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Noether's Theorem in Classical Field Theories and Gravitation

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Abstract Noether's theorem attains its maximum simplicity and depthwhen formulated in curved space-time, gravitation being included. Extension to curved space-times is here made simple by the use of a formulation, for the flat case, due to Jackiw. The exposition purports to be pedagogical.

1. INTRODUCTION

The theorem of Emmy Noether[1] connecting symmetries and conservation laws is a result of great importance and beauty. Modern treatises of theoretical physics, as e.g. the famous Landau-Lifshitz[2], make implicit (or explicit) use of it by defining conserved quantities like energy or momentum in terms of the symmetries to which they are associated.

Expositions of Noether's theorem are by no means scarce. The original paper (ref. (1)) is not easy to find. Shades of it can be gleaned from the pages of later editions of H. Weyl's *Raum, Zeit, Materie* (ref. (3)), and a rather successful English version was written by Hill[4] in the fifties. Modern remakes include Bogoliubov and Shirkov[5], Roman[6] and a learned, beautifully written version due to Gelfand and Fomin[7]. A sophisticate, hypermodern presentation is found in Thirring's lessons in Mathematical Physics[8]. When General Relativy reentered the limelight, the same sort of problem was studied in the more general situation of curved space-times. Two good references are Trautman[9] and Papapetrou[10].

In 1972 Roman Jackiw published a paper on Current Algebra[11] where, lost amidst more formidable subjects, a gem of a proof of Noether's theorem was presented which made the whole question transparent as it never had been before. This feat was obtained by concentrating the at-

*This is an invited review article.

tention on the dynamical variables (fields), instead of on coordinates, and by clearly defining what a symmetry is. The proof is, then, almost trivial.

In this paper I review Jackiw's proof and extend it to curved space-times, where the theorem attains its maximum simplicity and depth. In fact, by a posteriori specializing it to flat space-time, one is able to reveal the nature of a few remaining obscure questions of the Minkowskian special case.

In the second section I reproduce Jackiw's proof, for flatspace-time, and introduce a consistency condition which characterizes the admissible transformations for this particular geometry. A feature of that section is a much simpler reformulation of the Belinfante-Rosenfeld[12] construction of a symmetric energy-momentum tensor for electrodynamics, again inspired in Jackiw[13], but going a step farther. The third section contains the extension to curved space-times and the full interpretation of the above-mentioned consistency condition. Section 4 closes the paper with some comments and applications. To alleviate the presentation we consider only scalar fields and, when necessary, the metric tensor field. The extension to other spins is straightforward except for fermions on curved space-times, where *vierbeine* are required. This is left for the future.

2. THE FLAT CASE

The classical action which describes our system (for the moment restricted to flat space-time) is written as

$$S = \int d^4x L(\phi , \partial_\mu \phi) \quad , \tag{1}$$

$L(\phi,\partial_\mu\phi)$ being the Lagrangian density, a function of some fields ϕ and of their derivatives $\partial_\mu\phi$. To start with, ϕ will be a scalar, in order to reveal most clearly the structure of the theorem.

An infinitesimal transformation of the fields

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) \tag{2}$$

induces an infinitesimal variation $\delta L(x)$ in the Lagrangian. The transformation is a (continuous) symmetry when it can be shown, *without using the equations of motion*, that

$$\delta L(x) = \partial_\mu \Lambda^\mu \tag{3}$$

where Λ is some 4-vector.

Example I: $L = \lambda\phi^*\phi$, the star denoting complex conjugation. The infinitesimal transformation

$$\delta\phi(x) = i\alpha\phi(x) \qquad (\alpha \text{ real})$$

is a symetry. In fact,

$$\delta L = 0$$

meaning $\Lambda = 0$. This is called an internal symmetry.

