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A Matricial Approach for the Dirac-Kahler Formalism

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Abrtract A matricial approach for the Dirac-Kahler for;malism is considered. It is shown that the matricial approach i) brings a great computational simplification compared to the common use of differential forms and that ii) by an appropriate choice of notation, it can be extended to the lattice, including a matrix Dirac-Kahler equation.

1. INTRODUCTION

The Dirac-Kahler formalism has been investigated by some authors. The most extensive study on the subject has appeared in the work of Becher and Joos[1] where the formalism is discussed in the light of differential forms with very little attention to the matricial approach. In addition, the study of the Dirac-Kahler formalism on the lattice has not been considered in the matricial approach.

In this paper we intend to review the Dirac-Kahler formalism according to the matricial approach. It will be shown that this approach provides a great computational simplification. In particular, it is very useful in obtaining proofs and other results where long and tedious calculations are required by the differential form approach.

The Dirac-Kahler differential formalism, through the equivalence between the Dirac-Kahler equation and the usual Dirac equation, has assumed relevance by allowing the study of fermion fields using antisymmetrical tensors associated to differential forms[2,3]. But this consequence has become more important because the lattice fermion energy degeneracy[1,4,5] can be solved in a natural way in the lattice Dirac-Kahler formalism.

The lattice field theory has become very important mainly after applications to QCD[6], which offer many important results, as for example, quark confinement, through Monte Carlo simulations[7].

It is believed that the Dirac-Kahler formalism offers a proper mathematical structure for the lattice field theory developments.

There is a point to be mentioned: when we demonstrate the equivalente between the usual Dirac and the Dirac-Kahler equations, this equivalence is true only after the minimal left ideal reduction of the differential form space. This is because the four dimensional differential form space is equivalent to the 4×4 matrix space, and the minimal left ideal reduction corresponds to the column matrix reduction of the matrix space, each column representing a Dirac spinor.

We know that a general differential form defined in a four-dimensional space has 32 degrees of freedom, while a Dirac spinor has 8. However, the matrix equivalent to a general differential form has exactly the same 32 degrees of freedom, hence there is the necessíty tomake a reduction if we intend to represent a Dirac spinor by a differential form: Otherwise, we can profit from this degree of freedom redundance, as for example in the superfield formalism[8], where we may represent the N=8 extended supersymmetric charges by 4×4 complex matrices. This suggests that we can build the N=8 superfield on superspace where the spinor coordinates are elements of the 4×4 complex matrix. Actually, one purpose of this work, exalting the matrix representation, is just to prepare for the study of this possibility. We can foresee that from chiral reductions we can obtain the several N=8 supersymmetric representations, at same time that, from fermionic variables reduction, calculations may become feasible.

In the study of matrix representation, the possibility has appeared to extend it on the lattice, which hasn't been done at present because the lattice basis elements don't satisfy the Clifford algebra. However, we can extend the matrix representation to the lattice if we define the exterior product and contraction to act on Dirac matrices in much the same way as they act on differential forms. While it makes obscure the field geometric interpretation[9], there is simplification for calculations.

The plan of this paper is as follows: in section 2 the mathematical preliminaries are given, where definitions and notations are established. Section 3 is concerned with the study of the continuum Dirac-

Kahler formalism in a matricial approach. The Lorentz transformations, the scalar product and currents are also considered. In section 4, the matricial approach for the Dirac-Kahler formalism is extended to the lattice, and the matrix Dirac-Kahler field equations on the lattice are obtained.

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2. MATHEMATICAL PRELIMINARES

Let us consider the four dimensional Minkowski space with the metric $g^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. For the Dirac matrices, which as we know, satisfy the Clifford algebra

$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \tag{2.1}$$

and generate the 16 dimensional Clifford space, we shall use the Weyl representation. A suitable choice for the basis of this Clifford space is

$$\{\Gamma^H\} = \{1,\gamma^0,\gamma^i,\gamma^{\mu_1}\gamma^{\mu_2},\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3},\ldots \tag{2.2}$$

and so

$$\{\Gamma_H\} = \{1,\gamma_0,\gamma_i,\gamma_{\mu_2}\gamma_{\mu_1},\gamma_{\mu_3}\gamma_{\mu_2}\gamma_{\mu_1},\ldots \tag{2.3}$$

with

$$\mu_1 < \mu_2 < \mu_3 < \ldots$$

satisfy the orthogonality and completeness relations, respectively

$$\mathrm{tr}(\Gamma^H\Gamma_K) = 4\delta_{HK} \tag{2.4}$$

and

$$\sum_H (\Gamma^H)_{ab}(\Gamma_H)_{cd} = 4\ \delta_{cb}\ \delta_{ad} \tag{2.5}$$

Thus any 4×4 matrix, ψ, can be expanded in terms of Γ^H as

$$\psi = \sum_H \psi_H \Gamma^H = \sum_H \psi^H \Gamma_H \quad , \tag{2.6}$$

where

$$\psi_H = \frac{1}{4}\, tr(\psi\Gamma_H)$$

and (2.7)

$$\psi^H = \frac{1}{4}\, tr(\psi\Gamma^H)$$

Any differential form can be written as

$$\phi(x) = \sum_H \phi_H(x)dx^H = \sum_H \phi^H(x)dx_H \ , \tag{2.8}$$

where the bases $\{dx^H\}$ and $\{dx_H\}$ are defined as

$$\{dx^H\} = \{1,\ dx^\mu,\ dx^{\mu_1}\wedge dx^{\mu_2}\ ,\ \ldots\ \}$$

and (2.9)

$$\{dx_H\} = \{1,\ dx_\mu,\ dx_{\mu_2}\wedge dx_{\mu_1}\ ,\ \ldots$$

The coefficients $\phi_H(x)$ and $\phi^H(x)$ are given by

$$\{\phi_H\} = \{{}^0\phi, \phi_\mu, \phi_{\mu_1\mu_2}, \phi_{\mu_1\mu_2\mu_3}, \ldots$$

and (2.10)

$$\{\phi^H\} = \{{}^0\phi, \phi^\mu, \phi^{\mu_2\mu_1}, \phi^{\mu_3\mu_2\mu_1}, \ldots\ \}$$

