q)\xi^*(q)\hat{\xi}(q)\ dq \equiv \hat{H}(\hat{P}) \tag{2.1}$$

$$H(P)\ |\ (\tilde{H}_0=\hat{H}_e) = \int_{-\pi}^{\pi} \varepsilon(q)\quad (q)\xi^*(q)\tilde{\xi}(q)\ dq \equiv \tilde{H}(\tilde{P})\ .$$

Note that the vacuum vectors $\hat{\psi}$ and $\tilde{\psi}$ are ***not*** eigenvectors of H, P and that the spectrum of H,P has the structure of the lattice analog of the spectrum of a free massive scalar field theory with only an odd number of particles. In particular the vacuum and two-particle states are absent.

We define the time ***zero*** lattice field or spin operators by

$$\sigma_m^x = e^{-iPm}\ \sigma_0\ e^{iPm}$$

where

$$\sigma_0 = \frac{1}{\sqrt{2\pi}}\left(\alpha \int_{-\pi}^{\pi} e^{i\phi(q)}\ \tilde{\xi}(q)dq + \bar{\alpha}\int_{-\pi}^{\pi} e^{-i\phi(q)}\ \tilde{\xi}^*(q)dq\right)\ .$$

Furthermore a dense set of vectors in H is obtained from vectors of the form

$$\psi = \sum_{n \text{ odd}} \int dk_1 \ldots dk_n \; f_n(k_1,\ldots,k_n)\hat{\xi}^*(k_1)\ldots\hat{\xi}^*(k_n)\hat{\psi}$$
$$+ \sum_{m \text{ odd}} \int dk_1 \ldots dk_m \; g_m(k_1,\ldots,k_m)\tilde{\xi}^*(k_1)\ldots\tilde{\xi}^*(k_m)\tilde{\psi} \qquad (2.2)$$

where f_n, $g_n \in \mathcal{E}(R^n)$, the k integrals extending over $[-\pi,\pi]$.

3. INFINITE LATTICE QUANTUM FIELD THEORY

Conventionally, one defines an infinite lattice theory by taking the infinite lattice limit of correlation functions (c.f), i.e. the expectation of a product of spins at a finite number of points to obtain space-imaginary time translationally invariant c.f. The field theory is reconstructed from these c.f, through a lattice version of the O-S construction[5]. If one carries out this procedure for the a-p Ising model one expects on intuitive physical grounds to obtain the periodic c.f. or Schwinger functions of Thm. 3.1 of ref.1. Considering the finite W representation of the Schwinger c.f. of Corollary 1.5.1 and taking into account the operators defined in section 2 we formally have, by considering matrix elements of the spin operators after substituting complete sets of energy-momentum eivenvectors,

$$\lim \; \langle\sigma(n_1,m_1)\ldots\sigma(n_k,m_k)\rangle_M =$$
$$= \lim_{\varepsilon\to 0} (\psi_\varepsilon, \prod_{j=1}^{k-1} \sigma_0 \, e^{-H(n_j-n_{j-1})} \, e^{-iP(m_j-m_{j-1})} \, \sigma_0 \;\; \psi_\varepsilon)$$

where

$$\psi_\varepsilon = \tilde{\xi}^*(h_\varepsilon)\;\tilde{\psi}\;,\; |h_\varepsilon| = 1,\; \text{supp}\; h_\varepsilon \in [0,\varepsilon],\;\; h_\varepsilon > 0$$

so that $|\psi_\varepsilon| = 1$ (formally the $M\to\infty$ limit of $\xi_0^*\,\psi_{A_e}$ is $\lim_{\varepsilon\to 0} \tilde{\xi}^*(h_\varepsilon)\tilde{\psi})$. It is not obvious that the limit on the right side exists or that it is equal to the periodic c.f. If the periodic c.f.'s are obtained in this way than the spectrum of che resulting field theory will be different (even after overall additive changes) than the limit of the spectrum of the finite lattice a.p. Hamiltonian. In particular there will be no analog of the single particle hyperbola.

Onr construction of the infinite lattice theory will follow different lines. The representation space for our infinite latticetheory will be H and for any vector in H we define expectations of finite products of spin operators as limit of the analogous finite lattice objects. In the representation space H the energy-momentum and field operators are those defined in section 2 and neither the vacuum vector $\hat{\psi}$ nor $\tilde{\psi}$ are eigenvectors of the energy-rnomentum operators. The limit of the spectrum of the finite lattice a.p. Hamiltonian is equal tothe spectrum of the infinite lattice theory.

With ψ and ψ as in (1.1) and (2.2) we have

Thm. 3.1.

$$\lim_{M\to\infty} (\psi_M, \prod_{i=1}^{S} \tau^x_{m_i} \psi_M) = (\psi, \prod_{i=1}^{S} \sigma^s_{m_i} \psi) .$$

Furthermore, for the limit of the expectations of the state vector $\mu_m \psi_{A_e}$ we have

$$\lim_{M\to\infty} (\mu_m \psi_{A_e}, (\prod_{i=1}^{S} \tau^x_{n_i}) \mu_m \psi_{A_e}) =$$

$$= (e^{-iP_m} \tilde{\psi} \prod_{i=1}^{S} (e^{-iP_{n_i}} \sigma_0 e^{iP_{n_i}}) e^{-iP_m} \tilde{\psi}) =$$

$$= (\hat{d}_m \hat{\psi}, \prod_{i=1}^{S} \hat{s}_{n_i} \hat{d}_m \hat{\psi}) \tag{3.1}$$

where

$$\hat{d}_m = e^{-iP_m} U e^{i\hat{P}_m} \quad \text{and} \quad \hat{s}_n = e^{-i\hat{P}_n} \sigma_0 U e^{i\hat{P}_n} .$$

Prf. of Thm. 3.1: We give the proof for $s = 2$. By making the formal correspondence $(2M+1)^{1/2}\, e^{-iqM}\, \xi_q$ with $\tilde{\xi}(q)$ or $\tilde{\tilde{\xi}}(q)$ we show that all $e^{\pm iqM}$ factors are properly accounted for. By writing

$$(\psi_M, \tau_{m_1}\tau_{m_2}\psi_M) = ((2M+1)^{-m/2} \sum_{\{\ell_m\}} g_m\, \xi^*_{\ell_1}\ldots\xi^*_{\ell_m}\, \psi_{A_e}$$

$$+ (2M+1)^{-n/2} \sum_{\{k_n\}} f_n\, \xi_{k_1}\ldots\xi_{k_n}\, \psi_{A_0}\,,\ \underline{e^{-iP_M(M+m_1)}}\ \tau_{-M}$$

$$\underline{\underline{e^{iP_M(M+m_1)}\ e^{-iP_M(M+m_2)}}}\ \tau_{-M}\ \underline{e^{iP_M(M+m_2)}}$$

$$\{(2M+1)^{-m\ 1/2} \sum_{\{\ell' m'\}} g_{m'}\, \xi^*_{\ell'_1}\ldots\xi^*_{\ell'_{m'}}\, \psi_{A_e} +$$

$$+ (2M+1)^{-n\ 1/2} \sum_{\{k' n'\}} f_{n'}\, \xi^*_{k'_1}\ldots\xi^*_{k'_{n'}}\, \psi_{A_0}\})$$

the single-underlined $e^{\pm iP_M m}$ factors are absorbed by the $\{\xi^*_k\}$ and $\{\xi^*_\ell\}$. In the double underlined terms M of course cancels. To carry out be $M\to\infty$ limit proceed by the method of integral equations as in[1].

