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The Kustaanheimo-Stiefel Transformation in a Spinor Representation

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Abstract A spinor representation for the K-S transformation is derived by means of the Cartan spinor theory.

INTRODUCTION

It is well known that a regularization of the Kepler motion in the tridimensional space R^3 is developed by using a simple mapping of a four-dimensional space R^4 onto R^3. This simple mapping is known as the Kustaanheimo-Stiefel (K-S) transformation[1]. In R^4, the equations of the Kepler motion are linear differential equations with constant coefficients and remain completely regular at the center of attraction. These are equations of simple harmonic oscillator motions.

The K-S transformation is not a complete generalization of the two-dimensional Levi-Civita[2] transformation, but only a mapping of R^4 onto R^3 having the desired behaviour in order to get the regularization of the Kepler motion.

In this paper we use the Cartan spinor Theory[3] in order to get a spinor representation for the K-S transformation and reduce the equations of the Kepler motion in a spinor differential equation.

1. THE KS TRANSFORMATION

The K-S transformation relates a vector $\vec{x} \equiv (x_1, x_2, x_3) \in R^3$ to a vector $\vec{u} \equiv (u_1, u_2, u_3, u_4) \in R^4$ by the following relation

$$\binom{\vec{x}}{0} = A(\vec{u})\vec{u} \qquad (1.1)$$

where $A(\vec{u})$ is a 4x4 matrix

$$A(\vec{u}) = \left\| \begin{matrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 & -u_3 & u_2 & -u_1 \end{matrix} \right\| \qquad (1.2)$$

The main properties of this transformation are: column or line vectors of $A(u)$ are orthogonal to each other; $A^T(\vec{u})A(\vec{u}) = r 1_{4\times4}$; the length r of the position vector $\vec{x} \in R^3$ is given by

$$r = \left[\sum_{i=1}^{3} x_i^2\right]^{1/2} = \sum_{\alpha=1}^{4} u_\alpha^2 \tag{1.3}$$

One trouble of this transformation is that it is not one to one. If two vectors $\vec{u}$ and $\vec{v} \in R^4$ are related by

$$\begin{aligned} v_1 &= u_1 \cos\phi - u_4 \operatorname{sen}\phi & v_2 &= u_2 \cos\phi + u_3 \operatorname{sen}\phi \\ v_4 &= u_1 \operatorname{sen}\phi + u_4 \cos\phi & v_3 &= -u_2 \operatorname{sen}\phi + u_3 \cos\phi \end{aligned} \tag{1.4}$$

with arbitrary ϕ, they are mapped onto the same vector of R^3. The image of a point in R^3 is a circle of radius $[r]^{1/2}$ in the parametric space R^4.

Kustaanheimo and Stiefel showed that with R^4 parametrization we can analyse the map of a polarized R^2 plane onto R^3. If two vectors $\vec{u}$ and $\vec{v}$ are any pair of vectors of R^2 they satisfy the following bilinear relation

$$u_4 v_1 - u_3 v_2 + u_2 v_3 - u_1 v_4 = 0 \tag{1.5}$$

This R^2 plane is conformally mapped onto a plane of R^3 and the mapping is a of Levi-Civita's type: distances from origin are squared and angles at the origin are doubled. A given point in R^2 has a point image in R^3. The position vector in R^3 is given in a particular orthogonal basis, where the vectors of this basis are the column elements of the Cayley matrix: the well-known Cayley parametrization of the rotations in the 3-dimensional space.

It is important to note that if two vectors $\vec{u}$ and $\vec{v} \in R^4$ satisfy the bilinear relations eq. (1.5), then $A(\vec{u})\vec{v} = A(\vec{v})\vec{u}$.

These properties of the K-S transformation have one fundamental role in the quantum application of the K-S transformation in the Coulomb problem: the operators associated to $\vec{u}$ and $\vec{v}$ are quantum canonical conjugates and the operator associated to the bilinear relation determines a constraint condition in the determination of the wave function[4].

By using this transformation and a time transformation $d\sigma = dt/r$, Kustaanheimo-Stiefel showed that the equation of the Kepler motion in R^4 can be written in the form

$$\frac{d^2 u_\alpha}{d\sigma^2} + \omega^2 u_\alpha = 0 \qquad \alpha \equiv (1,2,3,4) \tag{1.6}$$

with $\omega^2 = M/4a_0$, where M is the product of the mass and the gravitational constant and a_0 is a semi-major axis of the orbit.