Example 2 translations. Change coordinates this way:

$$x'^\mu = x^\mu + \varepsilon$$

ε being an infinitesimal constant 4-vector. This transformation induces on $\phi(x)$ a transformation $\delta\phi(x)$ to be computed now. As $\phi(x)$ is a scalar,

$$\phi'(x') = \phi(x) \quad . \tag{4}$$

On the other hand, power expansion on ε^μ gives

$$X = x + x - x^\lambda \partial_\lambda \phi(x)$$

or

$$\phi'(x') = \phi'(x) + \varepsilon^\lambda \partial_\mu \phi(x) \tag{5}$$

which, combined with eq. (4), gives

$$\delta\phi(x) \equiv \phi'(x) - \phi(x) = -\varepsilon^\lambda \partial_\lambda \phi \quad . \tag{6}$$

Suppose

$$L = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2 \tag{7}$$

so that

$$\delta L = \partial^\mu\phi\partial_\mu\delta\phi - m^2\phi\delta\phi \tag{8}$$

. and, using eq.(6),

$$\delta L = -\varepsilon^\lambda\partial_\lambda\left\{\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{m^2}{2}\phi^2\right\},$$

$$\delta L = -\varepsilon^\lambda\partial_\lambda L \tag{9}$$

Finally, as ε^λ is a constant,

$$\delta L = \partial_\lambda(-\varepsilon^\lambda L) \tag{10}$$

which shows that translations are symmetries of the system described by the Lagrangian (7). This is space-time symnetry. Notice that in neither case did we make use of the equations of motion. This kind of variation, $\delta\phi(x)$, is called the form variation of the field.

Noether's theorem asserts that to each continuous symmetry there corresponds a current which satisfies a continuity equation or, equivalently, a quantity which is conserved. Furthermore, it gives an explicit expression for that current.

Suppose $\delta\phi$ is the symmetry transformation. Then there is Λ^μ such that

$$\delta L = \partial_\mu\Lambda^\mu \quad . \tag{3}$$

An independent computation of δL, *with the use of the equations of motion* will now be done.

$$\delta L = \frac{\partial L}{\partial\phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu\delta\phi \ . \tag{11}$$

The equations of motion are

$$\frac{\partial L}{\partial\phi} = \partial_\mu\frac{\partial L}{\partial(\partial_\mu\phi)} \tag{12}$$

and, used in eq.(4), give rise to

$$\delta L = \partial_\mu\left[\frac{\partial L}{\partial(\partial_\mu\phi)}\delta\phi\right] \tag{13}$$

Subtract eq. (13) from eq. (3) to get

$$\partial_\mu\left(\Lambda^\mu - \frac{\partial L}{\partial(\partial_\mu\phi)}\,\delta\phi\right) = 0 \ . \tag{14}$$

This is Noether's theorem. The quantity

$$J^\mu \equiv \Lambda^\mu - \frac{\partial L}{\partial(\partial_\mu\phi)}\,\delta\phi \tag{15}$$

is the Noether current associated to the symnetry $\delta\phi$.

As a simple example I compute now the quantities whose conservation follows from the fact that translations are symmetries of Lagrangian (7). From eq. (10) one has that

$$\Lambda^\mu = - \varepsilon^\mu L \tag{16}$$

and, from eq. (6),

$$\delta\phi(x) = - \varepsilon^\nu\partial_\nu\phi \ . \tag{17}$$

Inserted into eq. (15) these give rise to

$$J^\mu = - \varepsilon^\mu L + \varepsilon^\nu\,\partial_\nu\phi\,\frac{\partial L}{\partial(\partial_\mu\phi)}$$

or

$$J^\mu = \varepsilon^\nu\,\left\{\frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta^\mu_\nu L\right\} \ . \tag{18}$$

As ε^ν are arbitrary constants, the conservation law $\partial_\mu J^\mu = 0$ may be written

$$\partial_\mu T^\mu_\nu = 0 \tag{19}$$

where

$$T^\mu_\mu \equiv \frac{\partial L}{\partial(\partial_\mu\phi)}\,\partial_\nu\phi - \delta^\mu_\nu \tag{20}$$

is the (canonical) energy-momentum tensor of Lagrangian (7).

Example 3: infinitesimal torentz transformations are given by

$$x'^\mu = x^\mu + \omega^\mu{}_\nu\, x^\mu \quad \cdot \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \tag{21}$$

where the $\omega^{\mu\nu}$ are constants.