Prf, of Thm.3.3. Consider (note that $\mu_m = (c_m+c^*_m)\tau^x_m = \tau^x_m(c_m+c^*_m))$

$$\mu_m\,\psi_{A_e} = e^{-iP_M(M+m)}\,\tau^x_{-M}\, e^{iP_M(M+m)}\,(c_m+c^*_m)\,\psi_{A_e}$$

where p_M is replaced by $\bar{P}_M$. Express $c_m+c^*_m$ in terms of ξ_ℓ and ξ^*_ℓ. Noting that

$$\xi_\ell\,\psi_{A_e} = 0 \quad\text{and}\quad e^{i\bar{P}_M(M+m)^m}\,.(c_m + c^*_m)\, e^{-i\bar{P}_M(M-m)} = \tau^{x2}_{-M} = 1\,,$$

so that

$$\mu_m\,\psi_{A_e} = e^{-iP_M(M+m)}\,\psi_{A_e}$$

$$(\mu_m \psi_{A_e}, \prod_{i=1}^{S} \tau_{n_i} \mu_m \psi_{A_e}) =$$

$$= (e^{-iP_M m} \psi_{A_e}, \prod_{i=1}^{S} (e^{-iP_M n_i} \tau_{-M} e^{iP_M n_i}) e^{-iP_M m} \psi_{A_e})$$

and all e^{-iqM} factors are properly accounted for, The $M\to\infty$ limit is

$$(e^{-iPm} \tilde{\psi}, \prod_{i=1}^{S} (e^{-iPn_i} \sigma_0 e^{iPn_i}) e^{-iPm} \tilde{\psi}) =$$

$$= (e^{-i\hat{P}m} U \hat{\psi}, (\prod_{i=1}^{S} e^{-iPn_i} \sigma_0 e^{iPn_i}) e^{-i\hat{P}_m} U \hat{\psi}) .$$

Using $e^{i\tilde{P}n} = U e^{i\hat{P}n} U^{-1}$ in the above gives

$$(e^{i\hat{P}(n_1-m)} U \hat{\psi}, U e^{i\hat{P}(n_1-n_2)} \ldots \sigma_0 U e^{i\hat{P}(n_S-m)} U \hat{\psi}).$$

Noting that $e^{i\hat{P}m} U \hat{\psi} = e^{i\hat{P}m} U e^{-i\hat{P}m} \hat{\psi}$ the result follows.

4. DECOMPOSITION OF THE ALGEBRA OF OBSERVABLES

Similar to the periodic case we have the decomposition of the algebra of observables given by

Thm.4.1. $[\sigma_0, U] = [e^{imP}, U] = [e^{imH}, U] = 0$ and $P_{\mp} = (I\pm U)/2$ are mutually orthogonal self-adjoint projections. Thus if O is any finite products of σ_0,

$$\{ e^{in_j H} \} , \{ e^{im_k P} \}$$

then

$$O_\psi = O_{\psi+} + O_{\$-}$$

where $O_\psi = (\psi, O\psi)$ and $O_{\psi^\pm} = (\psi, P_\pm O P_\pm \psi)$.

We omit the proof which parallels that of Thm.4.1 of , but with H and P defined as in section 2 of this paper.

In particular we have for the decomposition of (3.1) of Thm.3.2.

Corollary 4.1.

a) $\left(e^{-iPm}\,\tilde{\psi},\; P_{\pm} \prod_{i=1}^{S} \left(e^{-iPn_i}\,\sigma_0\, e^{iPn_i}\right) P_{\pm}\, e^{-iPm}\,\tilde{\ }\right)$

$= \left(\hat{d}_m\,\hat{\psi},\; \prod_{i=1}^{S} \hat{s}_{n_i}\,\hat{d}_m\,\hat{\psi}\right)$ for s even.

b) same for s odd,

5. INFINITE LATTICE REPRESENTATIONS OF THE ORDER AND DISORDER OPERATORS

In this section we obtain, for $T < T_c$, an infinite lattice representation for the spin and disorder operators of section 1 (see eq. (1.2-4)). We obtain these representations in terms of the $M\to\infty$ limit of the operators associated with the finite lattice periodic Ising model from $-M \leqslant m \leqslant M$. In ref.1 the infinite lattice operators are constructed for the $M\to\infty$ limit of a lattice from $1 \leqslant m \leqslant M$. As an analogous diagonalization process of the finite lattice transfer matrix and construction of the infinite lattice theory holds for the $|-M,M|$ case we only give the definitions and results which parallel those of section 1 of[1]; the infinite lattice objects are those of[1].

We define

$$\tau^x_{-M} = c^x_{-M} = c_{-M} + c^*_{-M}$$

$$\tau^x_m = \left|\exp\left(\pi i \sum_{j=-M}^{m-1} c^*_j\, c_j\right)\right| c^x_m \;,\quad c^x_m = c_m + c^*_m \;,\quad -M < m \leqslant M$$

$$\tau^{\pm}_{M+1} = \tau^{\pm}_{-M}\;.$$

Furthermore we have $V = V_2^{1/2}\, V_1\, V_2^{1/2}$, $V = V^+ \oplus V^-$,

$$V^{+(-)} = V \upharpoonright H_{e(0)},\; c_{M+1} = \pm c_{-M},\; c^*_{M+1} = \pm c^*_{-M}$$

where

$c_m = (2m+1)^{-1/2}\, a \sum_q e^{iqm}\, \eta_q$, $a = e^{-i\pi/4}$, $-M \leq m \leq M$. For $c_{M+1} = -c_{-M}$,

$$S^+ = \{q \equiv \ell = \frac{\pi(2s+1)}{2M+1} \Big| \pm \frac{\pi}{2M+1}, \pm \frac{3\pi}{2M+1}, \ldots \frac{(2M-1)\pi}{2M+1}, \pi\},$$

for $c_{M+1} = c_{-M}$,

$$S^- = \{q \equiv k = \frac{2\pi s}{2M+1} \,|0, \pm \frac{2}{2M+1}, \pm \frac{4\pi}{2M+1}, \ldots \pm \frac{2M}{2M+1}\} .$$

The Bogolubov - Valatin transformation is

$$\xi_q = \cos\phi_q\, \eta_q + \sin\phi_q\, \eta^*_{-q} ,$$

$$\xi^x_{-q} = \cos\phi_q\, \eta^*_{-q} - \sin\phi_q\, \eta_q ,$$

in terms of which

$$c_m + c^*_m = (2M+1)^{-1/2} \sum_q e^{iqm}\, e^{i\phi q}\, \xi_q + \text{h.c.}) , \; a = e^{-i\pi/4},$$

$$\tau^x_{-M} = (2M+1)^{-1/2} (\alpha \sum_q e^{-iqM}\, e^{i\phi q}\, \xi_q + \text{h.c.}) .$$

We have

Thm.5.1. Let $B_M = V_2^{1/2}\, V \;\; V \qquad = (2\sinh 2K)^{(2M+1)/2}\, e^{-H_M}$. Then $H_M = H^+_M \oplus H^-_M$, $H^{+(-)}_M = H_M \restriction H_{e(0)}$, where $H^+_M = \sum_{\ell\epsilon S} \varepsilon_\ell (\xi^*_\ell\, \xi_\ell - \frac{1}{2})$ and $H^-_M = \sum_{k\epsilon S^-} \varepsilon_k (\xi^*_k\, \xi_k - \frac{1}{2})$ and similarly for P_M.

