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Propagator for a Charged Particle in Time-Dependent Electromagnetic Field and Quadratic Potential

BIN KAN CHENG
Departamento de Fhica, Universidade Federal do Paraná, Caixa PosD>I 19081, Curitiba, PR, Brasil

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Abstract Through a time-dependent linear transformation and the time substitution, we can evaluate exactly the propagator for a charged particle in a time-dependent electromagnetic field subjected to a time-dependent quadratic potential.

1. INTRODUCTION

It is well known that for a quadratic Lagrangian, the propagator is related to the classical action through the Van Uleck-Pauli formula[1,2]. However, the evaluation of the classical action is not always simple. The time-dependent linear coordinate transformations with new-time have been used by Junker and Inomata[3], and by Cheng[4] to transform the original quadratic action into a new quadratic action whose classical action can be evaluated exactly. Later several authors[5,6] derived such transformations in a broader sense by applying a non-linear superposition law of Ray and Reid[7]. In this paper we are able to deduce them from a Feynman path integral by considering the mid-point expansion[8-10] for each short-time action, and to obtain the propagator for a time-dependent harmonically bound charged particle in a time-dependent electromagnetic field.

For a time-dependent harmonically boundcharged particle of charge q and massmsubject to a time-dependent electrornagnetic field $\vec{E}(t)$ and $\vec{B}(t)$ (along the z direction), the Lagrangian has the form

$$L(\vec{r},\dot{\vec{r}},t) = L_{\parallel}(z,\dot{z},t) + L_{\perp}(\vec{r}_{\perp},\dot{\vec{r}}_{\perp},t) \qquad (1)$$

with

$$L_{\parallel}(z,\dot{z}.t) = \frac{m}{2}\left[\dot{z}^2-\omega_z^2(t)z^2\right]+qE_{\parallel}(t)z \qquad (2)$$

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$$L_\perp(\vec{r}_\perp,\dot{\vec{r}}_\perp,t) = \frac{m}{2}\{\dot{\vec{r}}_\perp^2 - [\omega_x^2(t)x^2 + \omega_y^2(t)y^2] + \omega(t)(x\dot{y}-y\dot{x})\} + q\vec{E}_\perp\cdot\vec{r}_\perp \quad (3)$$

where $\omega(t) = qB(t)/mc$ is the cyclotron frequency, $\vec{r}_\perp$ and $\vec{E}_\perp(t)$ denote the components of $\vec{r}$ and $\vec{E}(t)$ perpendicular to $\vec{B}(t)$. Here $\omega_x(t)$, $\omega_y(t)$ and $\omega_z(t)$ are, respectively, the oscillator frequencies along x, y and z directions. Since the z coordinate is separated from the $\vec{r}_\perp$ (x and y) coordinates in eq. (1), the propagator is of the form

$$K(\vec{r}'',t'';\vec{r}',t') = K_{\parallel}(z'',t'';z',t')K_\perp(\vec{r}_\perp'',t'';\vec{r}_\perp',t') \quad (4)$$

with

$$K_{\parallel}(z'',t'';z',t') = (m/2\pi i\hbar f(t'))^{1/2}\exp\{(-im/2\hbar f(t'))[z'^2\dot{f}(t') + 2z'z'' - z''^2\dot{g}(t'')]\}\exp\{(i/\hbar f(t'))[z'\int_{t'}^{t''}E_{\parallel}(t)f(t)dt + z''\int_{t'}^{t''}E_{\parallel}(t)g(t)dt - (1/m)\int_{t'}^{t''}\int_{t'}^{t}E_{\parallel}(t)f(t)E_{\parallel}(\theta)g(\theta)dtd\theta]\} \quad (5)$$

being the propagator[11] of a time-dependent harmonic oscillator. In eq.(5) the functions $f(t)$ and $g(t)$ satisfy the following differential equations

$$\ddot{f}(t) + \omega_z^2(t)f(t) = 0 \qquad f(t'') = 0 \quad \text{and} \quad \dot{f}(t'') = -1 \quad (6)$$

$$\ddot{g}(t) + \omega_z^2(t)g(t) = 0 \qquad g(t') = 0 \quad \text{and} \quad \dot{g}(t') = 1\,. \quad (7)$$

Now we are only left to evaluate the propagator of the Lagrangian (3), which will be carried out by using time-dependent linear coordinate transformation with new-time for a special case.