The eqs. (1.6) are linear differential equations with constant coefficients. The image-point moves as if it were connected with the origin by an elastic string of rigidity, ω^2. Its path is a conical section centered at the origin.

2. SPINOR REPRESENTATION FOR THE K S TRANSFORMATION

In order to get a spinor representation for the K-S transformation we introduce a null four-vector x^α in a Minkowski space with metric (+1,-1,-1,-1). The components of x^α are

$$x^i \equiv (x^1, x^2, x^3)$$

$$x^0 = r = \Big[\sum_i x_i^2\Big]^{1/2}$$

By the Cartan theory of spinor[3] we can associate to x^α a 2×2 complex matrix

$$x_\alpha \to T_{AB} = \frac{1}{2}\begin{Vmatrix} x^0 + x^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - x^1 \end{Vmatrix} \tag{2.1}$$

or equivalently, in a tensor form,

$$T_{AB} = \frac{1}{2}\,\tau_{\alpha AB}\,x^\alpha \tag{2.2}$$

where the $\tau_{\alpha AB}$ tensors are

$$\tau_{0AB} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \tau_{1AB} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\tau_{2AB} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \tau_{3AB} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

and have the followings properties

$$\tau^{\alpha AB} = \tau_{\alpha AB}, \quad \alpha = 0,1,2$$

and

$$\tau^{\alpha AB} = -\tau_{\alpha AB}, \quad \alpha = 3$$

By eq. (2.2), the components of the four-vector x^a are

$$x^{\alpha} = \tau^{\alpha AB}\, T_{AB} \tag{2.3}$$

and the length of x^a is given by the determinant of T_{AB} which is null. The four-vector x^a is associated with a singular matrix, hence by the Cartan spinor theory there exist two complex (1,1) spinors ψ_A and ψ_B such $T_{AB} = \psi_A\psi_{\bar{B}}$, where the bar means complex conjugate. Therefore we can write

$$x^{\alpha} = \tau^{\alpha AB}\, \psi_A\psi_{\bar{B}} \tag{2.4}$$

and by eq. (2.2), we have

$$\psi_A\psi_{\bar{B}} = \frac{1}{2}\,\tau_{\alpha AB}\, x^{\alpha} \tag{2.5}$$

or

$$2\psi_A\psi_{\bar{B}} = \tau_{iAB}\, x^{i} + x^{0}\,\tau_{0AB} \tag{2.6}$$

If the x^i components of the four-vector x^a are the components of a vector in R^3, then we can identify eq. (2.6) with the spinor transformation of Kustaanheimo[5].

In order to get the K-S transformation we make a linear correspondence between a vector $\overset{+}{u}$ in the R^4 parametric space and the complex spinor (1,1) ψ_A. We define our complex spinor ψ_A in terms of four real parameters

$$\psi_A = \begin{pmatrix} u_1 - iu_4 \\ u_2 + iu_3 \end{pmatrix} \tag{2.7}$$

This correspondence is linear

$$\begin{aligned} &\alpha\vec{u} \rightarrow \alpha\psi \\ &(\vec{u} + \vec{v}) \rightarrow (\psi + \phi) \end{aligned} \tag{2.8}$$

and the length of the vector is equal to the square root of the norm of the associated spinor.

By eq.(2.4), we have

$$\begin{aligned} x^1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2 \\ x^2 &= 2(u_1u_2 - u_3u_4) \\ x^3 &= 2(u_1u_3 + u_2u_4) \\ x^0 &= u_1^2 + u_2^2 + u_3^2 + u_4^2 \end{aligned} \tag{2.9}$$

which are the relations between R^3 and R' obtained by the K-S transformation.

The fact that the K-S transformation is not one to one can be easily reproduced in the spinor space by a simple gauge transformation. In fact, if two vectors $\vec{u}$ and $\vec{v} \in R^4$ are related by eq.(1.4), the spinor associated to $\vec{v}$ is given by

$$\psi_A(\vec{v}) = e^{-i\phi}\,\psi_A(\vec{u}) \tag{2.10}$$

The spinor gauge transformation $\psi_A \rightarrow \psi_A\, e^{i\phi}$ shows that the K-S transformation is not one to one.

We need to show that with eq. (2.4) we can reproduce the map of a polarized R^2 plane onto the R^3 and the mapping is of Levi-Civita's type.