The linear relation between ξ_ℓ and ξ_k is as in the anti-periodic case from $|-M,M|$, i.e. for $k \in S^-$,

$$\xi_k = \frac{2}{2M+1} \sum_{\ell\epsilon S^+} \{ \frac{e^{-i(\ell-k)M}}{1-e^{i(\ell-k)}} |\cos(\phi_k-\phi_\ell)\xi_\ell + \sin(\phi_k-\phi_\ell)\xi^*_{-\ell}|\}.$$

The vacuum vectors are $\psi_{S^+} \in H_e$, $\xi_\ell\, \psi_{S^+} = 0$ and $\psi_{S^-} \in H_0$, $\xi_k\, \psi_{S^-} = 0$, and ψ_{S^+} is the eigenvector associated to the lowest eigenvalue of H_M. The **momentum** operator P_M obeys

$$\tau_m^x = e^{-iP_M(M+m)}\,\tau_{-M}^x\, e^{iP_M(M+m)} \ , \quad e^{iP_M}\,\Omega = \Omega$$

$$e^{iP_M}\,\tau_m^x\, e^{-iP_M} = \tau_{m-1}^x \ , \quad -M < m \leq M$$

$$e^{iP_M}\,\tau_{-M}^x\, e^{-iP_M} = \tau_M^x \ .$$

Keeping in mind the formal correspondencer $\psi S+(S^-) \to \hat{\psi}(\tilde{\psi})$, $e^{ikM}\xi_k \to \tilde{\psi}(q)$, $e^{-i\ell M}\xi_\ell \to \hat{\xi}(q)$, $\tau_{-M}^x \to \sigma_0$ as $M \to \infty$ **we obtain the same infinite lattice** representation as in the $|1,M|$ periodic case, We have for the $M\to\infty$ **limit of the** $|-M,M|$ **theory.**

Thm.5.2. $S_M^2 = (\psi_S\, ,\ \tau_{m_1}\ \tau_{m_2}\ \psi_S+) \xrightarrow{M\to\infty} (\hat{\psi}, \sigma_0\, U\, e^{i\hat{P}(m_1-m_2)}\, U\sigma_0\,\hat{\psi})$

where

$$\sigma_0 = (2\pi)^{-1/2}\int \left[\alpha\, e^{i\phi q}\,\hat{\xi}(q) + \bar{\alpha}\, e^{-i\phi q}\,\hat{\xi}^*(q)\right]dq \ .$$

Remark. The n-**point** function limit **is** also the same as the $|1,M|$ **periodic case.**

Proof: $(\psi_S+,\ \tau_{m_1}\ \tau_{m_2}\ \psi_S+) = (\psi_S+,\ e^{-iP_M(M+m_2)}\,\tau_{-M}^x\, e^{iP_M(M+m_2)}\ .$

$$e^{-iP_M(M+m_2)}\ \tau_{-M}^x\ e^{iP_M(M+m_2)}\,\psi_S+) \ .$$

But $e^{iP_M}\,\psi_S+ = \psi_S+$ so

$$S_M^2 = (\psi_{S+}, \tau_{-M}\, e^{iP_M(m_1-m_2)}\,\tau_{-M}\,\psi_{S+}) \ ,$$

P_M **is** P_M^- and **by** the limit of $e^{-iqM}\,\xi_q$ using **formal** correspondences

$$S_M^2 \xrightarrow{M\to\infty} (\hat{\psi}, \sigma_0\, U\, e^{i\hat{P}(m_1-m_2)}\, U\,\sigma_0\,\psi) \ ,$$

We now obtain the infinite lattice representation for products of disorder operators μ_m of eq. 1.2.

Thm.5.3. $(\psi_{S^+},\mu_{m_1}\ \mu_{m_2}\ \psi_{S^+}) \xrightarrow{M\to\infty} (\hat\psi,\ U\ e^{i\hat P(m_1-m_2)}\ U\ \hat\psi)$

where $\mu_n = (c_m + c_m^*)\ \tau_M = \tau_M(c_m + c_m^*)$.

Remarks. 1. $U\hat\psi$ is in the odd Fermion number subspace.

2. A similar formula holds for a product of n μ's.

Prf. $(\psi_{S^+},\mu_{m_1}\ \mu_{m_2}\ \psi_{S^+}) = ((c_{m_1} + c^*_{m_1})\tau_{m_1}\ \psi_{S^+},\ (c_{m_2} + c^*_{m_2})\tau_{m_2}\ \psi_{S^+})$

$= ((c_{m_1} + c^*_{m_1})e^{-iP_M(M+m)}\ \tau_{-M}\ \psi_{S^+},\ (c_{m_2} + c^*_{m_2})\ e^{-iP_M(M+m_2)}\ \tau_{-M}\ \psi_{S^+})$.

New $P_M = P_M^-$ in the above so that

$(\psi_{S^+},\ \mu_{m_1}\ \ \mu_{m_2}\ \psi_{S^+}) = (e^{-iP_M^-(M+m_1)}\ \tau_{-M}\ \psi_{S^+},\ (c_{m_1} + c^*_{m_1})(c_{m_2} + c^*_{m_2})$.

$e^{-iP_M^-(M+m_2)}\ \tau_{-M}\ \psi_{S^+})$.

Use c_{m_i} in terms of the set S^- (k set) and note that

$$e^{-iP_M^-(M+m_1)}(c_{m_1} + c^*_{m_1})e^{-iP_M^-(M+m\ )} = (c_{-M} + c^*_{-M}).$$

Thus

$$(\psi_{S+},\ \mu_{m_1}\ \mu_{m_2}\ \psi_{S^+}) = (\tau_{-M}\ \psi_{S+},\ \tau_{-M}\ e^{iP_M^-(M+m_1)}\ e^{-iP_M^-(M+m_2)}\ \tau_{-M}\ \tau_{-M}\ \psi_{S^+})$$

$$= (\psi_{S^+},\ e^{iP_M^-(m_1-m_2)}\ \psi_{S^+}) \xrightarrow{M\to\infty}$$

$$(\hat\psi,\ e^{i\tilde P(m_1-m_2)}\ \hat\psi) = (\hat\psi,\ U\ e^{i\hat P(m_1-m_2)}\ U\ \hat\psi).$$

The following result from Thm.2.4 of[1] will play an important role in what follows:

$$\xi(q)(\xi(q)) = \cos\phi_q\ \hat\xi(q)((\tilde q)) - \sin\phi_q\ \hat\xi^*(-q)(\tilde\xi^*(-q)),$$
$$\tilde\eta(f) = \hat\eta(Hf).$$

We let

$$\hat{c}_m(\tilde{c}_m) = \frac{1}{\sqrt{2\pi}} e^{-i\pi/4} \int e^{iqm} \hat{\eta}(q)(\tilde{\eta}(q))dq \ .$$

Lemma 5.4. a) $\tilde{\tilde{c}}_m + \tilde{\tilde{c}}^*_m = \tilde{\tilde{\xi}}(g_m) + \tilde{\tilde{\xi}}^*(\ddot{g}_m)$, $g_m(q) = \alpha e^{imq} e^{i\phi(q)}$,

$= e^{-i\pi/4}$ and $\hat{c}_m = (\text{sgn } m)\ \tilde{c}_m$.

$$\text{b)} \quad (\tilde{\tilde{c}}_m + \tilde{\tilde{c}}^*_m)U = \begin{cases} U(\tilde{\tilde{c}}_m + \tilde{\tilde{c}}^*_m), & m \geq 0 \ . \\ -U(\tilde{\tilde{c}}_m + \tilde{\tilde{c}}^*_m), & m < 0 \ . \end{cases}$$