with $\mu_1 < \mu_2 < \mu_3 < \ldots$. If we introduce the symbol ε_{HK}, which is similar to a truncated completely antisymmetric Levi-Civita tensor with the value +1 for $\varepsilon_{\mu_1\mu_2\mu_3}\ldots$ when $\mu_1 < \mu_2 < \mu_3 < \ldots$ and inverting the sign for each permutation between two of the indices, we can define the exterior product and contraction as

$$dx^\mu \wedge dx^H = \varepsilon_{\mu,H}\ \ dx^{H\cup\mu}$$

and (2.11)

$$e_\mu \lrcorner\ dx^H \equiv dx_\mu \lrcorner\ dx^H = \varepsilon_{\mu,H\Delta\mu}\ \ dx^{H\setminus\mu},$$

respectively. Here, H and K are ordered sets of indices and $H\Delta K$ is the symmetrical difference which contains the elements of H and K that do not appear simultaneously in H and K. We also use the other familiar set theory operations.

Because the elements (indices) of H and K are ordered, then for H = { } or $K = \{\ \}$, we have

$$\varepsilon_{\phi,K} = \varepsilon_{H,\phi} = +1 \quad .$$

Also, the contraction with the upper indices gives

$$dx^\mu \lrcorner\, dx^H = e^\mu \lrcorner\, dx^H = g^{\mu\nu}\, e_\nu \lrcorner\, dx^H$$
$$= g^{\mu\nu}\, \varepsilon_{\nu,H\Delta\nu}\, dx^{H\setminus\nu} \quad , \tag{2.12}$$

which obviously satisfies the usual definition

$$e^\mu \lrcorner\, dx^\nu = g^{\mu\nu} \quad . \tag{2.13}$$

With the exterior product and contraction, let us define the Clifford product

$$dx^\mu \vee dx^H = dx^\mu \wedge dx^H + dx^\mu \lrcorner\, dx^H \quad . \tag{2.14}$$

It is easy to see that the elements dx^μ provided with the Clifford product, $\vee$, generate the Clifford algebra

$$dx^\mu \vee dx^\nu + dx^\nu \vee dx^\mu = 2g^{\mu\nu} \quad ,$$

in a similar way to the Dirac matrices, γ^μ (with the usual matrix multiplication).

At this point, it is convenient to define matrix operations analogous to the differential form operations. More details of this algebraic structure in the abstract vector space, referred to as the Kahler-Atiyah algebra, can be seen in the works of W. Graf[2] and Benn and Tucker[3]. With this purpose, we define the matrix exterior product

$$\gamma^{\mu} \wedge \Gamma^{H} = \varepsilon_{\mu,H}\, \Gamma^{H\cup\mu} \tag{2.15}$$

and contraction

$$\gamma_{\mu} \lrcorner\, \Gamma^{H} = \varepsilon_{\mu,H\Delta\mu}\, \Gamma^{H\setminus\mu} \tag{2.16}$$

The usual matrix product is given by the sum

$$\gamma^{\mu}\Gamma^{H} = \gamma^{\mu} \wedge \Gamma^{H} + \gamma^{\mu} \lrcorner\, \Gamma^{H} \quad , \tag{2.17}$$

which is equivalent to the differential form Clifford product.

The suitable arrangement for the bases

$$\Gamma^{H} = \gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3} \ldots$$

and

$$\Gamma_{H} = \ldots\, \gamma_{\mu_3}\gamma_{\mu_2}\gamma_{\mu_1} \;, \quad \mu_1 < \mu_2 < \mu_3 < \ldots$$

leads us to define the right matrix exterior product and contraction, re-spectively

$$\Gamma_{H} \wedge \gamma_{\mu} = \varepsilon_{\mu,H}\, \Gamma_{H\cup\mu} \tag{2.18}$$

and

$$\Gamma_{H} \llcorner\, \gamma_{\mu} = \varepsilon_{\mu,H\Delta\mu}\, \Gamma_{H\setminus\mu} \tag{2.19}$$

with the matrix product

$$\Gamma_{H}\gamma_{\mu} = \Gamma_{H} \wedge \gamma_{\mu} + \Gamma_{H} \llcorner\, \gamma_{\mu} \quad . \tag{2.20}$$

Having introduced these mathematical tools, let us see how the matricial approach can simplify rnatters.

3. THE CONTINUUM DIRACXAHLER FORMALISM

In this section we review some aspects of the matricial version of the continuum Dirac-Kahler formalism as introduced by Becher and Joos[1], whose framework we will follow.

The Dirac-Kahler field equations written in terms of differential forms are given by

$$(d-\delta)\phi = 0 \tag{3.1}$$

where

$$\phi(x) = \sum_H \phi_H(x) dx^H = \sum_H \phi^H(x) dx_H \tag{3.2}$$

is a general differential form, which contains

$$2 \cdot \sum_{n=0}^{4} \binom{4}{n} = 32 \tag{3.3}$$

degrees of freedom. The Dirac-Kahler operator is defined by the exterior differentiation

$$d = dx^\mu \wedge \partial_\mu \tag{3.4}$$

and its adjoint

$$\delta = -\star^{-1} d\star = - dx^\mu \lrcorner\, \partial_\mu \quad . \tag{3.5}$$

Thus

$$(d-\delta) = dx^\mu \wedge \partial_\mu + dx^\mu \lrcorner\, \partial_\mu = dx^\mu \vee \partial_\mu \tag{3.6}$$

The Hodge star operator $\star$ acts on the basis elements of the differential forms space, and the definition given by Becher and Joos[1] is enough for our purposes.

In some sense, apart from a higher number in the degrees of freedom, the differential Dirac-Kahler equation is equivalent to the usual Dirac equation. To show this equivalence, we use the auxiliary basis functions Z_{ab} which connect the differential form vector spaceand the matrix vector space,

$$Z = \sum_H (\Gamma_H)^T dx^H = \sum_H (\Gamma^H)^T dx_H \tag{3.7}$$

such that the general differential form (3.2) can be written as

$$\phi(x) = \sum_H \phi_H(x) dx^H = \sum_{a,b} \frac{1}{4} \psi_{ab}(x) Z_{ab} \tag{3.8}$$

Notice that the auxiliar basis function Z cannot be seen as a matrix while basis of differential form space, but as a set of differential forms Z_{ab} defined on each matrix component.

