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Equations of Motion of a Spinning Nucleon in a Scalar Field

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Abstract The equations of motion of a spinning nucleon interacting witk a scalar massless field are derived from energy-momentum conservation and charge conservation following the moment method of Papapetrou.

1. INTRODUCTION

The equation of motion of a particle, which will be called nucleon, interacting with a scalar field has been derived by Harish-Chandra[1] using the method that Dirac[2] developed for the electron, by calculating the flux pf the energy-momentum tensor and of the angular momentum tensor across a tube containing the world line of the particle. As in the case of Dirac, simplicity arguments were used to justify the eontribution of the basis elements of the tube. In this paper we will derive the equation of motion by the method of moments of the energy-momentum tensor. The method of moments was introduced by Fock[3] and elaborated by Papapetrou[4] to describe the motisn of a partiele with spin in a gravitational field. This method was recently used to obtain the equations of motion of a non-abelian charged spin particle in a Yang-Mills field[5].

Besides the fact that the equations of motion can be obtained in a straightforward and simple way, the methsd of moments allows one to obtain the expression of the energy-momentum tensor and ofthenucleonic current of the particle and of their moments.

2. THE EQUATION OF MOTION

The energy-momentum tensor of the scalar massless field Φ is well-known[6]

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$$T_F^{\alpha\beta} = \frac{1}{4}\,(\Phi^{\alpha}\Phi^{\beta} - \frac{1}{2}\,g^{\alpha\beta}\,\Phi^{\lambda}\Phi_{\lambda}) \tag{1}$$

where we use the notation $\Phi^{\alpha} = \partial\phi/\partial x_{\alpha} = \partial^{\alpha}\Phi$.

Calling $\bar{\rho} = dg/d\bar{V}$ the restdensity of nucleonic charge, dg being the quantity of nucleonic charge in the element of volume $d\bar{V}$ in the charge rest frame, the equation obeyed by the Φ field is

$$\partial^{\alpha}\partial_{\alpha}\Phi = -4\pi\bar{\rho} \quad . \tag{2}$$

Note that as Φ is a Lorentz scalar the appropriate density is the scalar, actually invariant scalar, $\bar{\rho}$. The density ρ of the moving element with velocity $\vec{v}$ is, in the system of units $c=1$

$$\rho = dg/dV = \bar{\rho}(1-v^2)^{-1/2} = \rho u^0 = \bar{\rho}\,\frac{dt}{ds} \quad , \tag{3}$$

where u^0 is the time component of the four-velocity.

Using eq. (2) we can calculate the divergence of eq. (1) with the result

$$\partial_{\beta}\, T_F^{\alpha\beta} = -\,\bar{\rho}\,\Phi^{\alpha} \quad . \tag{4}$$

Now we introduce the nucleonic energy-momenturn tensor $T^{\alpha\beta}$ by requiring energy-momentum conservation of the particle plus field system, which has t'he covariant expression

$$\partial_{\beta}(T_F^{\alpha\beta} + T^{\alpha\beta}) = 0 \quad . \tag{5}$$

From eq. (4) we see that $T^{\alpha\beta}$ satisfies the following equation:

$$\partial_{\beta}\, T^{\alpha\beta} = \bar{\rho}\,\Phi^{\alpha} \quad . \tag{6}$$

This equation is our starting point to obtain the equations of motion of the particle and of $T^{\alpha\beta}$ itself. The only thing that we demand is that $T^{\alpha\beta}$ be symmetric as $T_F^{\alpha\beta}$ is.