Prf. a) $$c_m + c^*_m = \frac{1}{\sqrt{2\pi}} e^{-i\pi/4} \left| \int dq \ \ q^{iqm} \left\{ \left(\frac{e^{i\phi_q} e^{-i\phi_q}}{2}\right)\xi^*(q) - \left(\frac{e^{i\phi_q} - e^{-i\phi_q}}{2i}\right)\xi^*(-q) \right\} \right| + \frac{1}{\sqrt{2\pi}} e^{i\pi/4} \left| \int dq \ e^{-iqm} \left(\frac{e^{-i\phi_q} e^{i\phi_q}}{2}\right)\xi^*(q) + \left(\frac{e^{-i\phi_q} - e^{i\phi_q}}{2i}\right)\xi^*(-q) \right\} \right|$$

and the results foliows.

$$\text{b)} \quad (\hat{c}_m + \hat{c}^*_m)U = (\hat{\xi}(g_m) + \hat{\xi}^*(\bar{g}_m))U$$

$$= U(\tilde{\xi}(g_m) + \tilde{\xi}^*(\bar{g}_m)) \ \text{ since } U\hat{\xi}(q)U^{-1} = \tilde{\xi}(q)$$

$$= U(\tilde{c}_m + \tilde{c}^*_m) \ .$$

By Thrn. 2,4 of[1] $\tilde{\eta}(f) = \hat{\eta}(Hf)$ so that the theorem follows.

We now obtain a representation for the infinite lattice spinand disorder operator and their commutation relations.

Thm.5.5. Let $s_m = e^{-i\hat{P}m} \sigma_0 U e^{i\hat{P}m}$, $d_m = e^{-i\hat{P}m} U e^{i\hat{P}m} \equiv U_m$,

$\hat{c}_m + \hat{c}^*_m \equiv e^X_m = e^{-i\hat{P}m}(c_0+c^*_0)\ e^{-i\hat{P}m}$ and assume $[s_m, s_n] = 0$, Then

$$s_m d_0 = \begin{cases} -d_0 s_m, & m \geq 0 \\ d_0 s_m, & m < 0 \ . \end{cases}$$

Remark. The s_m comute at least on the subspace D generated by

$$\hat{\Pi}_{i=1} s_{ij} \ldots s_{in} \hat{\Psi} \in H_e \text{ since } (\hat{\Psi}, \prod_{j=1}^{m} s_{n_j} \quad \hat{\chi})$$

is a correlation function.

Prf: Using $[s_m, s_0] = 0$ and $c_n^x s_n = s_n c_n^x$ we have $s_m d_0 = s_m c_0^x s_0 = s_m s_0 c_0^x = s_0 s_m c_0^x$. But $s_m = c_m^x U_m$ and $\{c_m^x, c_0^x\} = 0$ so that $s_m d_0 = -s_0 V_m c_0^x c_m^x$. From Thm.5.5, $e^{-iPm} U = (\text{sgn } m) U e^{-iPm}$ c_0 e so that $s_m d_0 = -(\text{sgn } m) s_0 \sigma_0 U_m c_m^x = (-\text{sgn } m) U_0 s_m$

$= -(\text{sgn } m) d_0 s_m$.

6. SCALING LIMIT

In a manner analogous to that of refs. 2 and 3 we can obtain seriss representations for $(\psi, \prod_{i=1}^{S} \sigma_{m_i}^x \psi)$ and (3.1). Scaling limit expectations are defined as in[3] with the modification that one factor of the magnetization m* divides the infinite expectation for each factor of U occurring; the infinite lattice expectations being expresses in terms of the $\xi(q)$ Fermion operators only.

7. ALGEBRAIC CONSTRUCTION OF THE SOLITON SECTOR

A. Heuristic considerations. To motivate our construction of the soliton sector we first give a formall ill-defined construction and then show how we make it well-defined. For temperatures below the critical temperature T_C the translationally invariant vaccum vector $\Omega+$ of the quantum field theory obtained from the correlation functions ofthetwo-dimensional Ising model with plus boundary conditions has spinspredominantely up and $(\Omega+, \sigma_z(i) \Omega+) > 0$. Formally the Hilbert space is an infinite tensor product of two-dimensional Hilbert spaces H_i, $i \in Z$ denoting the spatial lattice site, and A will denote the algebra of operators generated by linear combinations of tensor products of Pauli matrices. The "kink" operator

$$\mu = \prod_{i<0} \sigma_x(i) \ , \quad \mu^2 = 1$$

flips all spins to the left of zero. We let $H_+ (H_s)$ denote the vacuum (soliton) Hamiltonian and set $R_s = \mu R_+$; H_+ obeys inf spec $H_+ = 0$ and $H_+ \Omega_+ = 0$.

Ignoring boundary effects we define the dynamics of the Hilbert space vector $\psi_s = A\, \Omega_s$, $A \in \mathcal{A}$, by

$$e^{iH_s t} \psi_s = e^{iH_+ t} A\, \mu\, \Omega_+ .$$

Notice that the right side can be written as

$$e^{iH_+ t} A\mu\, \Omega_+ = \mu\, \mu\, e^{iH_+ t} A\, \mu\, \Omega_+$$

$$= \mu\{\mu\, e^{iH_+ t}\, \mu\, e^{-iH_+ t}\}\{e^{iH_+ t}\, \mu\, A\mu\, e^{-iH_+ t}\}\, \Omega_+$$

$$= \mu\, \Gamma(t)\, \tau_+(\rho(A))\Omega_+ = \mu\Gamma(t)\tau_t(\rho(A))\, \mu\, \Omega_s$$

$$= \rho[\Gamma(t)\tau_t(\rho(A))]\Omega_s$$

where

$$\Gamma(t) = \mu\, e^{iH_+ t}\, \mu\, e^{-iH_+ t} = \rho(e^{iH_+ t})e^{-iH_+ t} \quad (7.1a)$$

$$\tau_t(A) = e^{iH_+ t} A\, e^{-iH_+ t} \quad (7.1b)$$

$$\rho(A) = \mu\, A\, \mu \quad (7.1c)$$

so that

$$e^{iH_s t} \psi_s = \rho[\Gamma(t)\tau_t(\rho(A))]\Omega_s \quad . \quad (7.2)$$

[illegible] is a unitary group; $\Gamma(t)$ is unitary and abeys the property

$$\Gamma(t_1 + t_2) = \Gamma(t_1)\, \tau_{t_1}(\Gamma(t_2)) \; .$$

We now describe how we make the above formal construction well-

-defined. First we pass to imaginary discrete time $t \to in$, $n \in Z^+$. Due to the special properties of e^{-H_+} (e^{-H_+} is proportional to the transfer matrix), not possessed by the time-continuum limit Hamiltonian, and ignoring boundary conditions on H_+, $\Gamma(in)$, $n \in Z^+$, can be defined as a local object with finite norm (we do not know if $\Gamma(n)$ is a local object). Also if A is an element of the local algebra $A(O) \subset$ A, O bounded, then $\rho(A)$ of (7.1c) can be defined as an automorphism on $A(O)$ and $\tau_{in}(A)$ is a local object with finite norm. What we are exploiting here is the finite propagation speed of $e^{-H_+ n}$, $n \in Z^+$.

The Hilbert space $H_+ (H_s)$ of the vacuum (soliton) sector is obtained from the GNS construction applied to the state $\omega_+ (\omega_s = \omega_+ . \rho)$ and A. ω_+ is the infinite space lattice limit, of the vector state associated with the vacuum vector of the quantum field theory obtained from the transfer matrix with plus boundary conditions in the space direction. The soliton sector semi-group $e^{-H_s n}$ is the imaginary time version of (7.2). μ is defined as a map from to H_s, formally, with A $\Omega_+ \in H_+$, μ A Ω_+ = = $\mu A \mu\mu \Omega_+ = \rho(A)\ \Omega_s$.