This is a particular case of the general infinitesimal transformation

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \tag{22}$$

where $\varepsilon^{\mu}(x)$ is an infinitesimal 4-vector field. For a scalar we can just repeat the computation which leads to eq.(6), getting

$$\delta\phi(x) = -\varepsilon^{\lambda}(x)\partial_{\lambda}\phi(x) \quad . \tag{23}$$

As the Lagrangian is itself a scalar, one is led to guess that

$$\delta L(x) = -\varepsilon^{\lambda}(x)\partial_{\lambda}L(x) \quad . \tag{24}$$

This result turns out to be true, but requires a consistency condition, to be discussed below. For Lorentz transformations eqs. (23) and (24) lead to

$$\delta\phi(x) = -\omega^{\lambda}{}_{\nu}x^{\nu}\partial_{\lambda}\phi(x) \tag{25}$$

$$\delta L(x) = -\omega^{\lambda}{}_{\nu}x^{\nu}\partial_{\lambda}L \quad . \tag{26}$$

Because of the antisymmetry of $\omega^{\mu\nu}$ this may be written

$$\delta L(x) = \partial_{\lambda}(-\omega^{\lambda}{}_{\nu}x^{\nu}L) \quad . \tag{27}$$

The Noether current then satisfies

$$\partial_{\mu}\left\{ -\omega^{\mu}{}_{\nu}x^{\nu}L + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\omega^{\lambda}{}_{\beta}x^{\beta}\partial_{\lambda}\phi(x) \right\} = 0 \quad . \tag{28}$$

This can be written

$$\omega^{\lambda\beta}\partial_{\mu}\left\{x_{\beta}\,\frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\partial_{\lambda}\phi - \delta^{\mu}_{\lambda}k_{\beta}L\right\} = 0 \tag{29}$$

or, using eq. (20),

$$\omega^{\lambda\beta}\partial_{\mu}\{x_{\beta}T^{\mu}_{\lambda}\} = 0 \quad . \tag{30}$$

The $\omega^{\lambda\beta}$ are not entirely arbitrary, being anti-symmetric. Therefore, it follows from eq.(30) only that the anti-symmetric part (in $\lambda\beta$)

of the term inside brackets has a vanishing divergence:

$$\partial_\mu \{x_\beta T^\mu_\lambda - x_\lambda T^\mu_\beta\} = 0 \quad . \tag{31}$$

This is usually written

$$\partial_\mu M^\mu_{\beta\lambda} = 0 \tag{32}$$

where $M^\mu_{\beta\lambda}$ is the (4-dimensional) angular-momentum tensor

Example 4: Electromagnetism

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad .$$

In the canonical formalism the variables are the A_μ. Taking them to transform, under translations, as scalars, one has

$$\delta A_\mu(x) = -\varepsilon^\lambda \partial_\lambda A_\mu(x) \quad ,$$

and for the Lagrangian, itself a scalar,

$$\delta L = -\varepsilon^\lambda \partial_\lambda L = \partial_\lambda(-\varepsilon^\lambda L(x)) \equiv \partial_\lambda \Lambda^\lambda \quad .$$

Noether's current eq. (15) then reads

$$J^\mu = -\varepsilon^\mu L + \frac{\partial L}{\partial(\partial_\mu A_\nu)} \varepsilon^\lambda \partial_\lambda A_\nu \quad .$$

It is a simple matter to see that

$$\frac{\partial L}{\partial(\partial_\mu A_\nu)} = -F^{\mu\nu} \quad ,$$

so that

$$J^\mu = -\varepsilon^\mu L - F^{\mu\nu} \varepsilon^\lambda \partial_\mu A_\nu \quad .$$

The conservation law $\partial_\mu J^\mu = 0$ may be written

$$\partial_\mu J^\mu = \varepsilon^\lambda \partial_\mu (-F^{\mu\nu}\partial_\lambda A_\nu - \delta^\mu{}_\lambda L) = 0$$

and the second order tensor

$$T^\mu{}_\lambda = - F^{\mu\nu}\partial_\lambda A_\nu - \delta^\mu{}_\lambda L$$

is the canonical energy-momentum tensor.