Let $\vec{u}$ and $\vec{v}$ be two orthogonal unit-vectors in R^4 spanning a plane R^2 through the origin

$$u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0 \tag{2.11}$$

and building a cartesian coordinate-system in R^2. They also satisfy the bilinear relation given by eq.(1.5)

$$u_4v_1 - u_3v_2 + u_2v_3 - u_1v_4 \tag{2.12}$$

If ω is the R^2 polarization angle, then this system of two linear equations in the components of $\vec{v}$ has its solution in the form

$$\begin{array}{ll} v_1 = u_2\cos\omega + u_3 \operatorname{sen}\omega & v_2 = u_1 \cos\omega + u_4 \operatorname{sen}\omega \\ v_4 = -u_2 \operatorname{sen}\omega + u_3 \cos\omega & v_3 = u_1 \operatorname{sen}\omega - u_4 \cos\omega \end{array} \tag{2.13}$$

Consider a given point in R^2 having polar coordinates ρ and θ with respect to the basis $(\vec{u},\vec{v})$ in R^2. The vector representing this point is given by

$$\vec{\rho} = \rho \operatorname{sen}\theta\, \vec{u} + \rho \cos\theta\, \vec{v}$$

By the correspondence in eq. (2.7) and eq. (2.8) the spinor $\psi(\vec{\rho})$ associated to $\vec{\rho}$, is given by

$$\psi(\vec{\rho}) = \rho \operatorname{sen}\theta\psi(\vec{u}) + \rho\cos\theta\psi(\vec{v}) \tag{2.14}$$

By using eqs. (2.4) and (2.13) we can see, after some straighforward computation, that the image of this point in R^3 is given by

$$x^0 = \rho^2$$

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} = \rho^2 \cos 2\theta \begin{pmatrix} u_1^2 + u_4^2 - u_2^2 - u_3^2 \\ 2(u_1u_2 - u_3u_4) \\ 2(u_1u_3 + u_2u_4) \end{pmatrix}$$

$$+ \rho^2 \cos\omega \begin{pmatrix} 2(u_1u_2 + u_3u_4) \\ -u_1^2 + u_2^2 - u_3^2 + u_4^2 \\ 2(u_2u_3 - u_1u_4) \end{pmatrix} \operatorname{sen} 2\theta + \rho^2 \operatorname{sen}\omega \begin{pmatrix} 2(u_1u_3 - u_2u_4) \\ 2(u_1u_4 + u_2u_3) \\ -u_1^2 - u_2^2 + u_3^2 + u_4^2 \end{pmatrix} \operatorname{sen} 2\theta \tag{2.15}$$

or, in the Kustaanheimo-Stiefel abbreviation,

$$\vec{x} = \rho^2[\cos 2\theta \, \vec{a} + [\vec{b}\cos\omega + \vec{c}\,\mathrm{sen}\,\omega]\,\mathrm{sen}\,2\theta] \qquad (2.16)$$

The two vectors $\vec{a}$ and $\vec{b}\cos\omega + \vec{c}\,\mathrm{sen}\,\omega$ are orthogonal. This follows from the Cayley parametrization of the rotations in the 3-dimensional space. As the x^0 component is the length of the image point in R^3, we have that the mapping is of Levi-Civita's type.

We note that the spinor transformation given by eq.(2.4) contains also the eight significant real scalars of the K-S theory. If $\vec{u}$ and $\vec{v}$ are two vectors in R^4 and $\psi(\vec{u})$ and $\psi(\vec{v})$ are the corresponding spinors in the spinorial space, then by eq. (2.4) we can see that $\mathrm{Re}(x^0)$ is the scalar product $(\vec{u},\vec{v})$ and $-\mathrm{Im}(x^0)$ is the bilinear relation given in eq. (1.5) ; $\mathrm{Re}(x^i)$ are the first three components of $A(\vec{u})\vec{v}$ and $\mathrm{Im}(x^i)$ are the three components of the skew-symmetric cross product $\vec{u}\times\vec{v}$.

With this final verification we can conclude that with a linear correspondence between a vector in R^4 and a complex spinor (1,1) defined in terms of four real parameters by eq. (2.7), the K-S transformations are completely described by the spinor transformation given by eq. (2.4).

Using this particular spinor representation in eq.(1.6) we have that the four differential equations for the Kepler motion can be transformed into a two component spinor differential equation

$$\ddot{\psi}_A + \omega^2\psi_A = 0$$

where the dots means differentiation with respect a σ.

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Resumo

Deriva-se uma representação spinorial para a transformação K-S usando-se a teoria dos spinores de Cartan.