Finally we introduce soliton field operators $\mu(\ell,m)$, $\ell(m)$ a time (space) point, and show that for imaginary discrete time (Ω_s, $e^{iH_s t}$ Ω_s) is the two-point function for the Kadanott and Ceva disorder operator[10]. Formally,

$$(\Omega_s\ ,\ e^{-H_s \ell}\ \Omega_s) = (\Omega_+ ,\ \mu(0,0)\mu(i\ell,0)\Omega_+)$$

and the right side is shown to be equal to a modified partition function (anti-ferromagnetic along the line joining (0,0) and (ℓ,0) divided by the pertition function. We remark that by using chessboard estimates the static soliton energy can be bounded below by the surface tension (see also[7]). Also our construction can be carried out for other boundary conditions one obtains the anti-soliton (soliton-anti-soliton sector).

B. Hilbert space and algebra of observables

Let H'_0 be the pre-Hilbert space generated by elements of the tensor product $\prod_{i \in S \subset Z} H_i$, H_i = C , S arbitrary but finite. $\sigma_x(i)$, $\sigma_y(i)$,

$\sigma_z(i)$ are tensor product operators in H_0'; on H_i they are the Pauli matrices, on H_j, $j \neq i$, they are the identity. We let H denote the completion of H_0'.

For any bounded set O in Z, let $A(O)$ be the algebra generated by $\sigma_x(i)$ and $\sigma_z(i)$, $i \in O$ and let A, the qiiasi-local C* algebra of observables, denote the norm closure of $\underset{O}{U}\, A(O)$, i.e. $A = \overline{\underset{O}{U}\, A(O)}$.

C. Some real (imaginary) discrete rime automorphisms (transformations) of the algebra of òbservables

The finite M transfer matrix with plus boundary conditions in the spase direction for the $(-M,M) \times (-N,N)$ lattice Ising model is defined as

$$V'_{++}(M) = V_2^{1/2}\; V_1'\; V_2^{1/2}$$

where

$$V_2 = \exp\left[K \sum_{m=-M}^{M-1} \sigma_z(m)\sigma_z(m+1) + K(\sigma_z(-M) + \sigma_z(M))\right]$$

and

$$V_1' = (2 \sinh 2K)^{(2M+1)/2}\; V_1\,,\quad V_1 = \exp\left[-2K^* \sum_{m=-M}^{M} \sigma_x(m)\right].$$

As $V_{++} = V_2^{1/2}\, V\, V$ is self-adjoint and invertible we can write

$$V_{++} \equiv e^{-H_M}\,,\quad e^{-\hat{H}_M} \equiv e^{-(H_M - \inf sp\,(H_M))}\,.$$

e^{-H_M} and $e^{-\hat{H}_M}$ are also self-adjoint and invertible and $e^{-\hat{H}_M}$ is a contraction.

Lema 7.1. For $A \in A(O)$, $z \in C$, define $\tau_z : A(O) \to A$ by

$$\tau_z(A) = e^{iz\hat{H}_M}\, A\, e^{-iz\hat{H}_M}$$

Then

a) For z real τ_z extends by continuity to an automorphism of A.

b) For $z = in$, $n \in Z$, and M sufficiently large $\tau_{in}(A)$ is independent of M and bounded with $|\tau_{in}(A)|$ independent of M but possibly dependent on $\mathcal{O}$.

Prf. a) For z real, $|\tau_{in}(A)| \leq |A|$. b) follows from the product nature of $e^{-H_M n}$ and the commutation relations obeyed by $\sigma_x(i)$, $\sigma_z(i)$.

D. The vacuum sector.

The correlation functions of the Ising model on a $(2\ +1)\times(2\ +1)$ (space × imaginary time) lattice with plus (periodic) boundary conditions in the space (imaginary time) direction are given by

$$<\sigma(i_1,j_1)\ldots\sigma(i_n,j_n)>_{++}(N\times M) = \frac{\mathrm{Tr}(\sigma_z(i_1)e^{-\hat{H}_M j_1}\sigma_z(i_2)\ldots\sigma_z(i_n)e^{-\hat{H}_M(N-j_n)}}{\mathrm{Tr}\ e^{-H_M N}}$$

Taking the $N\to\infty$ limit and letting $R_+(M)$ denote the eigenvector of $e^{-\hat{H}_M}$ with eigenvalue 1 (of multiplicity 1 by the Perron-Frobenius theorem) we obtain the finite space lattice Feynmann-Kac formula.

$$<\sigma(i_1 j_1)\ldots (i_n j_n)>_{++}(M) =$$

$$= (\Omega_+(M),\ \sigma_z(i_1)\ e^{-\hat{H}_M(j_2-j_1)}\ \ldots\ e^{-\hat{H}_M(j_n-j_{n-1})}\sigma_z(i_n)\Omega_+(M)$$

Thm. 7.2. Let $O_M(z_1\ldots z_k) = \prod_{j=1}^{k} e^{iz_j\hat{H}_M}\sigma_z(n_j)\,e^{-iz_j\hat{H}_M}$. Then

a) $O_M(z_i = iM_i,\ i = 1,2,\ldots k) \in A(\mathcal{O})$ for some O and is independent of M for large M.

b) $\omega_+(A) = \lim_{M\to\infty}[\omega_M(A) \equiv (\Omega_M,\ A\ \Omega_M)]$ exists for $A \subset O_m(z_j = im_j),\ m_{i+1}>m_i$

c) ω_+ extends to a state on A by taking a suitable subsequence of ω_M.

We expect the vectors O_M to generate the full algebra A so that the extension of ω_+ is unique.

Prf. a) as in Lemma 7.1b.

b) For plus boundary conditions in the space and imaginary time direc- the correlation functions can be expressed in terms of the transfer matrix with the trace replaced by an appropriate inner product, The $N\to\infty$ limit gives the same F-K formula as above. For all plus boundary conditions Griffiths 2nd inequality implies the correlation functions are monotone decreasing in M so that the $M\to\infty$ limit exists.

c) Follows by taking a suitable diagonal subsequence.

E. Construction of the soliton sector,

For each $j_0 \in z$ we define the automorphism $\rho(j\ ) : A\to A$ as the continuous extension of

$$\rho(j_0)(\sigma_z(i)) = \begin{cases} \sigma_a(i), & i > j_0 \\ -\sigma_a(i), & i \leq j_0 \end{cases}$$

$\rho(j_0)(\sigma_x(i)) = \sigma_x(i)$ and we set $\rho(0) \equiv \rho$. To see that $|\rho(A)| \leqq |A|$ for A local, i.e. $A \in A(0)$, we use the fact that for such ,

$$\rho(A) = \prod_{i \in S \supset 0} \sigma_x(i)\ A \prod_{i \in S \supset 0} \sigma_x(i)$$

where $S(0 \subset S)$ is finite.

From Thm. 7.2 $\omega_s = \omega_+ . \rho$ also defines a state which we define as the soliton state and we let $H_+(H_s)$ denote the GNS Hilbert space constructed from $\omega_+(\omega_s)$ and A.