2. SPACE TRANSFORMATION AND TIME SUBSTITUTION

For the Lagrangian (3), the propagator can be expressed as the path integral

$$K_\perp(\vec{r}_\perp'',t'';\vec{r}_\perp',t') = \int\ldots\int\exp\{(i/\hbar)\int_{t'}^{t''}L_\perp(\vec{r}_\perp,\dot{\vec{r}}_\perp,t)dt\}Dx(t)Dy(t) \quad (8)$$

where $Dx(t)Dy(t)$ is the usual two-dimensional Feynman differential measure. Using Feynman's polygonal paths, the propagator (8) becomes

$$K_\perp(\vec{r}''_\perp,t'';\vec{r}'_\perp,t') = \lim_{\varepsilon_j\to 0}\left(\frac{m}{2\pi i\hbar\varepsilon_j}\right)^N \int\cdots\int \exp\{\frac{i}{\hbar}\sum_{j=1}^{N} S(\vec{r}_j,\vec{r}_{j-1};\varepsilon_j)\prod_{j=1}^{N-1} dx_j dy_j \tag{9}$$

with

$$S(\vec{r}_j,\vec{r}_{j-1};\varepsilon_j) = (m/2\varepsilon_j)\{(x_j-x_{j-1})^2+(y_j-y_{j-1})^2+(\varepsilon_j\omega_j/2)\left[(x_j+x_{j-1})(y_j-y_{j-1})\right.$$

$$\left. - (y_j+y_{j-1})(x_j-x_{j-1})\right]-\varepsilon_j^2(\omega_{xj}^2x_j^2+\omega_{yj}^2y_j^2)\} + q\varepsilon_j(E_{xj}x_j+E_{yj}y_j) \tag{10}$$

For later convenience we set $\varepsilon_j=t_j-t_{j-1}$ and $F_j=F(t_j)$ and $\bar{F}_j=F(\bar{t}_j)$ $(\bar{t}_j = (t_j+t_{j-1})/2)$ for any function $F(t)$.

Introducing the time-dependent linear transfarmations of space

$$x(t) = s_x(t)X(\tau) \quad , \quad y(t) = s_y(t)Y(\tau) \tag{11}$$

and the time substitution

$$d\tau = u(t)dt \quad , \tag{12}$$

we obtain the following relations[10]

$$x_j-x_{j-1} = s_{xj}\Delta X_j + \varepsilon_j\dot{\bar{s}}_{xj}\bar{X}_j \quad , \quad y_j-y_{j-1} = s_{yj}\Delta Y_j + \varepsilon_j\dot{\bar{s}}_{yj}\bar{Y}_j \quad , \tag{13}$$

by expanding eq.(11) about the mid-point $\bar{t}_j$ in the time interval $[t_{j-1},t_j]$ to terms of order ε_j. Here we have let $\Delta X_j = X(\tau_j) - X(\tau_{j-1})$, $\bar{X}_j = (X(\tau_j)+X(\tau_{j-1}))/2$ and similar for ΔY_j and $\bar{Y}_j$. Substituting eq.(13) into eq.(10), we have

$$S(\vec{r}_j,\vec{r}_{j-1};\varepsilon_j) = \Delta G_{j,j-1} + (m/2\varepsilon_j)\{s_{xj}^2(\Delta X_j)^2 + s_{yj}^2(\Delta Y_j)^2$$

$$+ \varepsilon_j\bar{\omega}_j\bar{s}_{xj}\bar{s}_{yj}(\bar{X}_j\Delta Y_j-\bar{Y}_j\Delta X_j) + \varepsilon_j\bar{\omega}_j\bar{X}_j\bar{Y}_j(\bar{s}_{xj}\dot{\bar{s}}_{yj}-\dot{\bar{s}}_{xj}\bar{s}_{yj}) - \varepsilon_j^2\left[\bar{s}_{xj}(\ddot{\bar{s}}_{xj}\right.$$

