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A Gauge Approach for the SO(3,1) ⊕ U(1) Group

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Abstract A sourceless gauge model for gravitation, coupled to Maxwell theory is examined. By considering the Lorentz group SO(3,1) in semi-direct product with the U(l) group in a bundle of linear frames, Maxwell equations and Yang's gravitational equation can be derived. This approach has a gauge-like Lagrangian which leads to the field equations.

1. INTRODUCTION

In spite of the analogies existing between Yang-Mill (YM) theory, at the classical level, and General Relativity (GR), the fact remains that the Einstein-Hilbert Lagrangian is not of the YM type, and the dynamical aspects of both theories are qualitatively different.

Gauge theories, besides their phenomenological successes, present also renormalizability as a basic formal property (is spite of the fact that not all gauge theories are automatically renormalized). This does not happen with Einstein's gravitational model.

Several attempts to renormalize the Einstein-Maxwell models have been made, without much success. It seems that any electro-gravitational approach, combining both Maxwell and Einstein theories will not give any new results, this being due to the impossibility of eliminating ghosts after quantization.

A gauge orthodox method for pure gravitation has been developed before[1], leading to a unique Lagrangian density, by considering the Poincaré group SO(3,1) ⊕ $T_{3,1}$. This approach is implemented by the use of weak constraints and requires viewing the Poincaré group as the Wigner-Inonu[2] contraction of the de Sitter group. Yang's and Einstein's gravitational equations are eventually obtained. In such a case, Yang-Mills equations are derived from Bianchi identities, by duality sym-

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metry, for the sourceless case. The absence of a nondegenerate Killing-Cartan metric on the group is not by itself an impediment: theories for the non semisimple groups are quite feasible through the use of general invariants of the adjoint representation.

We question here why not consider another gravitational theory, instead of GR, coupled to Maxwell's theory. The reason for such an inquiry is that it is possible, as demonstrated below, to have another gauge-like electro-gravitational model in which, instead of GR, Yang's gravitational theory is obtained.

The gauge model approach, developed in a fibre bundle P of linear frames[3] has for base manifold Minkowski space-time and for gauge group G the semidirect product between the Lorentz group and the unimodular group of rotations in the circle: $G = SO(3,1) \oslash U(1)$. Starting from YM equations, two independent sets of equations are obtained: Yang's equation for the Lorentz sector and Maxwell equations for the $U(1)$ sector. Finally, a Lagrangian is proposed which couples the above two theories. The formalism of differential forms will be used throughout.

2. THE GAUGE FIELD

We start by considering a linear connection Γ on the P-bundle of linear frames, represented by a matrix of 1-forms, written in a basis $\{dx^\mu\}$ of Minkowski's space-time, and with values in the Lorentz algebra $SO(3,1)$:

$$\Gamma = J_a{}^b \Gamma^a{}_{b\mu}\, dx^\mu \qquad (2.1)$$

Here, latin indices $a,b, \ldots = 1,\ldots,4$ correspond to the $SO(3,1)$ algebra, and greek indices $\mu,\nu,\ldots = 1,\ldots,4$ refer to space-time. The $J_a{}^b$ are the generators of the Lorentz algebra[4], and the components $\Gamma^a{}_{b\mu}$ of the connection form will be interpreted as a gauge potential[i]. The Γ connection defines covariant derivatives of tensors belonging to any representation of the Lorentz group. For simplicity, all forms will be considered as projected on the base manifold.

Next, we replace the Lorentz group by the extended group $G = SO(3,1) \oslash U(1)$. This allows the introduction of another linear connection A, written in the same basis $\{dx^\mu\}$ above, and with values in the U(1) algebra

$$A = I A_\mu dx^\mu \tag{2.2}$$

where I is the single generator of U(1). Our bundle P has now Minkowski space-time for base manifold and SO(3,1) ⊗ U(1) for symmetry group.