In order to construct the imaginary time semi-group in the soliton sector Hilbert space H_S we define $\Gamma(z) \in A$ by

$$\Gamma(z) = \rho(e^{izH_M^-})\ e^{-izH_M}$$

where H_M^- is obtained from H_M by replacing $\sigma_z(-M)$ by $-\sigma_z(-M)$. The above definition does not contradict the formal definition of (7.1a); the heuristic definition is so ignore boundary effects and as the next lemma shows this is exactly what the above definition does for imaginary discrete time. We have

Lemma. 7.3.

a) $\Gamma(in) = (e^{-k\sigma_z(0)\sigma_z(1)}\, e^{-\hat{H}_M}\, e^{-k\sigma_z(0)\sigma_z(1)})^n\, e^{\hat{H}_M n}$, $n \in Z$, for sufficiently large M and is independent of M. Furthermore $|\Gamma(in)| \leqq C(n) < \infty$ and $|\Gamma(i1)| \leqq \cosh K + e^{3K}\, e\, e^{8K^*} \sinh K \equiv C'$.

b) $\Gamma(z_1+z_2) = \Gamma(z_1)\tau_{z_1}(\Gamma(z_2)) = \Gamma(z_2)\tau_{z_2}(\Gamma(z_1))$,

$z_i = in_i$, i = 1,2, for sufficiently large M

Prf. a) The formula for $\Gamma(in)$ is obvious. For $n = 1$ by explicit calculation

$$e^{-H_M}\, e^{-K\sigma_z(0)\sigma_z(1)}\, e^{H_M} = \cosh K - [e^{1/2K \sum_{m=-1}^{1} \sigma_z(m)\sigma_z(m+1)}$$

$$. \, e^{-4K^*(\sigma_x(0)+\sigma_x(1))}\, e^{-1/2K \sum_{m=-1}^{1} \sigma_z(m)\sigma_z(m+1)}] . \sigma_z(0)\sigma_z(1) \sinh K$$

and the bound for $(\Gamma(i1))$ follows. For $n>1$ we can commute the n factors of e^{-H_M} to the right to cancel $e^{H_M n}$; what remains is some $B \in A(0)$ of finite norm.

b) Let $L = \exp|- K\sigma_z(0)\sigma_z(1)|$; then by a)

$$\Gamma(i(n_1+n_2)) = \rho(e^{-(n_1+n_2)H_M})e^{(n_1+n_2)H_M}$$

$$= \{(L\, e^{-H_M} L)^{n_1}\, e^{H_M n_1}\} . \{e^{-H_M n_1}(L\, e^{-H_M} L)^{n_2}\, e^{H_M n_2}\, e^{H_M n_1}\}$$

$$= \Gamma(in_1)\tau_{in_2}(\Gamma(in_2)).$$

We denote the inner product in $H_+ (H_S)$ by $(\;,\;)_+ \; ((\;,\;)_S)$

and for notational simplicity we do not distinguish between $A \in \mathcal{A}$ and A considered as a representative of the Hilbert space vector equivalence class obtained from the GNS construction. Thus if $A, B \in \mathcal{A}$

$$(A,B)_+ = \omega_+(A^* B)$$

$$(A,B)_s = \omega_s(A^* B) = \omega_+(\rho(A^*B)) = (\rho(A),\rho(B))_+ \ .$$

We will also denote by $\Omega_+(\Omega_s)$ the vector associated with I in $H_+(H_s)$.

The vacuum sector semi-group $T_+(n)$ of imaginary time lattice translations is defined as the closure of the operator associated with the quadratic form, with $A, B \in A(O)$,

$$(B,T_+(n)A)_+ \equiv \omega_+(B^* \tau_{in}(A)) \ .$$

Lema 7.4. $T_+(n), n \in Z$, is a self-adjoint contraction semi-group and $T_+(n)1 = 1$, or $T_+(n)\ R_+ = R_+$.

Prf. This clearly holds for ω_M, M sufficiently large, by lemma 7.1 and thus also for the subsequential $M \to \infty$ limit.

Define a map μ; $H_+ \to H_s$ as the closure of

$$\mu(A) \equiv \rho(A), \quad A \in A(O) \ .$$

In particular $\mu(I) = \rho(I) = ■$ or $\mu(\Omega_+) = R_S$.

Lemma 7.5. μ is isometric and onto.

Prf, μ is isometric as

$$(\mu(A),\ \mu(B))_s = (\rho(A),\ \rho(B))_s = \omega_s(\rho(A)^*\ \rho(B))$$

$$= \omega_+(\rho.\rho(A^*B)) = \omega_+(A^*B) = (A,B)_+$$

and as finite linear combinations of $\rho(A)$, $A \in A(O)$, are dense in H_s μ is onto.

The soliton sector semi-group $T_s(n)$, $n \in Z^+$, is defined as the closure of the operator associated with the form

$$(B,T_s(n)A)_z \equiv (B,\rho|\Gamma(in)\tau_{in}(\rho(A))|)_s$$

where A, B $\in A(0)$.

Thm.7.6. $T_s(n)$, $n \in Z^+$, is a self-adjoint semi-group and there is a constant $c < \infty$ such that $|T_s(n)| \leq e^{cn}$.

Remark. $T_s(n)$ is expected to be a contraction semi-group with $|T_s(n)| \leq e^{-m_s n}$, where m_s is the soliton mass, but this is difficult to prove in our abstract framework.

Prf. a) Boundedness. We have

$$T_s(n)\ A\ \Omega_s = \rho[\Gamma(it)\tau_{it}(\rho(A))]\ \Omega_s$$

so that

$$(B,T(n)A)_s = (\rho(B),\ |\Gamma(in)\tau_{in}(\rho(A))|)_+$$

$$= \omega_+(\rho(B)^*\ \Gamma(in)\tau_{in}(\rho(A)))$$

$$= \lim_{M\to\infty} \omega_M(..).$$

But for M large

$$|\omega_M(..)| = |(\Omega_M,\rho(B)^*\Gamma(in)e^{-\hat{H}_M n}\rho(A)\ e^{\hat{H}_M n}\ \Omega_M)|$$

$$\leq |\rho(B)\Omega_M|\ |\Gamma(in)|\ |\rho(A)\Omega_M|$$

$$\leq |\Gamma(in)|\ |\rho(B)|\ \ |\rho(A)| \leq |\Gamma(in)|\ |B|_s\ \ |A|_s$$

As $|\Gamma(in)| \leq C(n)$ by lemma 7.3d boundedness of $T(n)$ follows. By the semi-group property (shown below) $|T_S(n)| \leq |T_S(1)|^n \leq |\Gamma(i1)|^n \leq e^{Cn}$ by lemma 7.3d.

b) Semi-group property. For A, $B \in A(O)$ we have

$$(B\Omega_s, T_s(n_1+n_2)A\Omega_s)_s = \omega_+\{\rho(B^*)\Gamma(i(n_1+n_2))\tau_{i(n_1+n_2)}(\rho(A))\}$$

$$= \omega_+\{\rho(B^*)\Gamma(in_1)\tau_{in_1}(\Gamma(in_2))\tau_{i(n_1+n_2)}(\rho(A))\}$$

$$= \lim_{M\to\infty}(\Omega(M),\rho(B^*)\Gamma(in_1)\ \tau_{in_1}[\Gamma(in_2)e^{-H_M n_2}(\rho.A)e^{H_M n_2}]\ \Omega(M))$$

$$= \omega_+\{\rho(B^x)\Gamma(in_1)\ \tau_{in_1}[\Gamma(in_2)\ \tau_{in_2}(\rho.A)]\}\ . \tag{7.3}$$

On the other hand

$$(B\Omega_s, T_s(n_1)(T_s(n_2)A\Omega_s))_s =$$

$$\omega_+\{\rho(B^*)\Gamma(in_1)\tau_{in_1}(\rho(T_s(in_2)A))\}$$

$$= \omega_+\{\rho(B^*)\ \Gamma(in_1)\ \tau_{in}\ [\Gamma(in_2)\tau_{in_2}(A))]\}\ . \tag{7.4}$$