From the orthogonality eq. (2.4), we obtain

$$dx^H = \frac{1}{4}\,\mathrm{tr}\left[(\Gamma^H)^T Z\right] = \frac{1}{4}\sum_{a,b}\Gamma^H_{ab} Z_{ab} \tag{3.9}$$

which inserted into eq. (3.8) results

$$\psi(x) = \sum_H \phi_H(x)\;\Gamma^H \tag{3.10}$$

where

$$\phi_H(x) = \frac{1}{4}\,\mathrm{tr}\left[\Gamma_H\;\psi(x)\right] \quad . \tag{3.11}$$

Notice the similarity between the matricial expansion and the general differential form (3.2). Here ψ is a 4×4 matrix with the same degreesof freedom as the general differential form, Suppose that the 4×4 matrix $\psi(x)$ is equivalent to the differential form $\phi(x)$; from now on we will denote this equivalence by

$$\phi(x) \sim \psi(x) \quad . \tag{3.12}$$

Fixing H and defining $\phi_H = 1$ in eq.(3.10), we obtain the equivalence

$$dx^H \sim \Gamma^H \tag{3.13}$$

and also From earlier definitions,

$$\begin{aligned} dx^\mu \wedge dx^H &\sim \gamma^\mu \wedge \Gamma^H \quad , \\ dx^\mu \lrcorner\; dx^H &\sim \gamma^\mu \lrcorner\; \Gamma^H \quad , \\ dx^\mu \vee dx^H &\sim \gamma^\mu\, \Gamma^H \quad . \end{aligned} \tag{3.14}$$

Using these equivalence in the Dirac-Kahler eq. (3.1), we obtain

$$(d-\delta)\phi = (dx^\mu \vee \partial_\mu)\phi = \sum_H \partial_\mu\phi_H dx^\mu \vee dx^H \sim$$
$$\sim \sum_H \partial_\mu\phi_H\gamma^\mu\Gamma^H = \sum_H \gamma^\mu\partial_\mu\phi_H\Gamma^H = \gamma^\mu\partial_\mu\psi \quad ,$$

which shows the equivalence between the Dirac matrix equation and the Dirac-Kahler differential form equation

$$(d-\delta)\phi = 0 \sim \gamma^\mu\partial_\mu\psi = 0 \quad . \tag{3.15}$$

Including a mass term, we have

$$(d-\delta+im)\phi = 0 \sim (i\gamma^\mu\partial_\mu - m)\psi = 0 \quad . \tag{3.16}$$

Notice that the equivalence (3.12) and (3.15) are between differential forms and 4×4 matrices. This is easy to understand when we take into account that the differential form space difined by the 16 dimensional basis $\{dx^H\}$ is a Clifford space analogous to the 16 dimensional matrix space defined by the basis $\{\Gamma^H\}$.

If we wish to take into account a one-to-one equivalence between the Dirac spinors and differential forms, we must take the minimal left ideals reduced differential form space, equivalent to the column matrix space. So, the general differential form and its 4×4 matrix equivalent contain four Dirac spinors, to which we refer as the spinor four multiplicity representation.

3.1 - The Lorentz transformations

In this subsection we will study the properties of the Lorentz transformations** of the matrix ψ which isequivalent to the differential form ϕ. Actually, we transpose to matrix formalism the treatment which has been done by Becher and Joos[1] on the subject in the differential form formalism.

The coordinate transformations are

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu \quad , \tag{3.17}$$

where, due to the metric invariance, we have

$$\Lambda_\mu{}^\nu = (\Lambda^{-1})^\nu{}_\mu \quad . \tag{3.18}$$

By construction, the differential form is a Lorentz scalar,

$$\phi'(x') = \phi(x) \quad , \tag{3.19}$$

thus

$$\begin{aligned} \phi'(x) &= {}^0\phi(\Lambda^{-1}x) + \Lambda_\mu{}^\nu\phi_\nu(\Lambda^{-1}x)dx \\ &+ \frac{1}{2}\Lambda_\mu{}^\rho\Lambda_\nu{}^\sigma\phi_{\rho\sigma}(\Lambda^{-1}x)dx^\mu \wedge dx^\nu + \ldots \\ &= {}^0\phi(\Lambda^{-1}x) + \phi_\nu(\Lambda^{-1}x)(\Lambda^{-1})^\nu{}_\mu dx^\mu + \\ &+ \frac{1}{2}\phi_{\rho\sigma}(\Lambda^{-1}x)(\Lambda^{-1})^\rho{}_\mu dx^\mu \wedge (\Lambda^{-1})^\sigma{}_\mu dx^\nu + \ldots \\ &= \phi(\Lambda^{-1}x) \quad , \end{aligned} \tag{3.20}$$

which is equivalent to the matrix expansion

$$\begin{aligned} \psi'(x) &= {}^0\phi(\Lambda^{-1}x) + \phi_\nu(\Lambda^{-1}x)(\Lambda^{-1})^\nu{}_\mu\gamma^\mu + \\ &+ \frac{1}{2}\phi_{\rho\sigma}(\Lambda^{-1}x)(\Lambda^{-1})^\rho{}_\mu\gamma^\mu \wedge (\Lambda^{-1})^\sigma{}_\nu\gamma^\nu + \ldots \end{aligned} \tag{3.21}$$

Using the equality

$$\Lambda^\mu{}_\nu\gamma^\nu = S^{-1}\gamma^\mu S \quad , \tag{3.22}$$

we obtain

$$\psi'(x) = S\psi(\Lambda^{-1}x)S^{-1} \quad . \tag{3.23}$$

Although eq. (3.12) shows an equivalence under Lorentz transformations, the differential forms and the matrices transform in different ways, see eqs. (3.20) and (3.23). This is so because in the change of the coordinates, the elements dx_μ transform as

$$dx_\mu \rightarrow dx'_\mu = \Lambda_\mu{}^\nu \, dx_\nu \quad , \tag{3.24}$$

while the Dirac matrices γ^μ are chosen to remain in the same representation.