We shall consider our system as an extended body pcupying a volume V, which will tend to zero at the'end of the calculations to describe our point particle. Following Papapetrou we start by integration eq.(6) over V to obtain

$$\int (\partial_t T^{\alpha 0} + \partial_i T^{\alpha i})\, dV = \int \bar{\rho}\, \Phi^\alpha\, dV \;. \tag{7}$$

By Gauss's theorem the three-divergence term can be converted into a surface term, which is zero just outside the system, and we obtain the result

$$\frac{d}{dt} \int T^{\alpha 0}\, dV = \int \bar{\rho}\, \Phi^\alpha\, dV \;. \tag{8}$$

Now, we choose a reference point X^α inside the system, which can be its center of mass as will be discussed later. Calling δx^α the distance of x^α to X^α we have

$$x^\alpha = X^\alpha \text{ a } \delta x^\alpha \;. \tag{9}$$

This choice is being made at equal times, that is, $t = X^0$ or $\delta x^0 = 0$.

Next, we expand $\bar{\Phi}^\alpha(x)$ in eq.(8) around X to obtain

$$\bar{\Phi}^\alpha(x) = \bar{\Phi}^\alpha(X) + \delta x^\beta\, \partial_\beta\, \bar{\Phi}^\alpha(X) + \ldots \;. \tag{10}$$

We shall work first to zeroth order in δx^α. To that order eq.(8) becomes, after using eq.(3),

$$\frac{dp^\alpha}{ds} = g\, \bar{\Phi}^\alpha \tag{11}$$

where Φ^α is now caleulated at point X^α,

$$p^\alpha = \int T^{\alpha 0}\, dV \tag{12}$$

is the momentum of our system and

$$g = \int \rho\, dV \tag{13}$$

its nucleonic charge. Of course this quantity is a constant as follows by integrating the equation of continuity for the nucleonic charge over the volume V. We have

$$\partial_\alpha J^\alpha = 0 \quad . \tag{14}$$

By Gauss's theorem

$$\frac{d}{dt} \int J^0 \, dV = 0 = \frac{dg}{dt} \quad . \tag{15}$$

To obtain more information on p^a we consider the divergence of $x^\beta T^{\alpha\lambda}$. With eq.(6) we obtain

$$\partial_\lambda (x^\beta T^{\alpha\lambda}) = T^{\alpha\beta} + x^\beta \bar{\rho} \Phi^\alpha \quad . \tag{16}$$

Integration over the volume of our system gives

$$\frac{d}{dt} \int x^\beta \, T^{\alpha 0} dV = \int T^{\alpha\beta} \, dV + \int \bar{\rho} \, x^\beta \, \Phi^\alpha \, dV \quad . \tag{17}$$

Using eqs. (8) and (9) we obtain

$$\frac{dX^\beta}{dt} \int T^{a0} \, dV + \frac{d}{dt} \int \delta x^\beta T^{\alpha 0} \, dV = \int T^{\alpha\beta} \, dV + \int \bar{\rho} \, \delta x^\beta \, \Phi^a \, dV \cdot \tag{18}$$

To zeroth order in δx^β this equation becomes'

$$\frac{u^\beta}{u^0} \int T^{\alpha 0} \, dV = \int T^{\alpha\beta} \, dV \tag{19}$$

where

$$u^\beta = \frac{dX^\beta}{ds} \quad . \tag{20}$$

Putting a = 0 in eq.(19) we obtain with eq.(12)

$$p^\beta = \frac{p^0}{u^0} \; u^\beta \; . \tag{21}$$

Substituting this equation into equation (11) we have

$$\frac{d}{ds} \left(\frac{p^0}{u^0}\right) u^\alpha + \frac{p^0}{u^0} \frac{du^\alpha}{ds} = g \; \partial^\alpha \Phi \tag{22}$$

Now we contract this equation with u^α. Noting that $u_\alpha u^\alpha = 1$, $u_\alpha du^\alpha/ds = 0$ and $u_\alpha \partial^\alpha \Phi = d\Phi/ds$, we obtain

$$\frac{d}{ds}\left(\frac{p^0}{u^0}\right) = g\,\frac{d\Phi}{ds} . \tag{23}$$

From this we obtain a constant of motion

$$\frac{p^0}{u^0} - g\Phi = \text{constant} = m \tag{24}$$

which we identify with the mass of our particle. Returning to eq. (22) we finaly obtain the explicit form of the particle equation of motion

$$m\,\frac{du^\alpha}{ds} = g\,\partial^\alpha\Phi - g\,\frac{d(\Phi u^\alpha)}{ds} . \tag{25}$$