Since P is a bundle of linear frames, it is possible to take a linear connection[5]

$$W = \Gamma + A \tag{2.3}$$

with values in the Lie algebra of G. Now, the components of W will be interpreted as the total electro-gravitational potential, while the components $\Gamma^a{}_{b\mu}$ and A_μ will be interpreted as partial gauge potentials, corresponding to each one of the two algebra sectors.

The Lie algebra of G has the commutation rules[6]

$$[J_a{}^b, J_c{}^d] = \frac{1}{2}(\eta_a{}^d J_c{}^b - \eta_{ac} J^{bd} + \eta^b{}_c J_a{}^d - \eta^{bd} J_{ac}) \tag{2.4}$$

$$[J_a{}^b, I] = 0$$

where $\eta = (+,+,+,-)$ is the Minkowski metric.

The curvature $\mathcal{F}$ of the W connection is

$$\mathcal{F} = dW + W \wedge W \tag{2.5}$$

and can be decomposed into

$$\mathcal{F} = F + f \tag{2.6}$$

where (on account of the commutation rules eq. (2.4)), the curvature F, corresponding to the Lorentz sector is

$$F = d\Gamma + \Gamma \wedge \Gamma \tag{2.7}$$

and the curvature f, corresponding to the U(1) sector, is

$$f = dA \tag{2.8}$$

The interpretation of the linear connections W, Γ and A as gauge potentials makes of the corresponding curvatures F, F and f the gauge fields of the model. Thus, the components

$$F^{a}{}_{b\mu\nu} = \partial_\mu \Gamma^{a}{}_{b\nu} - \partial_\nu \Gamma^{a}{}_{b\mu} + \Gamma^{a}{}_{c\mu}\Gamma^{c}{}_{b\nu} - \Gamma^{a}{}_{c\nu}\Gamma^{c}{}_{b\mu} \qquad (2.9)$$

represent the gauge field of the Lorentz algebra sector, while the components

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad (2.10)$$

represent the gauge field of the U(1) algebra sector.

From eq. (2.8) we have

$$df = 0 \qquad (2.11)$$

which corresponds to the homogeneous pair of Maxwell equations[7].

3. FIELD EQUATIONS

Differentiation of eq. (2.5) furnishes Bianchi's identity

$$dF + W,F = 0 \qquad (3.1)$$

which guarantees that the covariant derivative of the total gauge field F is automatically zero[3]. Moreover, since gauge theories exhibit duality symmetry, the dynamical YM field equations (for the sourceless case) are eq. (3.1), but written in terms of the dual of F[1]

$$d * F + [W, * F] = 0 \qquad (3.2)$$

where $*$ is Hodge's differential star operator[7]. Here, we shall deal with the sourceless case only. After separating eq. (3.2) for each algebra sector we are led to

$$\delta f = 0 \qquad (3.3)$$

for the U(1) sector, and

$$\delta F + *^{-1} [\Gamma, * F] = 0 \qquad (3.4)$$

for the Lorentz sector. Here, δ stands for the coderivative operator[7].

As f is a 2-form, we have in the cartesian basis $\{dx^\mu\}$

$$f = \frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{3.5}$$

and

$$\delta f = \frac{1}{4}\, \partial_\lambda\, f_{\mu\nu}\, \varepsilon^{\mu\nu}{}_{\alpha\beta}\, *^{-1} (dx^\lambda \wedge dx^\alpha \wedge dx^\beta) =$$

$$= \partial^\nu f_{\mu\nu}\, dx^\mu \ .$$

So, eq.(3.3) is, in terms of its components

$$\partial^\alpha f_{\mu\alpha} = 0 \ . \tag{3.6}$$

Similarly, in order to write eq.(3.4) in terms of its com- nents, we consider

$$F = \frac{1}{2}\, J_a{}^b F^a{}_{b\mu\nu}\, dx^\mu \wedge dx^\nu \tag{3.7}$$

then

$$\delta F = *^{-1} d{*F} = \frac{1}{4}\, \partial_\lambda\, J_a{}^b F^a{}_{b\mu\nu} \varepsilon^{\mu\nu}{}_{\alpha\beta} \varepsilon^{\lambda\alpha\beta\sigma} dx_\sigma =$$