The left matrix transformation

$$\psi'(x') = S\psi(x) \tag{3.25}$$

acts on a matrix row,

$$(S\psi)_{ab} = S_{ac}\,\psi_{cb} \quad , \tag{3.26}$$

corresponding to a spinor transformation if we consider the 4×4 matrix $\psi(x)$ containing four Dirac spinors in its four columns, each column equivalent to the minimal left ideal decomposed differential form. Now, the right matrix multiplication is a *flavor* transformation which mixes the four columns, that is, the minimal left ideal subspaces,

$$(\psi S^{-1})_{ab} = \psi_{ac}\, S^{-1}_{cb} = (S^{-1})^T_{bc}\, \psi_{ac}. \tag{3.27}$$

In this sense, the Lorentz coordinate transformations induce internal transformations on the *flavor* space. If we consider the Dirac space as element of some reduced minimal left ideal, the *flavor* transformation is of no matter. On the other hand, if we consider the four multiplicities, we will have an internal simmetry transformation, because the right matrix multiplication does not affect the Dirac-Kahler field equation. Nevertheless, it is not needed that each column should represent precisely a Dirac spinor if we take into account the whole Lorentz transformations. That is, we could represent Dirac spinors without making the minimal left ideal decomposition. For example, we could make the four left and right chiral decompositions.

3.2 - Scalar products and currents

The matrix representation of differential forms is very useful for practical calculations. Here we will express the scalar productsand the conserved currents in a matrix notation. Let us consider two differential forms,

$$\begin{aligned}\phi(x) &= \sum_H \phi_H(x) dx^H = \sum_H \phi^H(x) dx_H \\ &= {}^0\phi(x) + \phi_\mu(x) dx^\mu + \frac{1}{2}\phi_{\mu\nu}(x) dx^\mu \wedge dx^\nu + \dots\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}Z(x) &= \sum_H Z_H(x) dx^H = \sum_H Z^H(x) dx_H \\ &= {}^0\xi(x) + \xi_\mu(x) dx^\mu + \frac{1}{2}\xi_{\mu\nu}(x) dx^\mu \wedge dx^\nu + \dots\end{aligned} \tag{3.29}$$

with the matrix representations

$$\psi(x) = \sum_H \phi_H(x)\ \Gamma^H \tag{3.30}$$

and

$$\chi(x) = \sum_H Z_H(x)\ \Gamma^H \tag{3.31}$$

respectively.

The scalar component of the Clifford product between $\phi(x)$ and $Z(x)$ is

$$\begin{aligned}{}^0(\phi \vee Z) &= \sum_H \phi_H(x) Z^H(x) = {}^0\phi\,{}^0\xi + \phi_\mu \xi^\mu + \frac{1}{2}\phi_{\mu\nu}\xi^{\nu\mu} \\ &+ \frac{1}{3!}\phi_{\mu\nu\rho}\xi^{\rho\nu\mu} + \phi_{0123}\xi^{3210}\end{aligned} \tag{3.32}$$

When we use eq. (3.11), we have

$$\begin{aligned}{}^0(\phi \vee Z) &= \sum_{a,b,c,d}\sum_H \frac{1}{4}\psi_{ab}(x)\,\frac{1}{4}\chi_{cd}(x)\,(\Gamma_H)_{ab}(\Gamma^H)_{cd} \\ &= \frac{1}{4}\sum_{a,b}\psi_{ab}(x)\chi_{ba}(x) = \frac{1}{4}\,\mathrm{tr}\,\{\psi(x)\chi(x)\} \\ &= \frac{1}{4}\sum_{H,K}\phi_H(x)\ Z_K(x)\ \mathrm{tr}(\Gamma^H\Gamma^K)\end{aligned} \tag{3.33}$$

Fixing $\phi_H = Z_K = 1$ in the above sum, we have

$$ {}^{0}(dx^{H} \vee dx^{K}) = \frac{1}{4}\,\mathrm{tr}(\Gamma^{H}\Gamma^{K}) \;, \tag{3.34} $$

or, if $\phi_{H} = Z^{K} = 1$,

$$ (dx^{H} \vee dx_{K}) = \frac{1}{4}\,\mathrm{tr}(\Gamma^{H}\Gamma_{K}) = \delta_{HK} \;, \tag{3.35} $$

and so,

$$ {}^{0}(dx \;\vee dx^{H} \vee dx^{K}) = \frac{1}{4}\,\mathrm{tr}(\gamma^{\mu}\Gamma^{H}\Gamma^{K}) \;. \tag{3.36} $$

The generalization of eq. (3.36) is straightforward.

The Lorentz invariant scalar product can be conveniently defined as

$$ \begin{aligned} (\phi,Z)_{0} &= (\beta\phi \vee Z) \wedge \varepsilon = \frac{1}{4}\,\mathrm{tr}\{(\beta\psi)\chi\}\varepsilon \\ &= ({}^{0}\phi\,{}^{0}\xi + \phi_{\mu}\xi^{\mu} + \frac{1}{2}\,\phi_{\mu\nu}\xi^{\mu\nu} + \frac{1}{3!}\,\phi_{\mu\nu\rho}\xi^{\mu\nu\rho} + \phi_{0123}\xi^{0123})\varepsilon \end{aligned} \tag{3.37} $$

where

$$ \mathrm{E} = \&c^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} \tag{3.38} $$

This expression for the Lorentz scalar product has been introduced by E. Kahler in his original work[12]. He also introduced the generalized p--product $(\phi,Z)_{p}$.