This is the known equation for a particle interacting with a scalar field, which can be derived from the lagrangian $L = -(m+g\Phi)\times$ $\times\,(u^0)^{-1}$. As it was mentioned before, the moment method also gives the form of the energy-momentum tensor of the particle. Frsm eqs. (12),(19), (21) and (24) we have

$$\int T^{\alpha\beta}\,dV = \frac{1}{u^0}\,(m+g\Phi)u^\alpha u^\beta . \tag{26}$$

This is the integrated form of the energy-momentum of our particle.

From this we obtain

$$T^{\alpha\beta} = (\bar\sigma + \bar\rho\Phi)u^\alpha u^\beta \tag{27}$$

where $\bar\sigma$ is the rest mass density of the particle and $\bar\rho$ its charge density

$$\sigma(\vec{x}) = \frac{1}{u^0}\,m\,\delta(\vec{x}-\vec{X}), \quad \rho(\vec{x}) = \frac{1}{u^0}\,g\,\delta(\vec{x}-\vec{X}) . \tag{28}$$

3. THE EQUATIONS OF MOTION FOR A SPINNING PARTICLE

After illustrating the procedure for a spinless point particle we turn to the case of a spinning particle with a nucleonic dipole moment. For this we substitute eq. (10) into eqs. (8) and (18) and we keep terms to first order in δx^α. From the first equation we obtain

$$\frac{dp^\alpha}{ds} = g\Phi^\alpha + D^\beta \partial_\beta \Phi^\alpha \tag{29}$$

where

$$D^\alpha = \int \rho \delta x^\alpha \, dV \tag{30}$$

is the nucleonic dipole moment of our system.

From eq. (18) we obtain, after multiplication by u^0,

$$u^\beta p^\alpha + \frac{d}{ds} \int \delta x^\beta \, T^{\alpha 0} \, dV = u^0 \int T^{\alpha\beta} \, dV + D^\beta \Phi^\alpha . \tag{31}$$

Subtracting from this equation the one obtained by interchanging α and β we obtain the equation

$$\frac{dS^{\alpha\beta}}{ds} = p^\alpha u^\beta - p^\beta u^\alpha + D^\alpha \Phi^\beta - D^\beta \Phi^\alpha \, , \tag{32}$$

where

$$S^{\alpha\beta} = \int (\delta x^\alpha \, T^{\beta 0} - \delta x^\beta \, T^{\alpha 0}) \, dV \tag{33}$$

is the spin tensor of the particle.

Eqs. (29) and (32) are the equations of motion of a nucleon with spin and dipole moment. They coincide with those obtained by Harish-Chandra by the method of Dirac. These equations are approximate equations of motion for an extended system and are exact for a point particle with spin and dipole moment, defined as the limit of the extended syçtem when the volume or δx^α tend to zero, with ρ and $T^{\alpha\beta}$ going to infinity in such a way that the integrals (12) and (13) remain finite, just as one defines the usual electric charge multipoles.

Considering that g and D^a are given we have fourteen quantities to be determined, that is p^α, u^α and $S^{\alpha\beta}$. As eqs. (29) and (32) add to only ten equations we still need four more equations to solve for the unknown quantities. These extra subsidiary equations are related to the choice of the ,particular point X for the extended system. One possibility is to choose $S^{\alpha 0} = 0$ in the system's rest frame. From eq. (33) this cheice gives the following expression for the reference point

$$X^{\alpha} = \frac{\int x^{\alpha}\, T^{00} dV}{\int T^{00}\, dV} \quad . \tag{34}$$

As we see the choice $S^{\alpha 0} = 0$ is equivalent to the choice of X^{α} as the center of energy of the extended system. In an arbitrary system the choice $S^{\alpha 0} = 0$ in the rest system becomes

$$u_{\alpha}\, S^{\alpha\beta} = 0 \tag{35}$$

This gives the four additional equations that we need to specify our problem.