$$= J_a{}^b \partial^\nu F^a{}_{b\mu\nu} dx^\mu \ . \tag{3.8}$$

Taking into account the first commutation rule from eq. (2.4) we obtain

$$[\Gamma, *F] = \frac{1}{4}\, (J_c{}^b \Gamma^c{}_a\, F^a{}_{b\mu\nu} \varepsilon^{\mu\nu}{}_{\lambda\sigma} - J_a{}^d \Gamma^b{}_d\, F^a{}_{b\mu\nu} \varepsilon^{\mu\nu}{}_{\lambda\sigma}) dx^\rho \wedge dx^\lambda \wedge dx^\sigma$$

and

$$*^{-1} [\Gamma, *F] = J_c{}^b \Gamma^c{}_a{}^\mu F^a{}_{b\nu p} dx^\nu - J_a{}^d \Gamma^b{}_d F^a{}_{b\mu\nu} dx^\nu \tag{3.9}$$

By rearranging the terms of eqs. (3.8) and (3.9), we have for the components of eq.(3.4)

$$\partial^\nu F^a{}_{b\mu\nu} + \Gamma^a{}_c{}^\nu F^c{}_{b\mu\nu} - \Gamma^c{}_b{}^\nu F^a{}_{c\,\mu\nu} = 0 \ . \tag{3.10}$$

Interpreting f as the electromagnetic field, eq. (3.6) becomes Maxwell dynamical equations in the sourceless case. In eq. (3.10) the components $F^{a}{}_{b\mu\nu}$ areskew-symmetrical in μ, ν, and this equation establishes that the covariant derivative of $F^{a}{}_{b\mu\nu}$ is null, for the algebra indices.

By means of the vierbein fields h^{a}_{a} and their inverse h^{α}_{a}, which ratisfy the conditions $h^{a}_{\alpha}\, h^{\alpha}_{b} = \delta^{a}_{b}$, $h^{a}_{a} h^{a}_{\beta} = \delta^{\alpha}_{\beta}$, we stablish an isomorphisrn between the bundle space and the base space of the fiberrnanifold, so that we may take the linear connection Γ on the bundle of linear frames as a Levi-Civita connection on the base space

$$R^{\alpha}{}_{\beta\mu\nu} = h^{\alpha}_{a}\, h^{b}_{\beta}\, F^{a}{}_{b\mu\nu} \tag{3.11}$$

which we now take as the components of the Riemann curvature tensor Thus, in spite of the fact that we started with a *flat* space-time base manifold (Minkoswski space-time), the gauge field of the Lorentzalgebra sector, F, generates a curvature in that base space through eq.(3.11).

By choosing a basis such that $h^{a}_{a} = \delta^{a}_{\alpha}$, we can write the field equations for the Lorentz sector, eq. (3.10) as

$$R^{\alpha}{}_{\beta\mu\nu}{}^{;\mu} = 0 \tag{3.12}$$

due to the skew-symmetry $R^{\alpha}{}_{\beta\mu\nu} = -R^{\alpha}{}_{\beta\nu\mu}$. This last equation can be written as[8]

$$g^{\sigma\alpha} g^{\rho\lambda} R_{\lambda\mu\sigma\beta;\rho} = g^{\sigma\alpha} R^{\rho}{}_{\mu\sigma\beta;\rho} = g^{\sigma\alpha}(R^{\sigma}{}_{\mu\rho\beta;\sigma} - R^{\rho}{}_{\mu\rho\sigma;\beta}) = 0$$

which leads to Yang's gravitational equation[9]

$$R_{\alpha\beta;\mu} - R_{\alpha\mu;\beta} = 0 \ . \tag{3.13}$$

These equations, which have been proposed by Popov and Daikhin[10] on the basis of a heuristic argument, have, as a very particular solution, Einstein's sourceless equation

$$R_{\alpha\beta} = 0 \ . \tag{3.14}$$

4. LAGRANGIAN FORMALISM

In order to write the total Lagrangian, we have to keep in mind that G is not a semisimple group, so it does not admit a Killing-Cartan metric[4]. In this case, we can deal with a new technique to obtain invariants and such invariants may be taken as Lagrangians.