The operator β^{1} is an anti-automorphism,

$$ \beta(dx^{H} \vee dx^{K}) = \beta dx^{K} \vee \beta dx^{H} \;, \tag{3.39} $$

so

$$ \beta(dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge \ldots) = \ldots dx^{\rho} \wedge dx^{\nu} \wedge dx^{\mu} \;. \tag{3.40} $$

In this work we define the anti-automorphism β to act in much the same way on the Dirac matrices γ^{μ},

$$ \beta(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\ldots) = \ldots\gamma^{\rho}\gamma^{\nu}\gamma^{\mu} \;. \tag{3.41} $$

The scalar product defined in eq. (3.37) is a 4-form. A general p-product $(\phi,Z)_p$ is defined to give a $(d-p)$-form, where d is the space-time dimension. (In our case, $d=4$.). We are considering here only the cases $p=0$, scalar product, and $p=1$, *vector* product. The latter is defined as follows

$$\begin{aligned}(\phi,Z)_1 &= e_\mu \lrcorner (dx^\mu \vee \phi, Z)_0 = e_\mu \lrcorner (\beta(dx^\mu \vee \phi) \vee Z) \quad \varepsilon \\ &= {}^0(\beta\phi \vee dx^\mu \vee Z)e_\mu \quad \varepsilon \qquad (3.42)\\ &= \frac{1}{4}\,\mathrm{tr}\{(\beta\psi)\gamma^\mu\chi\} e_\mu \lrcorner \varepsilon \quad .\end{aligned}$$

Let us consider the action of the Hodge star operator on the contraction,

$$\star^{-1}\, e_\mu \lrcorner \varepsilon = -dx_\mu \quad .$$

Then

$$\star^{-1}(\phi,Z)_1 = -\frac{1}{4}\,\mathrm{tr}\{(\beta\psi)\gamma^\mu\chi\}dx_\mu \quad . \qquad (3.43)$$

From this. we can define a current'

$$\begin{aligned} j &= j^\mu(x)dx_\mu = -\star^{-1}(\phi,Z)_1 \\ &= \frac{1}{4}\,\mathrm{tr}\left[(\beta\psi)\,\gamma^\mu\,\chi\right]dx_\mu \ , \qquad (3.44)\end{aligned}$$

with vector components

$$j^\mu(x) = \frac{1}{4}\,\mathrm{tr}\left[(\beta\psi)\gamma^\mu\chi\right] = {}^0(\beta\phi \vee dx^\mu \vee Z) \quad . \qquad (3.45)$$

The action of the adjoint operator 6, defined in eq.(3.5), on the currents gives

$$\begin{aligned}\delta j &= -e^\mu \lrcorner\, \partial_\mu j = -\partial_\mu j^\mu \\ &= -\star^{-1}d\star j = -\star^{-1}d(\phi,Z)_1 \ , \qquad (3.46)\end{aligned}$$

which, to be conserved, requires

$$d(\phi,Z)_1 = 0 \quad . \tag{3.47}$$

From eq.(3.42) we have

$$d(\phi,Z)_1 = dx^\mu \wedge \partial_\mu \frac{1}{4}\,\mathrm{tr}\{(\beta\psi)\gamma^\nu\chi\}e_\nu \lrcorner\, \varepsilon$$

$$= \frac{1}{4}\,\mathrm{tr}\ \partial_\mu\{(\beta\psi)\gamma^\nu\psi\}dx^\mu \wedge (e_\nu \lrcorner\, \varepsilon) \quad .$$

A simple inspection shows that

$$dx^\mu \wedge (e_\nu \lrcorner\, \varepsilon) = g^\mu{}_\nu \varepsilon \quad ,$$

hence

$$d(\phi,Z)_1 = \frac{1}{4}\,\mathrm{tr}\{\beta(\gamma^\mu\partial_\mu\psi)\chi + (\beta\psi)\gamma^\mu\partial_\mu\chi\}\varepsilon \quad . \tag{3.48}$$

Also, from the definition of the scalar product (3.37), we have

$$((d-\delta)\phi,Z)_0 = \frac{1}{4}\,\mathrm{tr}\{[\beta(\gamma^\mu\partial_\mu\psi)]\chi\}\varepsilon \tag{3.49}$$

and

$$(\phi,(d-\delta)Z)_0 = \frac{1}{4}\,\mathrm{tr}\{(\beta\psi)\gamma^\mu\partial_\mu\chi\}\varepsilon \ , \tag{3.50}$$

leading to the result

$$d(\phi,Z)_1 = ((d-\delta)\phi,Z)_0 + (\phi,(d-\delta)Z)_0 \ . \tag{3.51}$$

This is the so-called Green's formula. Hence, the condition for the current conservation demands, from eq. (3.51), that the fields $\phi(x)$ and $Z(x)$ both satisfy the Dirac-Kahler equation, This is a known result [2]; however the matricial approach has been shown to be very handy in obtaining it. As another example of the usefulness of the rnatricial approach let us consider the product

$$(\phi \vee Z) = \sum_{H,K} \phi_H(x) Z_K(x) dx^H \vee dx^K \quad .$$

The calculation of the Clifford product in differential forms is a very hard task, though with the aid of the matrix representation, we see that it is

$$(\phi \vee Z) = \sum_H (\phi \vee Z)_H dx^H = \frac{1}{4} \sum_H \mathrm{tr}\{\Gamma_H \psi \chi\} dx^H , \qquad (3.52)$$

which is a simpler expression.

Here, $\psi(x)$ and $\chi(x)$ are the matrix representations of the differential forms $\phi(x)$ and $Z(x)$, respectively. The trace is on the Dirac matrices products, and its evaluation is well known.

4. THE LATTICE DIRAC-KAHLER FORMALISM

For the sake of simplicity, we use the eartesian coordinate system, and we present some current fundamental definitions. For a complete and more precise definitions we refer the reader to Becher and Joos's work [1]. We must mention also that the lattice is better defined on Euclidian space-time.

The lattice analogous of the differential form (see eq. (3.2)) is the co-chain, which has the general form

$$\phi = \sum_x \phi(x) = \sum_{x,H} \phi(x,H) d^{x,H} , \qquad (4.1)$$

where the elements $d^{x,H}$ are the elementary co-chains, dual to the elementary chain $[x, H]$, and defined to satisfy

$$d^{x,H}[y,K] = \delta_{xy}\ \delta_{HK} , . \qquad (4.2)$$

just to allow the integration (sum) of the co-chains on the lattice space.