As we shall see now, we can also determine the energy-momentum tensos and the nucleonic current and their moments. Using eq.(33) wesee that eq.(31) can be written as

$$u^0 \int T^{\alpha\beta} dV = u^{\beta} p^{\alpha} - D^{\beta}\bar{\Phi}^{\alpha} - \frac{1}{2}\frac{dS^{\alpha\beta}}{ds} + \frac{1}{2}\frac{d}{ds}\int(\delta x^{\beta} T^{\alpha 0} + \delta x^{\alpha} T^{\beta 0})dV \quad . \tag{36}$$

To determine the last term in terms of u^{α} and $S^{\alpha\beta}$ we consider the divergence of $x^{\alpha}x^{\beta}T^{\lambda\nu}$

$$\partial_{\nu}(x^{\alpha}x^{\beta}T^{\lambda\nu}) = x^{\beta}T^{\lambda\alpha} + x^{\alpha}T^{\lambda\beta} + x^{\alpha}x^{\beta}\,\bar{\rho}\Phi^{\lambda} \quad . \tag{37}$$

Integration of this equation over the system gives

$$\frac{d}{dt}\int x^{\alpha}x^{\beta}T^{\lambda 0}dV = \int(x^{\beta}T^{\lambda\alpha} + x^{\alpha}T^{\lambda\beta})dV + \int x^{\alpha}x^{\beta}\,\bar{\rho}\Phi^{\lambda}\, dV \quad . \tag{38}$$

Using eqs. (9), (17) and (18) we obtain to order δx^{a},

$$\frac{dX^{\alpha}}{dt}\int\delta x^{\beta}T^{\lambda 0}dV + \frac{dX^{\beta}}{dt}\int\delta x^{\alpha}T^{\lambda 0}dV = \int(\delta x^{\alpha}T^{\lambda\beta} + \delta x^{\beta}T^{\lambda\alpha})dV \quad . \tag{39}$$

Now we add to this equation the one resulting from the interchange of a and A and subtract the one comming from the interchange of β and A. The result is

$$2\int \delta x^{\beta}T^{\lambda\alpha}dV = \frac{dX^{\lambda}}{dt}S^{\beta\alpha} + \frac{dX^{\alpha}}{dt}S^{\beta\lambda} + \frac{dX^{\beta}}{dt}\int(\delta x^{\alpha}T^{\lambda 0} + \delta x^{\lambda}T^{\alpha 0})dV . \quad (40)$$

Putting $\beta = 0$ in this equation and recalling that $\delta x^0 = 0$ we obtain

$$0 = \frac{dX^{\lambda}}{dt}S^{0\alpha} + \frac{dX^{\alpha}}{dt}S^{0\lambda} + \int(\delta x^{\alpha}T^{\lambda 0} + \delta x^{\lambda}T^{\alpha 0})dV . \quad (41)$$

Using this equation in (36) we obtain

$$u^0 \int T^{\alpha\beta}dV = u^{\beta}p^{\alpha} - D^{\beta}\Phi^{\alpha} - \frac{1}{2}\frac{dS^{\alpha\beta}}{ds} + \frac{1}{2}\frac{d}{ds}\left(\frac{u^{\beta}}{u^0}S^{\alpha 0} + \frac{u^{\alpha}}{u^0}S^{\beta 0}\right) . \quad (42)$$

This is the integrated form of the energy-momentum tensor of the system.