It is, however, not symmetric, and this makes it almost useless, particularly in General Relativity. To circumvent this problem, Belinfante and Rosenfeld[12] introduced another tensor which is symmetric and has the same conserved quantities as the canonical energy- momentum tensor. Their method is wellknown (see, for instance, ref.(2)) and rather involved. We present here a different way of obtaining the Belinfante-Rosenfeld tensor which is much more intuitive. It relies on gauge invariance. Consider again the relation

$$\delta A_\nu (x) = -\varepsilon^\lambda \partial_\lambda A_\nu (x)$$

and add and substract $-\varepsilon^\lambda \partial_\nu A_\lambda (x)$, so as to get $F_{\lambda\nu}$ in the second member:

$$\delta A_\nu (x) = - \varepsilon^\lambda F_{\lambda\nu} - \varepsilon^\lambda \partial_\nu A_\lambda (x) \ .$$

Inserting this into the expression for the Noether current we will now have

$$J^\mu = - \varepsilon^\mu - \varepsilon^\lambda F^{\mu\nu}(F_{\lambda\nu} + \partial_\nu A_\lambda (x)) \ .$$

The conservation law reads

$$0 = \partial_\mu J^\mu = \varepsilon^\lambda \partial_\mu (-F^{\mu\nu}F_{\lambda\nu} - \delta^\mu{}_\lambda L) - \varepsilon^\lambda \partial_\mu (F^{\mu\nu}\partial_\nu A_\lambda) \ .$$

Now, the last term is zero, as $\partial_\mu F^{\mu\nu} = 0$ (Maxwell equations) and $F^{\mu\nu}\partial_\mu \partial_\nu A_\lambda = 0$ (antisymmetry of $F^{\mu\nu}$). Then, the tensor

$$T^\mu{}_\lambda = - F^{\mu\nu}F_{\lambda\nu} - \delta^\mu{}_\lambda L$$

satisfies

$$\partial_\mu T^\mu{}_\lambda = 0$$

and

$$T^{\mu\lambda} = T^{\lambda\mu}$$

As a matter of fact, it is precisely the Belinfante-Rosenfeld tensor.

We now turn back to eq. (24). It expresses the fact that L is a scalar. If we explicitly check its validity, we obtain restrictions on $\varepsilon^{\mu}(x)$ which must characterize those transformations under which L is a scalar. Taking, for instance, the Lagrangian of eq. (7) and considering just the troublesome part which contains derivatives, one has'

$$\delta L = \partial^{\mu}\phi\partial_{\mu}(-\varepsilon^{\lambda}(x)\partial_{\lambda}\phi) = -\partial_{\mu}\varepsilon^{\lambda}(x)\partial^{\mu}\phi(x)\partial_{\lambda}\phi(x) - $$
$$- \varepsilon^{\lambda}(x)\partial^{\mu}\phi(x)\partial_{\lambda}\partial_{\mu}\phi(x)$$

$$\delta L = -\partial^{\mu}\varepsilon^{\lambda}\partial_{\mu}\phi\partial_{\nu}\phi - \varepsilon^{\lambda}\partial_{\lambda} L(x)$$

$$\delta L = -\frac{1}{2}(\partial^{\mu}\varepsilon^{\lambda} + \partial^{\lambda}\varepsilon^{\mu})\partial_{\mu}\phi\partial_{\nu}\phi - \varepsilon^{\lambda}(x)\partial_{\lambda} L(x) . \qquad (34)$$

So, eq. (24) is true provided that the transformations

$$x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad (35)$$

satisfy the relation

$$\partial^{\mu}\varepsilon^{\nu} + \partial^{\nu}\varepsilon^{\mu} = 0 \qquad (36)$$

which should be considered a consistency condition. A consequence of it is

$$\partial^{\mu}\varepsilon_{\mu}(x) = 0 \qquad (37)$$

Combining now eq. (24) and eq. (37) it is seen that the transformations obeying eq. (36) exhaust the space-time symmetries of our Lagrangian*. This result can be easily extended to all reasonable Lagrangians of special relativity. The extension of Noether's

*We qualify this statement. What we call space-time symmetry is a transformation which, besides being a symmetry in the sense we defined above, does not alter the nature of the space-time. A flat space-time is kept flat. This excludes, for instance, conformal transforrnations (see eq. (37)). These will be included in another note. The proof of the statement should be obvious by the end of the paper.

theorem to curved space-times will reveal a deeper interpretation of this consistency condition.