The lattice version of the Dirac-Kahler equation is

$$(\check{\Delta}-\check{\nabla})\phi \equiv d^{\mu} \vee \Delta^{-}_{\mu}\phi = 0 , \qquad (4.3)$$

where

$$\Delta^{+}_{\mu}\, \phi(x) = \frac{1}{e^{\mu}} \left[\phi(x+e^{\mu}) - \phi(x)\right] \qquad (4.4)$$

and

$$\Delta_\mu^- \phi(x) = \frac{1}{e^\mu}\left[\phi(x) - \phi(x-e^\mu)\right] \qquad (4.5)$$

are the *up* and *down* finite differences, both approaching the usual continuum derivative ∂^μ in the continuum limit, and the element d^μ is defined as the sum

$$d^\mu = \sum_x d^{x,\mu} \quad , \qquad (4.6)$$

or in general,

$$d^H = \sum_x d^{x,H} \quad . \qquad (4.7)$$

The symbol v means the Clifford product operator, defined as the sum of the exterior product and the contraction. The lattice exterior product is defined as a non-local operation by

$$d^{x,H} \wedge d^{y,K} = \varepsilon_{HK}\, \delta^{x+e^H,y}\, d^{x,H\cup K} \quad , \qquad (4.8)$$

which leads to

$$d^\mu \wedge d^{x,H} = \sum_y d^{y,\mu} \wedge d^{x,H} = \varepsilon_{\mu,H}\, d^{x-e^\mu,H\cup\mu} \qquad (4.9)$$

and the contraction by

$$e_\mu \lrcorner\, d^{x,H} = \varepsilon_{\mu,H\Delta\mu}\, d^{x,H\setminus\mu} \qquad (4.10)$$

From this, we have

$$(\overset{\vee}{\Delta}-\overset{\vee}{\nabla})\phi = d^\mu \vee \Delta_\mu^-\phi = \sum_{x,H} \Delta_\mu^- \phi(x,H) d^\mu \vee d^{x,H} \qquad (4.11)$$

$$= \sum_{x,H}\left[\varepsilon_{\mu,H\Delta\mu}\Delta_\mu^+ \phi(x,H\setminus\mu) + \varepsilon_{\mu,H}\Delta_\mu^- \phi(x,H\cup\mu)\right] d^{x,H}$$

where we have used

$$\Delta_\mu^- \phi(x+e^\mu) = \Delta_\mu^+ \phi(x) \qquad (4.12)$$

Due to the non locality of the exterior product the lattice elements d^μ together with the Clifford product v do not satisfy the

Clifford algebra. Thus we cannot define any lattice Dirac equation in matrix representation founded only on the usual matrix product. However, we have a way to introduce a rnatrix representation on the laticce: let us define the auxiliary lattice basis functions given by

$$Z^x = d^x + \gamma_\mu^T d^{x,\mu} + \frac{1}{2}\ \gamma_\mu^T \gamma_\nu^T\ d^{x,\mu\nu} + \ldots$$

$$= \sum_H\ (\Gamma_H)^T\ d^{x,H} \quad , \tag{4.13}$$

where

$$\{d^{x,H}\} = \{d^x, d^{x,\mu}, d^{x,\mu\nu}, \ldots\} \quad . \tag{4.14}$$

Here, we write

$$\phi(x,H) = \phi_H(x)$$

with the lower Lorentz indices as defined in eq. (2.10), and the lattice elements $d^{x,H}$ with the upper indices, analogous to the continuum elrnents dx^H defined in eq.(2.9).

After a sirnple manipulation, we obtain

$$d^{x,H} = \frac{1}{4}\ \mathrm{tr}(\Gamma^H)^T\ Z^x \tag{4.15}$$

and, in the new basis eq.(4.13).

$$\phi(x) = \sum_H\ \phi(x,H) d^{x,H} = \sum_{a,b} \frac{1}{4}\ \psi_{ab}(x) Z^x_{ab}$$

$$= \frac{1}{4} \sum_{a,b}\ \psi_{ab}(x)\,(\Gamma_H)_{ba}\ d^{x,H} \quad , \tag{4.16}$$

that is,

$$\phi(x,H) = \frac{1}{4}\ \mathrm{tr}\{\psi(x)\Gamma_H\} \tag{4.17}$$

On the other hand, by eq. (4.15),

$$\phi(x) = \sum_H\ \phi(x,H) d^{x,H} = \frac{1}{4} \sum_{a,b} \sum_H\ \phi(x,H)\,(\Gamma^H)_{ab}\ Z^x_{ab} \tag{4.18}$$

which when compared with eq. (4.16) above gives

$$\psi(x) = \sum_{H} \phi(x,H)\ \Gamma^{H} \tag{4.19}$$

This is the matrix representation of the co-chain.

Inserting these results into the lattice Dirac-Kahler eq. (4.3), we obtain

$$(\check{\Delta}-\check{\nabla})\phi = \sum_{x,H} \varepsilon_{\mu,H}\ \Delta^{+}_{\mu}\ \phi(x,H) d^{x,H\cup\mu} + \varepsilon_{\mu,H\Delta\mu}\ \Delta^{-}_{\mu}\ \phi(x,H) d^{x,H\backslash\mu}$$

$$= \sum_{x,H} \varepsilon_{\mu,H}\ \Delta^{+}_{\mu}\ \phi(x,H)\ \frac{1}{4}\ \Gamma^{H\cup\mu}_{ab} Z^{\ x}_{ab} \quad +$$

$$+\ \varepsilon_{\mu,H\Delta\mu} \Delta^{-}_{\mu}\ \phi(x,H)\ \frac{1}{4}\ \Gamma^{H\backslash\mu}_{ab}\ Z^{\ x}_{ab}$$

$$= \sum_{x,H} \frac{1}{4}\ [\Delta^{+}_{\mu}\ \phi(x,H)\ \ (\gamma^{\mu} \wedge \Gamma^{H})_{ab} +$$

$$+\ \Delta^{-}_{\mu}\ \phi(x,H)\ \ (\gamma^{\mu} \lrcorner\, \Gamma^{H})_{ab}]\ Z^{\ x}_{ab}$$

$$= \frac{1}{4} \sum_{x,H} [\gamma^{\mu} \wedge \Delta^{+}_{\mu}\ \ \psi(x) + \gamma^{\mu} \lrcorner\, \Delta^{-}_{\mu}\psi(x)]_{ab}\ Z^{\ x}_{ab}\ , \tag{4.20}$$

which gives the matricial Dirac-Kahler equation for the lattice,

$$\gamma^{\mu} \wedge\ \Delta^{+}_{\mu}\ \psi(x)\ +\ \gamma^{\mu} \lrcorner\ \Delta^{-}_{\mu}\ \psi(x)\ =\ 0\ , \tag{4.21}$$

where $\psi(x)$ is a 4×4 matrix. We can introduce the gauge interaction by making the substitution $\Delta^{\pm}_{\mu} \to D^{\pm}_{\mu}$, where $D^{\pm}_{\mu}$ are the gauge covariant derivatives. Then, with the gauge interaction, the matricial Dirac-Kahler equation for the lattice becomes

$$\gamma^{\mu} \wedge\ D^{+}_{\mu}\ \psi(x)\ +\ \gamma^{\mu} \lrcorner\ D^{-}_{\mu}\ \psi(x)\ =\ 0\ . \tag{4.22}$$