Using eqs. (41) in (40) we also obtain the integrated first moment of the energy-momentum tensor

$$\int \delta x^{\beta}T^{\lambda\alpha}dV = \frac{1}{2}\left(\frac{u^{\lambda}}{u^0}S^{\beta\alpha} + \frac{u^{\alpha}}{u^0}S^{\beta\lambda}\right) + \frac{1}{2}\frac{u^{\beta}}{u^0}\left(\frac{u^{\lambda}}{u^0}S^{\alpha 0} + \frac{u^{\alpha}}{u^0}S^{\lambda 0}\right) . \quad (43)$$

Now we turn to the nucleonic current. With eq. (14) we have the identity

$$\partial_{\lambda}(x^{\alpha}J^{\lambda}) = J^{\alpha} . \quad (44)$$

Integration of this equation over the volume of our system gives

$$\frac{d}{dt}\int x^{\alpha}J^0 dV = \int J^{\alpha}dV . \quad (45)$$

Using eqs. (9) and (13) we obtain

$$\frac{dX^{\alpha}}{dt}g + \frac{d}{dt}\int \delta x^{\alpha}J^0 dV = \int J^{\alpha}dV . \quad (46)$$

Therefore

$$\int J^{\alpha}dV = g\frac{u^{\alpha}}{u^0} + \frac{1}{u^0}\frac{dD^{\alpha}}{ds} \quad (47)$$

which is the integrated form of the current. Besides the convection term we also have a contribution from the dipole moment.

4 EQUATIONS OF MOTION FOR CONSTANT SPIN AND DIPOLE MOMENT

Let us now see in what physical circumstances we can simplify the equations of motion.

First we note that as $D^0 = 0$ we have

$$u_\alpha D^\alpha = 0 \tag{48}$$

since this is valid in the particle rest frame. Also, as $dD^0/ds = 0$, we also have

$$u_\alpha \dot{D}^\alpha = 0 \tag{49}$$

where the dot designates d/ds.

From eqs. (48) and (49) it follows that

$$\dot{u}_\alpha D^\alpha = 0 \quad . \tag{50}$$

From eq.(35) we have

$$u_\alpha \dot{S}^{\alpha\beta} = - \dot{u}_\alpha S^{\alpha\beta} \quad . \tag{51}$$

Contracting eq.(32) with u_β and using eqs. (48) and (51) we have the relation

$$p^\alpha = (u_\beta p^\beta) u^\alpha - D^\alpha \dot{\Phi} + \dot{u}_\beta S^{\beta\alpha} \quad . \tag{52}$$

We shall now try to obtain $(u_\alpha p^a)$. Contracting eq.(52) with $\dot{u}_\alpha$ and using eq.(50) together with the antisymmetric character of $S_{\alpha\beta}$, we obtain

$$\dot{u}_\alpha p^\alpha = 0 \quad . \tag{53}$$

Contraction of eq.(29) with u_α gives

$$u_\alpha \dot{p}^\alpha = g\dot{\Phi} + D^\beta \dot{\Phi}_\beta \quad . \tag{54}$$

Adding these last two equations we obtain

$$\frac{d}{ds}\left(u_\beta p^\beta - g\Phi - D^\beta \Phi_\beta\right) = - \dot{D}^\beta \Phi_\beta \quad . \tag{55}$$

As we shall see now, the right-hand side of this equation is zero if we postulate that both the spin and the dipole moment of the particle have constant-magnitudes, that is

$$S^{\alpha\beta} S_{\alpha\beta} = \text{constant} \tag{56}$$

$$D^\alpha D_\alpha = \text{constant} \quad . \tag{57}$$

Contracting eq.(32) with $S_{\alpha\beta}$ and using eqs. (35) and (56) we have $S_{\alpha\beta} D^\beta \Phi^\alpha = 0$ and, therefore, as Φ^α is arbitrary

$$S_{\alpha\beta} D^\beta = 0 \quad . \tag{58}$$

From this we have

$$\dot{S}_{\alpha\beta} D^\beta = - S_{\alpha\beta} \dot{D}^\beta \quad . \tag{59}$$