3. THE CURVED CASE

An infinitesimal coordinate transformation in curved space-time*

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \tag{38}$$

induces on a scalar field $\phi(x)$ the same form variation we have met before,

$$\delta\phi(x) = - \xi^{\lambda}(x)\partial_{\lambda}\phi \quad . \tag{39}$$

Let us compute the form variation induced on the metric tensor $g^{\mu\nu}(x)$. From

$$g'^{\mu\nu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} g^{\alpha\beta}(x) \tag{40}$$

which characterizes it as a second order tensor field, it follows, for the infinitesimal transformations (38),

$$g'^{\mu\nu}(x') = g^{\mu\nu}(x) + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{41}$$

whereas, from a Taylor expansion,

$$g'^{\mu\nu}(x) = g'^{\mu\nu}(x) + \xi^{\lambda}(x)\partial_{\lambda}g^{\mu\nu}(x) \quad . \tag{42}$$

Using both eqs. (41) and (42), one arrives at

$$\delta g^{\mu\nu}(x) = - \xi^{\lambda}\partial_{\lambda}g^{\mu\nu} + \partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} \tag{43}$$

or, equivalently,

$$\delta g^{\mu\nu}(x) = \xi^{\mu;\nu} + \varepsilon^{\nu;\mu} \quad , \tag{44}$$

the semicolon standing for covariant differentiation.

Vector fields $\xi^{\mu}(x)$ which satisfy $\delta g^{\mu\nu} = 0$, that is, which generate transformations (38) which do not change the form of the metric

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* Up to the paragraph containing eq. (54) everything is true also for flat spacetimes with curvilinear coordinates. We choose to ignore this case as the curved situation, involving gravity, far outweights the flat one, in physical motivation.

field, are called Killing fields. Therefore, a Killing field is characterized by

$$\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \ . \tag{45}$$

Suppose two observers are connected by a transformation (38) in which $\xi^{\mu}(x)$ is a Killing field. It follows that both observe precisely the same gravitational field. The systematic study of the Killing fields of a metric is therefore equivalent to the study of the geometric symmetries of the corresponding space-time. This is particularly important in cosmology, where assumptions about these symmetries, namely the cosmological principle, are made a priori. A rather complete study of these symmetries is found in refs. (2) and (14).

With a Lagrangian which is a scalar under general coordinate transformations, an invariant action may be constructed,

$$S = \int d^4x \sqrt{-g}\, L(g^{\mu\nu},\phi,\partial_{\mu}\phi) \tag{46}$$

whose response to variations of the field reads

$$\delta S = \int d^4x \sqrt{-g} \left\{\frac{\partial L}{\partial\phi}\,\delta\phi + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\partial_{\mu}\delta\phi\right\} + \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\,\delta g^{\mu\nu} \ . \tag{47}$$

For the last piece consult, for instance, ref. (2) and ref. (14). $T_{\mu\nu}$ is the symmetric energy-momentum tensor. For solutions of the equations of motion, $T^{\mu\nu}{}_{;\nu} = 0$, as a consequence of invariance under general transformations. Using eqs. (39) and (42) one has

$$\delta S = \int d^4x \sqrt{-g} \left\{- \xi^{\lambda}\left[\frac{\partial L}{\partial\phi}\,\partial_{\lambda}\phi + \frac{\partial L}{\partial(\partial_{\mu}\phi)}\,\partial_{\lambda}\partial_{\mu}\phi\right] - \right.$$

$$\left. -(\partial^{\mu}\xi^{\lambda} + \partial^{\lambda}\xi^{\mu})\,\frac{\partial L}{\partial(\partial^{\mu}\phi)}\,\partial_{\lambda}\phi\right\} + \frac{1}{2}\int d^4x \sqrt{-g}\, T_{\mu\nu}\,\delta g^{\mu\nu}. \tag{48}$$