We notice that the gauge covariant derivatives as referred to above are well defined only in the adjoint representation[14], and their precise geometrical interpretations are considered by Aratyn and Zimerman[15].

4.1 - Scalar products and currents

Let us consider two general co-chains denoted as ϕ and Z, respectively,

$$\phi = \sum_{x,H} \phi(x,H) d^{x,H}$$

$$= \sum_{x} {}^{0}\phi(x) d^{x} + \phi_{\mu}(x) d^{x,\mu} + \frac{1}{2}\phi_{\mu\nu}(x) d^{x,\mu\nu} + \ldots \qquad (4.23)$$

and

$$Z = \sum_{x,H} Z(x,H) d^{x,H}$$

$$= \sum_{x} {}^{0}\xi(x) d^{x} + \xi_{\mu}(x) d^{x,\mu} + \frac{1}{2}\xi_{\mu\nu}(x) d^{x,\mu\nu} + \ldots \qquad (4.24)$$

We wish to define a scalar product between them in the usualway, that is, as a Lorentz invariant expression

$$(\phi,Z)_0 = {}^{0}\phi\,{}^{0}\xi + \phi_{\mu}\xi^{\mu} + \frac{1}{2}\phi_{\mu\nu}\xi^{\mu\nu} + \ldots , \qquad (4.25)$$

as in the continuum. In the lattice we cannot define the scalar product as $(\beta\phi \vee Z) \wedge \varepsilon$ because of the non-locality of the Clifford product. However, it can be well defined if we adopt the matrix representation directly

$$(\phi,Z)_0 = \frac{1}{4}\sum_{x} \mathrm{tr}\{(\beta\psi)\psi\}\ \varepsilon \quad , \qquad (4.26)$$

where $\psi(x)$ and $\chi(x)$ are the matrix equivalents of the co-chains ϕ and Z, respectively, with

$$\varepsilon = d^{x,1234} \qquad (4.27)$$

From this definition, we can see that the scalar product is symmetric in ϕ and Z,

$$(\phi,Z)_0 = (Z,\phi)_0 \quad . \qquad (4.28)$$

We wish to define a vector product $(\phi, Z)_1$, in order to obtain a simple expression for the conserved current. Notice that if we use the analogous expression on the lattice obtained from the continuurn,

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$$e_\mu \lrcorner \, (d^{x,\mu} \vee \phi, Z)_0 \quad ,$$

we do not have the syrnmetry between the ϕ and Z fields, due to the non locality of the Clifford product. Also, the direct use of the matrix representation, as in the scalar product (4.26), is not useful because we cannot show current conservation in a simple manner.

A better way to define the vector product $(\phi,Z)_1$ would be such that it were symnetric on the lattice as well. In the continuum, this symmetry is a consequence of the equalities

$$(dx^\mu \wedge \phi, Z)_0 = (\phi, e^\mu \lrcorner Z)_0 \tag{4.29}$$

and

$$(e^\mu \lrcorner \phi, Z)_0 = (\phi, dx^\mu \wedge Z)_0 \quad , \tag{4.30}$$

which give

$$(dx^\mu \vee \phi, Z)_0 = (\phi, dx^\mu \vee Z)_0 \; . \tag{4.31}$$

Because of these, we can define the vector product in the continuum as

$$(\phi,Z)_1 = e^\mu \lrcorner (dx^\mu \wedge \phi, Z)_0 + e^\mu \lrcorner (\phi, dx^\mu \wedge Z)_0 \tag{4.32}$$

or

$$(\phi,Z)_1 = e^\mu (e^\mu \lrcorner \phi, Z)_0 + e^\mu \lrcorner (\phi, e^\mu \lrcorner Z)_0 \quad , \tag{4.33}$$

where the symmetry between ϕ and Z fields appears explicitly. In the continuum, we have the equivalences

$$dx^\mu \wedge \phi(x) \sim \gamma^\mu \wedge \psi(x)$$

and

$$e^\mu \lrcorner \phi(x) \sim \gamma^\mu \lrcorner \psi(x) \quad ,$$

(see eq. (3.14)). On the lattice, we have

$$d^\mu \wedge \phi \sim \gamma^\mu \wedge \psi(x+e^\mu) \tag{4.34}$$

and

$$e^\mu \lrcorner \phi \sim \gamma^\mu \lrcorner \psi(x) \tag{4.35}$$

It is the non locality of the exterior product $d^{\mu}\wedge\phi$ that makes eqs. (4.29) and (4.30) fail on the lattice. To correct them, we can use the translation operator T_{μ} defined as

$$T_{\mu}d^{x,H} \equiv d^{x-e^{\mu},H} \quad , \tag{4.36}$$

which provides the equivalence

$$T_{-\mu}d^{\mu}\wedge\phi \sim \gamma^{\mu}\wedge\psi(x) \quad . \tag{4.37}$$

Thus

$$(T_{-\mu}d^{\mu}\wedge\phi,Z)_0 = (\phi,e^{\mu}\lrcorner Z)_0 \tag{4.38}$$

and

$$(e^{\mu}\lrcorner\phi,Z)_0 = (\phi,\ T_{-\mu}d^{\mu}\wedge Z)_0 \ . \tag{4.39}$$

The symmetric vector product $(\phi,Z)_1$ can be defined now as

$$(\phi,Z)_1 = T_{-\mu}\ e^{\mu}\ \lrcorner\ [(d^{\mu}\wedge\phi,Z)_0 + (\phi,d^{\mu}\wedge Z)_0] , \tag{4.40}$$

where the symmetry of ϕ and Z fields is evident.