Now contract eq.(32) with $D_\alpha \dot{D}_\beta$. Using eq.(59) and the antysimmetric nature of $S^{\alpha\beta}$ together with eqs. (48), (49) and (57) we obtain the relation

$$0 = D_\alpha D^\alpha \Phi^\beta \dot{D}_\beta \quad . \tag{60}$$

From here it follows that

$$\Phi^\beta \dot{D}_\beta = 0 \quad . \tag{61}$$

In this way eq.(55) gives us the following constant of motion:

$$u^\beta p_\beta - g\Phi - D^\beta \Phi_\beta = \text{constant} = m \tag{62}$$

which we identify with the mass of particle.

From tkis we obtain

$$u_\beta p^\beta = m + g\Phi + D^\alpha \overset{a}{\Phi}_\alpha \quad . \tag{63}$$

Substituting this result into eq.(52) we obtain

$$p^\alpha = (m + g\Phi + D^\beta \Phi_\beta) u^\alpha - D^\alpha \dot{\Phi} + \dot{u}_\beta S^{\beta\alpha} \quad . \tag{64}$$

From eqs. (35) and (58) we see that spin tensor is perpendicular to both the four-velocity and the dipole moment, that is, we can write the relation

$$S^{\alpha\beta} = I \ \varepsilon^{\alpha\beta\lambda\nu} \ u_\lambda D_\nu \ . \tag{65}$$

Contracting this equation with itself and using eq.(48) we obtain

$$S^{\alpha\beta} S_{\alpha\beta} = -2 \ I^2 \ D^\lambda D_\lambda \ . \tag{66}$$

This shows that I is a constant and that the magnitude of the spin and the dipole moment are related to each other.

Eqs. (64) and (65) coincide with the expression given by Harish-Chandra[1] who uses eq.(65) to prove eq. (64).

Substituting eq. (65) into (64) we obtain

$$p^\alpha = (m + g\Phi + D^\beta \Phi_\beta) u^\alpha - D^\alpha \dot{\Phi} + I \ \varepsilon^{\alpha\lambda\beta\nu} \ u_\lambda \dot{u}_\beta D_\nu \ . \tag{67}$$

Using this result in eq. (29) we obtain

$$\frac{d}{ds} \left[(m + g\Phi + D^\beta \Phi_\beta) u^\alpha - D^\alpha \dot{\Phi} + I \varepsilon^{\alpha\lambda\beta\nu} \ u_\lambda \dot{u}_\beta D_\nu\right] = g\Phi^\alpha + D^\beta \ \partial_\beta \ \Phi^\alpha \ . \tag{68}$$

Substitute now eqs. (65) and (67) into eq. (32). After using the equality

$$u^\alpha \varepsilon^{\beta\lambda\nu\rho} \ u_\lambda \dot{u}_\nu D_\rho - u^\beta \ \varepsilon^{\alpha\lambda\nu\rho} \ u_\lambda \dot{u}_\nu D_\rho = \varepsilon^{\alpha\beta\lambda\nu} \ D_\lambda \dot{u}_\nu$$

wich can be verified by contraction of the right-hand side with $\varepsilon_{\alpha\beta\tau\sigma}$, we obtain

$$I\ \varepsilon^{\alpha\beta\lambda\nu}\ u_\lambda \dot{D}_\nu = D^\alpha(\Phi^\beta - u^\beta\dot{\Phi}) - D^\beta(\Phi^\alpha - u^\alpha\dot{\Phi}) \tag{69}$$

Eqs. (68) and (69), which contain two arbitrary constantsmand I, are the equations of motíon of the nucleon with spin and dipole moment constant in magnitude.

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Resumo

As equações de movimento de um nucleon com spin em interação com um campo escalar sem massa são obtidas a partir da conservação da energia-momentum e da carga, pelo método dos momentos de Papapetrou.