If $\xi^{\lambda}(x)$ is a Killing field, then $\delta g^{\mu\nu} = 0$, or

$$\partial^{\mu}\xi^{\nu} + \partial^{\nu}\xi^{\mu} = \xi^{\lambda}\partial_{\lambda}g^{\mu\nu}$$

so that eq. (48) becomes

$$\delta S = -\int d^4x \sqrt{-g}\, \xi^\lambda \left\{ \frac{\partial L}{\partial \phi} \partial_\lambda \phi + \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\lambda \partial_\mu \phi + \partial_\lambda g^{\mu\nu} \frac{\partial L}{\partial(\partial^\mu \phi)} \partial_\lambda \phi \right\}$$

or, still simpler,

$$\delta S = -\int d^4x \sqrt{-g}\, \xi^\lambda \partial_\lambda L \quad . \tag{49}$$

The Killing equations $\xi^{\mu;\nu} + \xi^{\nu;\mu} = 0$ lead to $\xi^\mu{}_{;\mu} = 0$, that is,

$$\partial_\lambda (\sqrt{-g}\, \xi^\lambda) = 0 \tag{50}$$

so that we can rewrite δS, after a partial integration, as

$$\delta S = \int d^4x\, \partial_\lambda (-\sqrt{-g}\, \xi^\lambda L) \quad . \tag{51}$$

In eq. (3) of section 2 we defined a symmetry in the case of flat space-time. The extension to a curved space-time is simple. An infinitesimal transformation of the fields

$$\begin{aligned} \phi(x) &\rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \\ g^{\mu\nu}(x) &\rightarrow g'^{\mu\nu}(x) = g^{\mu\nu}(x) + \delta g^{\mu\nu}(x) \end{aligned} \tag{52}$$

is a (continuous) symmetry if the induced variation δS of the action can be written, *without the use of equations of motion,* as

$$\delta S = \int d^4x\, \partial_\mu \Lambda^\mu \tag{53}$$

where Λ^μ is some vector density. We immediately see from eq.(51) that all Killing fields of the metric $g_{\mu\nu}$ generate symmetries of the action S, provided L is a scalar.

The general case is obtained by adding to the action the gravitational contribution. Let it read

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{16\pi k} R + L \right\} \quad . \tag{54}$$

Its variatíon is

$$\delta S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{16\pi k} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R) + \frac{1}{2} T_{\mu\nu}\right\} \delta g^{\mu\nu}$$

$$+ \int d^4x \sqrt{-g} \ (- \xi^\lambda \partial_\lambda L - \frac{\partial L}{\partial(\partial^\mu\phi)} \partial_\nu \phi \delta g^{\mu\nu}) \quad .$$

So, if ξ^λ is a Killing field,

$$\delta S = \int d^4x \ \partial_\lambda (-\sqrt{-g} \ \xi^\lambda L) \ , \tag{55}$$

and the transformation

$$x'^\mu = x^\mu + \xi^\mu(x) \quad ; \quad \xi^{\mu;\nu} + \xi^{\nu;\mu} = 0 \tag{56}$$

is a symmetry of the complete action. As no other transformation can be a symmetry of the gravitational part, the transformations **(56)** exhaust the space-time symmetries of any action which includes gravitation.

As the last step, we construct the Noether currents. Rewriting δS as

$$\delta S = \int d^4x \sqrt{-g} \left\{ -\frac{1}{16\pi k} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R) + \frac{1}{2} T_{\mu\nu}\right\} \delta g^{\mu\nu}$$

$$+ \int d^4x \sqrt{-g} \left[\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)} \partial_\mu \ \delta\phi \right] \tag{57}$$

and finally using the equations of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}R = 8\pi k \ T_{\mu\nu} \tag{58}$$

and

$$\frac{\partial L}{\partial \phi} = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} \frac{\partial L}{\partial(\partial_\mu\phi)} \right) \tag{59}$$

one arrives at

$$\delta S = \int d^4x \ \partial_\mu \left\{ \sqrt{-g} \ \frac{\partial L}{\partial(\partial_\mu\phi)} \delta\phi \right\} \ . \tag{60}$$