From the definition of the scalar produc (4.26), we have

$$(\phi,Z)_1 = \frac{1}{4}\sum_x \mathrm{tr}\{\beta[\gamma^{\mu}\wedge\psi(x+e^{\mu})]\ .\ \chi(x)\ +$$

$$+\ \beta\psi(x)[\gamma^{\mu}\wedge\chi(x+e^{\mu})]\}T_{-\mu}e_{\mu}\lrcorner\,\varepsilon \tag{4.41}$$

If we apply the operator

$$\check{\Delta} \equiv d^{\mu}\wedge\Delta_{\mu}^{-}$$

to the left hand side of eq. (4.41) above, and after some manipulations, we get

$$\check{\Delta}(\phi,Z)_1 = d^\mu \wedge \Delta^-_\mu (\phi,Z)_1$$

$$= \frac{1}{4}\sum_x \mathrm{tr}\{\beta[\gamma^\mu \wedge \Delta^+_\mu \psi(x)]\ \chi(x) \ +$$

$$+ \beta[\gamma^\mu \wedge \psi(x)]\Delta^-_\mu \chi(x) + \beta\Delta^-_\mu \psi(x)[\gamma^\mu \wedge \chi(x)] +$$

$$\beta\psi(x)[\gamma^\mu \wedge \Delta^+_\mu \chi(x)]\}d^{x,1234} \tag{4.42}$$

Now the definition of the scalar product and the equivalences

$$\gamma^\mu \wedge \Delta^+_\mu \psi(x) \sim d^\mu \wedge \Delta^-_\mu \phi = \check{\Delta}\phi \ ,$$
$$\gamma^\mu \wedge \psi(x) \sim T_{-\mu} d^\mu \wedge \phi \ , \tag{4.43}$$

yield

$$\check{\Delta}(\phi,Z)_1 = (\check{\Delta}\phi,Z)_0 + (T_{-\mu}d^\mu \wedge \phi,\Delta^-_\mu Z)_0 + (\Delta^-_\mu\phi,T_{-\mu}d^\mu \wedge Z)_0 + (\phi,\check{\Delta}Z)_0$$

The equalities (4.39) and (4.40) give

$$(T_{-\mu}d^\mu \wedge \phi,\Delta^-_\mu Z)_0 = (\phi,e^\mu \lrcorner \Delta^-_\mu Z)_0 = -(\phi,\check{\Delta}Z)_0 \ ,$$

$$(\Delta^-_\mu \phi, T_{-\mu} d^\mu \wedge Z)_0 = (e^\mu \lrcorner \Delta^-_\mu \phi, Z)_0 = -(\check{\Delta}\phi,Z)_0 \ ,$$

and therefore

$$\check{\Delta}(\phi,Z)_1 = ((\check{\Delta}-\check{\nabla})\phi,Z)_0 + (\phi,(\check{\Delta}-\check{\nabla})Z)_0 \ . \tag{4.44}$$

This is the lattice version of the Green's formula (3.51).

The Hodge star operator on lattice is defined[1] by

$$\star d^{x,H} = \varepsilon_{H,CH}\ \beta d^{x+e^H,CH} \ ,$$
$$\star^{-1} d^{x,H} = \varepsilon_{CH,H}\ \beta d^{x-e^{CH},CH} \ , \tag{4.45}$$

where CH is the cornplementary set of H, which in the four dimensional space is

$$CH = H_5 \setminus H = \{1234\} \setminus H$$

Then

$$\star^{-1}(e^{\mu} \lrcorner\, \varepsilon) = d^{x-e^{\mu},\mu} \tag{4.46}$$

We can define the current as

$$j = \sum_{x} j_{\mu}(x) d^{x,\mu} = \star^{-1}(\phi, Z)_{1} \tag{4.47}$$

with components

$$j_{\mu}(x) = \frac{1}{4} \operatorname{tr}\{\beta[\gamma^{\mu} \wedge \psi(x+e^{\mu})] \chi(x) + + \beta\psi(x)[\gamma^{\mu} \wedge \chi(x+e^{\mu})]\} \quad . \tag{4.48}$$

The divergence of this current is

$$\overset{\vee}{\nabla} j = -e^{\mu} \lrcorner\, \Delta_{\mu}^{-} j = -\sum_{x} \Delta_{\mu}^{-} j_{\nu}(x)\, e^{\mu} \lrcorner\, d^{x,\nu} = - \sum_{x} \Delta_{\mu}^{-} j^{\mu}(x) d^{x} \quad , \tag{4.49}$$

or, in terms of the Hodge star operator,

$$\overset{\vee}{\nabla} j = \star^{-1} \overset{\vee}{\Delta} \star j = \star^{-1} \overset{\vee}{\Delta}(\phi, Z)_{1} \quad . \tag{4.50}$$

The current is conserved iff

$$\Delta_{\mu}^{-} j^{\mu}(x) = 0 \iff \overset{\vee}{\Delta}(\phi, Z)_{1} = 0 \quad , \tag{4.51}$$

that is, if the fields ϕ and Z both satisfy the Dirac-Kahler equation, as shown by Green's formula (4.44).

5. CONCLUSION

In the Dirac-Kahler formalism, the very hard task are mathematical calculations which involve Clifford products within differential forms space. To avoid this, the suggestion is to work with the matricial approach, which actually simplifies these caiculations, as we have shown in this paper.

One of the main results obtained in this paper is the introduction of the exterior product and contraction defined on the matrix space in analogy with these operations defined on the differential form space. Through these operations it is possible to obtain a matricial Dirac-Kahler equation on the lattice.

This work is an attempt to show the power fulness of the matricial Dirac-Kahler formalism and as such many more developments are needed in this direction.

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Resumo

Uma versão matricial do formalismo de Dirac-Kahler é apresentada. Pode-se mostrar que a versão matricial i) permite uma grande simplificado nos cálculos comparado com a versão diferencial e que ii) por uma escolha adequada de notaçao, pode ser estendida para a rede, inclusive com uma equação matricial de Dirac-Kahler.