For non semisimple groups we can build up invariants without using a metric for the group manifold[1]. We can take a matrix $X = J_a X^a$ in the adjoint representation of the linear group $GL(n,R)$ and develop the expression

$$\det(I + X) \tag{4.1}$$

where I is the identity matrix and λ is a parameter. By expanding (4.1) in the polinomial form in λ we have

$$\det(I+X) = \sum_{n=0}^{\infty} \lambda^n I_n = 1 + \frac{\lambda}{1!} \operatorname{tr} X + \frac{\lambda^2}{2!} \left[(\operatorname{tr} X)^2 - \operatorname{tr} X^2\right]$$
$$+ \frac{\lambda^3}{3!} \left[(\operatorname{tr} X)^3 - 3(\operatorname{tr} X)(\operatorname{tr} X^2) + 2 \operatorname{tr} X^3\right] + \dots \tag{4.2}$$

where I_n is the n-th invariant. The first invariant in eq.(4.2) is $I_1 = \operatorname{tr} X$ and the n-th invariant is $I_n = \det X$. In classical electromagnetic theory the Lagrangian is equivalent to the invariant $\operatorname{tr}(f \wedge *f)$, where f is the field strengh, because in this case $\operatorname{tr} f = 0$. The same happens to Yang-Mills theory, when we write the field in the adjoint representation of the SU(2) algebra. In the case we are dealing with, we can take as invariant the quantity $\operatorname{tr}(F \wedge *F)$ and the action integral ís chosen to be

$$S = -\frac{1}{4} \int d^4x \operatorname{tr}(F \wedge *F) . \tag{4.3}$$

By considering the functional dependences $F = F(\Gamma, d\Gamma, dA)$ and by setting the extremal condition $\delta S = 0$, we obtain Euler - Lagrange equations

$$\left[F^{a}{}_{b}{}^{\mu\nu} \frac{\delta F^{b}{}_{a\mu\nu}}{\delta \Gamma^{c}{}_{d\sigma}} - \partial_\rho \frac{\delta F^{b}{}_{a\mu\nu}}{\delta(\partial_\rho \Gamma^{c}{}_{d\sigma})}\right]$$
$$+ \left[f^{\mu\nu} \frac{\delta f_{\mu\nu}}{\delta A_\sigma} - \partial_\lambda f^{\mu\nu} \frac{\delta f_{\mu\nu}}{\delta(\partial_\lambda A_\sigma)}\right] \delta A_\sigma = 0 \tag{4.4}$$

for each group algebra separately. The terms in F lead to eq. (3.10) and the terms in f to eq.(3.6).

5. CONCLUSION

We have proposed a geometrical setting for a gauge model of gravitation coupled to Maxwell's theory. The model presented here leads to Yang's gravitational theory, instead of GR, suggesting that a unified approach to an electro-gravitational gauge model can be obtained if we are willing to replace GR by Yang's theory. We were able also to write down a gauge-like Lagrangian which leads to the field equations of each algebra sector separately. Further developments to be considered would be to introduce sources in eq. (3.2) and also to analise the possibility of a quantization procedure.

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Resumo

Propõe-se um formalismo de gauge para a gravitação, acoplada à teoria de Maxwell, desenvolvida num espaço fibrado sobre o espaço-tempo de Minkowski e com o grupo de gauge $SO(3,1) \otimes U(1)$. As equações de Maxwell e a equação gravitacional de Yang surgem de maneira natural. O presente modelo apresenta uma Lagrangeana de gauge típica, que conduz às equações de campo correspondentes de Maxwell e de Yang.