$$\left.+ \omega_{xj}^2\bar{s}_{xj})\bar{X}_j^2 + \bar{s}_{yj}(\ddot{\bar{s}}_{yj}+\omega_{yj}^2\bar{s}_{yj})\bar{Y}_j^2\right]\} + q\varepsilon_j(\bar{s}_{xj}\bar{E}_{xj}\bar{X}_j+\bar{s}_{yj}\bar{E}_{yj}\bar{Y}_j) \tag{14}$$

with

$$\Delta G_{j,j-1} = \left\{\left(\frac{\dot{\bar{s}}_{xj}x_j^2}{2\bar{s}_{xj}} + \frac{\dot{\bar{s}}_{yj}y_j^2}{2\bar{s}_{yj}}\right) - \left(\frac{\dot{\bar{s}}_{xj-1}x_{j-1}^2}{2\bar{s}_{xj-1}} + \frac{\dot{\bar{s}}_{yj-1}y_{j-1}^2}{2\bar{s}_{yj-1}}\right)\right\} \qquad (15)$$

after simplifications.

In order to get rid of the $\bar{X}_j\bar{Y}_j$ term in eq. (14), we must let

$$\bar{s}_{xj}\dot{\bar{s}}_{yj} - \dot{\bar{s}}_{xj}\bar{s}_{yj} = 0 \quad . \qquad (16)$$

We now choose

$$\ddot{\bar{s}}_{xj} + \omega_{xj}^2\bar{s}_{xj} = 0 \quad \text{and} \quad \ddot{\bar{s}}_{yj} + \omega_{yj}^2\bar{s}_{yj} = 0 \quad . \qquad (17)$$

In order to satisfy both eq. (16) and eq. (17), we have to consider the following special case hereafter:

$$\bar{s}_j = \bar{s}_{xj} = \bar{s}_{yj} \quad , \quad \Omega_j^2 = \omega_{xj}^2 = \omega_{yj}^2 \qquad (18)$$

Using the time substitution eq.(12) or $\sigma_j = \tau_j - \tau_{j-1} = \bar{u}_j\varepsilon_j$, we obtain from eqs. (14) and (17)

$$S(\vec{r}_j,\vec{r}_{j-1};\sigma_j) = \Delta G_{j,j-1} + (m\bar{u}_j\bar{s}_j^2/2\sigma_j)\{(\Delta X_j)^2 + (\Delta Y_j)^2 + \sigma_j\bar{\omega}_j(\bar{X}_j\Delta Y_j - \bar{Y}_j\Delta X_j)/\bar{u}_j\} + q\bar{s}_j\sigma_j(\bar{E}_{xj}\bar{X}_j + \bar{E}_{yj}\bar{Y}_j)/\bar{u}_j \quad . \qquad (19)$$

Choosing

$$\bar{s}_j^{-2}\bar{u}_j = 1 \quad , \quad \bar{s}_j^{-2}\bar{\omega}_j = \omega_0 \qquad (20)$$

with ω being a constant, eq. (19) becomes

$$S(\vec{r}_j,\vec{r}_{j-1};\sigma_j) = \Delta G_{j,j-1} + (m/2\sigma_j)\ \{(\Delta X_j)^2 + (\Delta Y_j)^2 + \sigma_j\omega_0(\bar{X}_j\Delta Y_j - \bar{Y}_j\Delta X_j)\} + q\bar{s}_j^{3}\sigma_j(\bar{E}_{xj}\bar{X}_j + \bar{E}_{yj}\bar{Y}_j) \quad . \qquad (21)$$