Thus (7.3) equals (7.4), i.e. $T_S(n_1+n_2) = T_S(n_1)T_S(n_2) = T_S(n_2)T_S(n_1)$.

c) Self-adjointness. By definition, for $C \in A(O)$, we have

$$T_s(n)\ C\ \Omega_s = \rho\ [\Gamma(in)\tau_{in}(\rho(C))]\Omega_s$$

and thus, for A, $B \in A(O')$ for some O'.

$$(B\Omega_s, T_s(n)\ A\Omega_s) = \omega_+\ \{\rho(B^*)\Gamma(in)\tau_{in}(\rho(A))\}$$

$$= \lim_{M\to\infty}\ (\Omega(M),\ \rho(B^*)\ J^n\ \ \rho(A)\ \Omega(M))$$

where $J = e^{-K\sigma_z(0)\sigma_z(1)}\ e^{-\hat{H}_M}\ e^{-K\sigma_z(0)\sigma_z(1)}$, On the other hand

$$(T_s(n)\ B\Omega_s, A\Omega_s) = (\rho[\Gamma(in)\tau_{in}(\rho(B))]\ \ \Omega_s, A\Omega_s)_s$$

$$= \omega_+\ \{[\Gamma(in)\ \tau_{in}(\rho(B))]^*\ \ \rho(A)\}$$

$$= \lim_{M\to\infty}\ (J^n\ \rho(B)\Omega(M),\ \rho(A)\Omega(M))$$

and as $J = J^*$ the result follows.

In order to define space translations in H_S it is convenient to define $\alpha_j \in A$ by

$$\alpha_j = \begin{cases} \prod_{\ell=1}^{j} \sigma_x(\ell), & j > 0 \\ \prod_{\ell=j+1}^{0} \sigma_x(\ell), & j < 0 \end{cases} .$$

Thus $\rho_j(A) = \rho(\alpha_j A \alpha_j) = \alpha_j \, \rho(A)\alpha_j$

We let $\tau_j' : A \to A$ denote the automorphism of space translations which when restricted to the generators of A is defined by

$$\tau_j'(\sigma_H(i)) = \sigma_H(i + j), H = x \text{ or } z .$$

Also let $\mu_j : H_+ \to H_s$ denote the map

$$\mu_j \, A\Omega_+ = \mu \, A \, \alpha_j \, \Omega_+$$

and define the unitary soliton sector space translation operator

$$U_s(0,j) : H_s \to H_s$$

by

$$U_s(0,j)A \, \Omega_s = \mu_j \, \tau_j'(\rho_{-j} A) \, \Omega_+ = \mu \, \tau_j'(\rho_{-j}A)\alpha_j \, \Omega_+ .$$

We have

Thm.7.2. $U_s(0,j)$, $j \in Z$, is a unitary group.

Prf. a) Unitary. To see that $U_s(0,j)$ is isometric we calculate

$$(U_s(0,j)A\Omega_s, \, U_s(0,j)B\Omega_s)_s = (\tau_j'(\rho_{-j}A)\alpha_j \, \Omega_+,$$
$$\tau_j'(\rho_{-j}B)\alpha_j\Omega_+)_+ = (\alpha_j\tau_j'(\rho(A)\Omega_+, \, \alpha_j\tau_j'(\rho B)\Omega_+)_+$$
$$= \omega_+ \, . \, \rho(A^*B) = (\Omega_s, \, A^*B \, \Omega_s)_s$$

where we have used $\alpha^2 = I$ and that ω_+ is translationally invariant in the third equality. As $U_s(0,j)$ is invertible unitarity follows.

b) The group property follows by direct calculation.

F) Soliton field operator and the Kadanoff and Ceva disorder operator.

Assuming that the null space of $T_S(n)$ $(T_+(n))$ is zero we can write $T_s(n) = e^{-H_S n}$ $(T_+(n) = e^{-H_+ n})$ and define the real time unitary translational operator on $H_s(H_+)$ by $U_s(t,0) = e^{-H_S t}$ $(U_+(t,0) = e^{-H_+ t})$. Thus $U_s(t,j) \equiv U_s(t,0)\, U_s(0,j)$ is the lattice space-time translation operator. Furthermore we define the soliton field operator $\mu(i,j): H_+ \to H_s$ by

$$\mu(i,j)A\Omega_+ = U_s(i,j)\mu(0,0)U_+(i,j)^* A\Omega_+$$

where $\mu(0,0) \equiv \mu$.

We now obtain a formula for the imaginary time soliton field two-point function in terms of a ratio of partition functions which is shown to be the disorder operator two-point function defined by Kadanoff and Ceva[10]. For simplicity let the two points lie along a line in the time direction. Using the definitions.

$$\mu(t,0)\Omega_+ = U_s(t,0)\mu\, U_+^*(t,0)\Omega_+ = U_s(t,0)\mu\Omega_+ ,$$

$$\mu(0,0)\Omega_+ = \mu\Omega_+ = \Omega_s ,$$

$$(\mu(0,0)\Omega_+ , \mu(t,0)\Omega_+)_s$$

continued to imaginary time is

$$(\mu(0,0)\Omega_+, T_s(in)\mu(0,0)\Omega_+)_s = (\Omega_s, T_s(in)\Omega_s)_s = (\Omega_+, \Gamma(in)\Omega_+)_+$$

so that the imaginary time two-point function is

$$(\Omega_+, \Gamma(i\ell)\Omega_+)_+ = \lim_{M\to\infty} (\Omega\,(M), \hat{M}^{\ell}\, e^{\hat{H}_M \ell}\, \Omega_+(M))$$

where

$$\hat{M} = e^{-K\sigma_z(0)\sigma_z(1)}\, e^{-\hat{H}_M}\, e^{-K\sigma_z(0)\sigma_z(1)}$$

Since

$$\lim_{M\to\infty} \hat{M}^{\ell}\, e^{-\hat{H}_M(N-\ell)} = (\Omega_+(M),\, \hat{M}^{\ell}\, \Omega_+(M))$$

and

$$\lim_{M\to\infty} \mathrm{Tr}\, e^{-\hat{H}_M N} = (\Omega_+(M),\, \Omega_+(M))$$

we have, letting

$$M = e^{-K\sigma_z(0)\sigma_z(1)}\, e^{-H_M}\, e^{-K\sigma_z(0)\sigma_z(1)} ,$$

$$(\Omega_+,\, \Gamma(i\ell)\Omega_+)_+ = \lim_{M\to\infty}\, \lim_{N\to\infty} \frac{\mathrm{Tr}(M^{\ell}\, e^{-H_M(N-\ell)})}{\mathrm{Tr}\, e^{-H_M N}}$$

which is the limit of the modified partition function (anti-ferromagnetic along the line joining $(0,0)$ and $(\ell,0)$ divided by the partition function.