Subtracting eq. (60) from eq. (55) one has

$$\partial_\mu\left\{\sqrt{-g}\ \xi^\lambda\left[\frac{\partial L}{\partial(\partial_\mu\phi)}\ \partial_\lambda\phi - \delta^\mu_\lambda L\right]\right\} = 0 \qquad (61)$$

where use was made of eq. (39). The object within curly brackets is the Noether current. Recognizing the expression within parentheses as the energy-momentum tensor, we rewrite eq.(61) as

$$\begin{aligned} &\partial_\mu(\sqrt{-g}\ \xi^\lambda\ T^\mu_\lambda) = 0 \\ &\xi^{\lambda;\mu} + \xi^{\mu;\lambda} = 0 \ . \end{aligned} \qquad (62)$$

We can now have a better understanding of the consistency condition in eq. (36). It is just the flat space version of the condition that ξ^h be a Killing field. Flat space-time is peculiar in that the covariance group, that is, the group of transformations under which the main physical quantities are tensors, coincides with the group of space-time symmetries. In the general case (curved space-times) the covariance group is the group of general coordinate transformations, whereas the group of symmetries may even be restricted to the sole identity transformation. This peculiarity of Minkowski space-time made it difficult even for Einstein to find the way to a *general relativity*.

4. COMMENTS AND APPLICATIONS

Noether's theorem finds its natural ground when gravity is included among the physical agents: the results contained in eqs. (62) can be proved in an extremely simple way as follows. The (symmetric) energy-momentum tensor satisfies, due to general covariance, the relation

$$T^{\mu\lambda}{}_{;\lambda} = 0 \ .$$

Because of this,

$$(\xi_\lambda\ T^{\mu\lambda})_{;\mu} = \xi_{\lambda;\mu}\ T^{\mu\lambda} = \frac{1}{2}\ (\xi_{\lambda;\mu} + \xi_{\mu;\lambda})T^{\mu\lambda} = 0$$

as ξ^λ is Killing. Now,

$$(\xi_\lambda T^{\mu\lambda})_{;\mu} = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g}\, \xi_\lambda T^{\mu\lambda})$$

and eq.(62) follows. This is so simple that the Noetherian currents could even have been guessed. Arriving at them through Noether's theorem adds, however, the information that all Noetherian current have the form eq. (62). Other conservation laws may, of course, exist, connected to discrete symmetries, topology, etc..

We mentioned in the introduction the possibility of defining conserved quantities (as energy and momentum) through their connection with certain symmetries. Energy, for instance, may be defined as that quantity whose conservation is a consequence of invariance under time translatíons. In this way we can construct directly the energy'density, say, of the electromagnetic field, instead of having to infer its expression from particular instances in which the field exchanges energy with mechanical systems. We can also understand why energy is a problematic concept in General Relativity. In fact, translations are constant Killing fields, and, from

$$\xi^\lambda \partial_\lambda g^{\mu\nu} = \partial^\mu \xi^\nu + \partial^\nu \xi^\mu$$

it follows that a space-time with arbitrary time-like translations must have (in some coordinate system) a time-independent metric tensor. This excludes, among others, the space-times of the standard cosmology (ref. (14)), where, in fact, energy is not conserved (ref. (15)). The energy problem in General Relativity has a satisfactory solution only for space-times that are asymptotically flat (see ref. (16)).

It is perhaps worth remarking that the usual presentations distinguish between a first Noether theorem, for symmetry groups with a finite number of parameters (global symmetries) and a second Noether theorem, for groups with infínitely many parameters (local symmetries). It is a virtue of the present treatment, which uses the fundamental relation $\delta L = \partial_\mu h^\mu$ to define a symmetry (*quasi-invariance* of the Lagrangian), that the two theorems are unified. Examples 1, 2 and 4 of the text exhibit global symmetries, whereas space-time symmetries (and local gauge symmetries) are local ones. They are all encompassed in the same formulation.

Once the extension to curved space-times is learned, one cannot help feeling that the method is being used below its full capacity. In fact, extensions of Noether's theorem to more complicated situations, as for theories employing non-Riemannian manifolds (e.g. Einstein-Cartan), or for spinorial formalisms, are reasonably straightforward.

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Resumo

O teorema de Noether alcança sua máxima simplicidade e profundidade quando é formulado em espaços-tempos curvos com a inclusão da gravidade. Sua extensão para este caso é aqui obtida de modo simples pelo uso de uma formulação, para o caso plano, devido a Jackiw. A exposição pretende ser pedagógica.