Using eqs. (11), (12) and (18), the Feynman path differential measure is given

$$(m/2\pi i\hbar\varepsilon_j)^N \prod_{j=1}^{N-1} dx_j dy_j = (s's'')^{-1} (m/2\pi i\hbar\sigma_j)^N \prod_{j=1}^{N-1} dX_j dY_j \qquad (22)$$

by symmetrizing about the end points in the time interval $[t_{j-1},t_j]$. Combining eq.(19) with eq.(22), we obtain our principal result

$$K_\perp(\vec{r}_\perp'',t'';\vec{r}_\perp',t') = (s's'')^{-1} G_s(\vec{r}_\perp',\vec{r}_\perp'') K(\vec{R}_\perp'',\tau'';\vec{R}_\perp',\tau') \qquad (\vec{R}_\perp = (X,Y)) \qquad (23)$$

with

$$G_s(\vec{r}_\perp',\vec{r}_\perp'') = \exp\{-(im/2\hbar) \left[(\dot{s}''/s'')(x''^2+y''^2) - (\dot{s}'/s')(x'^2+y'^2)\right]\} \qquad (24)$$

where $K(R_\perp'',\tau'';R_\perp',\tau')$ is the propagator of a charged particle in a constant magnetic field with the cyclotron frequency ω_0 and in a time-dependent electric field $\vec{\varepsilon}(\tau) = \vec{E}(t(\tau))s^3(t(\tau))$, which has been evaluated by us[13].

Without loss of generality we now consider the case of $E_x(t)=0$ or $\varepsilon(\tau) = \varepsilon_y(\tau)$ hereafter. We then have[13] $(T = \tau'' - \tau')$

$$\begin{aligned} K(R_\perp'',\tau'';R_\perp',\tau') &= (m/2\pi i\hbar T)(\omega_0 T/2 \sin(\omega_0 T/2)) \\ &\times \exp\{(im\omega_0/2\hbar)([\cot(\omega_0 T/2)/2][(X''-X')^2+(Y''-Y')^2] + (X'Y''-X''Y'))\} \\ &\times \exp\{(iq/\hbar \sin(\omega_0 T))[(Y''\varepsilon_b + Y'\varepsilon_a) - (q/m\omega_0)\varepsilon_0]\} \\ &\times \exp\{(iqm\omega_0/4\hbar \tan(\omega_0 T/2))\varepsilon_{ab}(q\varepsilon_{ab} - 2[(X'-X'')H + (Y'+Y'')\tan(\omega_0 T/2)])\} \end{aligned} \qquad (25)$$

with

$$\begin{aligned} \varepsilon_a &= \int_{\tau'}^{\tau''} \varepsilon_y(\tau)\sin[\omega_0(\tau''-\tau)]d\tau \ , \quad \varepsilon_b = \int_{\tau'}^{\tau''} \varepsilon_y(\tau)\sin[\omega_0(\tau-\tau')]d\tau \\ \varepsilon_0 &= \int_{\tau'}^{\tau''} \varepsilon_y(\tau)\sin[\omega_0(\tau''-\tau)]d\tau \int_{\tau'}^{\tau} \varepsilon_y(\theta)\sin[\omega_0(\theta-\tau')]d\theta \ , \end{aligned} \qquad (26)$$

$\varepsilon_{ab} = (\varepsilon_a+\varepsilon_b)/m \sin(\omega_0 T)$ and $H = 1 - 2\tan(\omega_0 T/2)/\omega_0 T$.

Combining eq. (5) with eq. (23) we have the propagator eq. (4) as our final result. As a final remark we should mention that since $s(t)=s_x(t) = s_Y(t)$ and $\Omega(t) = \omega_x(t) = w_Y(t)$, we have in the continuum case

$$s^2(t)\omega(t) = \omega_0, \quad s^2(t)u(t) = 1 \quad \text{and} \quad \ddot{s}(t) + \Omega^2(t)s(t) = 0 \tag{27}$$

as we expect[5,6]. Unfortunately, the present method can not be applied to the case of $\omega_x(t) \neq w_y(t)$, which will be studied in the near future.

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Resumo

Achando a transformação linear da coordenada dependente do tempo e a substituição do tempo, podemos calcular exatamente o propagador para uma particular carregada, no campo eletromagnético dependente do tempo, e com um potencial quadrático também dependente do tempo.