APPENDIX: Zero temperature finite lattice Hamiltonian

We give a pedagogical discussion of the qualitative difference, for $T < T_c$, between the periodic and anti-periodic Hamiltonians which can be understood by dropping the kinetic energy-like term $\sum_{i=-M}^{M} \tau_i^z$ from the partition function and considering the zero temperature periodic (anti-periodic Hamiltonian

$$V_{0P(A)} = -g \sum_{i=-M}^{M} \tau_i^x\, \tau_{i+1}^x$$

where

$$\tau_{M+1}^x = \tau_{-M}^x (\tau_{M+1} = -\tau_{-M}) \quad \text{and} \quad g > 0 .$$

V, acts in H_M and the vectors

$$\chi_{-M}^{s_{-M}} \times \ldots \times \chi_M^{s_M} ,$$

$s_i = \pm 1$, form an orthonormal basis for H_M where

$$\chi_m^{\pm 1} = \frac{1}{\sqrt{2}} \binom{1}{\pm 1} \quad \text{and} \quad \tau_m^x \chi_{-M}^{s_M} \ldots \chi_M^{s_M} = s_m \chi_{-M}^{s_{-M}} \ldots \chi_M^{s_M} ,$$

First let us consider the translation operator for the more general Hamiltonian

$$L_{P(A)} = T + V_{0P(A)} , \; T = \sum_{j=-M}^{M} \tau_j^z .$$

For the periodic case define the translation operator, S, by

$$S \chi_{-M}^{s_{-M}} \ldots \chi_M^{s_M} = \chi_{-M}^{s_M} \chi_{-M+1}^{s_{-M}} \ldots \chi_M^{s_{M-1}} .$$

For the a-p case define the 'approximate' translation operator, A, by

$$A \chi_{-M}^{s_{-M}} \ldots \chi_M^{s_M} = \chi_{-M}^{-s_M} \chi_{-M+1}^{s_{-M}} \ldots \chi_M^{s_{M-1}} .$$

Then

$$|T,S| = |T,A| = 0, \quad |S,LP| = |S,V_{0P}| = 0 , \; |A,L_A| = |A,V_{0A}| = 0 .$$

We now explicitly compute, in terms of the above basis (rather than the non-intuitive Fermion basis), the ground state eigenvectors and associated eigenvalue. Periodic: For the Hamiltonian V_{0P} the ground state is doubly degenerate with eigenvectors

$$\psi_\pm = \chi_{-M}^{\pm} \ldots \chi_M^{\pm}$$

and eigenvalue $-g(2M+1) \equiv E_0$. The next highest eigenvalue is at E_0+2g. Also

$$(\psi_\pm, \tau_{-M}^x \tau_M^x \psi_\pm) = 1 \quad \text{and} \quad (\psi_\pm, \tau_i^x \psi_\pm) = \pm 1 .$$

Anti-periodic: For the Hamiltonian V_{0A} we consider the 2(2 +1) dirnensional subspace H_0 of H_M ordered as follows: $\psi_s^{-M}, \ldots \psi_s^{M-1}, \psi_+, \psi_A^{-M} \ldots \psi_A^{M-1}$, ψ_- where

$$\psi_{S(A)}^i = \chi_{-M}^{-(+)} \ldots \chi_i^{-(+)} \chi_{i+1}^{+(-)} \ldots \chi_M^{+(-)}$$

are soliton (anti-soliton) type vectors. A and V_{0A} leave H_0 invariant and for any $\chi \in H_0$, $V_{0A}\chi = (E_0 + g)\chi$. However, A is not diagonal in H; it has the representation

$$A = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$$

where

$$B = \begin{pmatrix} 0 & & & & \\ 1 & 0 & 0 & & \\ & 1 & & & \\ & & & 0 & \\ & 0 & & 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & & 0 & 1 \\ & & & 0 \\ & 0 & & \\ 1 & & & 0 \end{pmatrix}$$

B and C are $2M+1$ dirnensional square matrices.

The simultaneous eigenfunctions of A and V_{0A} are found to be

$$\psi_k = \frac{1}{\sqrt{2(2M+1)}} (r_k, r_k, \dots r_k^{2(2M+1)}) ,$$

where $r_k = e^{-2\pi i k/2(2M+1)}$, $k = 0, \pm 1, 2, \dots, , 2M+1$ and $A\psi_k = e^{-2\pi i k/2(2M+1)} \psi_k$. The next highest eigenvalue of V_{0A} is at E_0+3 and the ground state of V_{0A} is precisely $2(2M+1)$ fold degenerate. Furtherrnore as

$$\tau_M \psi_k = (r_k, r_k^2, \dots r_k^{2M}, -r_k^{2M+1}, -r_k^{2M+2}, \dots, -r_k^{2(2M+1)-1}, r_k^{2(2M+1)})$$

and

$$\tau_{-M} \psi_k = (-r_k, -r_k^2, \dots, -r_k^{2M+1}, r_k^{2M+2}, \dots, r_k^{2(2M+1)-1}, r_k^{2(2M+1)})$$

we find $(\psi_k, \tau_{-M} \tau_M \psi_k) = -(1-(2M+1)^{-1})$. Also $(\psi_k, \tau_i \psi_k) = 0$.

For $T < T_c$ ψ_0 becornes the unique ground state (the other levels are raised in energy) and as $M \to \infty$ the other eigenvalues of $V_{0A} \wedge H_0$ fill in the infinite lattice anolog of the one-particle hyperbola.

To rnake contact with the Fermion solution recall that $\xi_0^* \psi_{A_e}$ is the unique ground state of H_{M}, the a-p Hamiltonian. Thus in the limitas $T \to 0$, $\xi_0^* \psi_{A_e}$ is proportional to ψ_0. To rnotivate the choice of the vector ψ_{A_e} in Thm. 3.2 we see that

$$(\mu_m \psi_{A_e}, \tau_{-M} \tau_M \mu_m \psi_{A_e}) = -(\psi_{A_e}, \tau_{-M} \tau_M \psi_{A_e})$$

for $-M < m < M$. Since $\xi_0 \psi_{A_e} = 0$ note that

$$(\psi_{A_e}, \tau_{-M} \tau_M \psi_{A_e}) = (\psi_{A_e}, \tau_{-M} \tau_M \xi_0 \xi_0^* \psi_{A_e})$$

and since $\xi^* = (2M+1)^{-1/2} \bar{a} \sum_m c_m$ we see that $|\tau_M, \xi_0| \approx 0$ and $\{\tau_{-M}, \xi_0\} \simeq 0$ (up to terms of $(2M+1)^{-1/2}$ times an operator of norm 1) so that

$$(\psi_{A_e}, \tau_{-M} \tau_M \psi_{A_e}) \approx -(\xi_0 \psi_{A_e}, \tau_{-M} \tau_M \xi_0^* \psi_{A_e}) = -(\psi_0, \tau_{-M} \tau_M \psi_0) = 1$$

Thus

$$(\mu_m \psi_{A_e}, \tau_{-M} \tau_M \mu_m \psi_{A_e}) \approx -1 .$$

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Resumo

Definimos uma Hamiltoniana anti-periódica numa rede finita através da matriz transferência do modelo de Ising bi-dimensional com condições de fronteira anti-periódicas na direção espacial e periódicas na direção temporal imaginária. Uma teoria quântica de campos numa rede infinita é obtida tomando-se limites de estados vetoriais na álgebra de observáveis gerada por produtos finitos de operadores de spin. Representações explícitas dos operadores de spin e energia- momentum são obtidos em termos de Fermions livres agindo num espaço de Fock Fermiônico. O espectro do operador energia-momentum na rede infinita é análogo ao espectro ímpar de uma teoria escalar livre e massiva: em particular, o vácuo e o estados de duas partículas estão ausentes. A álgebra de observáveis admite uma decomposição em duas subálgebras correspondentes ao soliton e antisoliton. Estados vetoriais particulares mostram comportamento tipo soliton. O limite de escala também é obtido. Uma construção algébrica do setor de solitons é dada e operadores de campo de solitons são definidos. Mostramos que a função de dois pontos de solitons para tempos imaginários coincide com a função de dois pontos do operador de desordem introduzido por Kadanoff e Ceva.