Revista Brasileira de Fkica, Vol. 16, nº 4, 1986

Dispersion Relation of the Collective Modes of Alfvén Wave Resonant Heating

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Recebido em 06 de maio de 1986

Abstract By considering the magnetic cornpressibility it is shown that, for a theta pinch type plasma, the m=l collective mode could be cut off. This is in contrast with the results based on an incompressible fluid model. This restricts the collective mode to a small region of k space near zero.

Alfvén wave plasma heating using the continuum spectrum[1,2,3] relies on the resonant absorption at the singular Alfvén layer by exciting a collective mode. Early works[1,2] considered an incompressible fluid so that the collective mode is a surface mode. With the inclusion of magnetic compressibility, the collective mode becomes a niagetosonic bulk mode whose first radial mode resembles the surface mode. Balet, Appert, and Vaclavik[4] solved the linearized magnetohydrodynamic (MHD) eigenvalue equation numerically and compared the heating characteristics of the first and second radial mode. They concluded that the first radial mode is far more efficient. Ott, Wersinger and Bonoli[3] solved the magnetosonic mode with poloidal mode number $m=0$ and Nozaki, Fried snd Morales[5] treated the case of a high-β semi-infinite plasma. Both of them used WKB analysis and slab geometry. Besides, in the case $m=0$, the Alfvén mode is actually decoupled from the magnetosonic mode. Here we consider a theta pinch type high-β plasma and solve the radial eigenvalue equation for $m\neq 0$.

We consider a cylindrical plasma with a magnetic field in the $\hat{z}$ direction. Both the plasma density and magnetic field are constant up to $r=a$ (first region) followed by linear profiles of decreasing density and increasing rnagnetic pressure up to $r=b$ (second region) and then jointed by a very low density plasma (third region) limited by a conducting wall at $r=R$. We obtain the solutions in each region and connect them up by boundary conditions. A vacuum third region would lead to the same dispersion relation.

Taking the Fourier component of the perturbations of the form $\exp i(kz+m\theta-\omega t)$, the equation for the radial displacement ξ_r is

$$\frac{\partial}{\partial r}\left\{\frac{B^2\left(k^2-\frac{\omega^2}{V_A^2}\right)}{\left(k^2+\frac{m^2}{r^2}\right)-\frac{\omega^2}{V_A^2}}\,\frac{1}{r}\frac{\partial}{\partial r}\,(r\xi_r)\right\}-B^2\left(k^2-\frac{\omega^2}{V_A^2}\right)\xi_r = 0$$

We now normalize the distance by a and magnetic field by its vacuum value, B_V. Multiplying the above equation by a/B_V^2, we have

$$\frac{\partial}{\partial r}\left\{\frac{\alpha^2\left(k^2-\frac{\omega^2}{\alpha^2\Omega_t^2}\frac{n}{n_0}\right)}{\left(k^2+\frac{m^2}{r^2}\right)-\frac{\omega^2}{\alpha^2\Omega_t^2}\frac{n}{n_0}}\,\frac{1}{r}\frac{\partial}{\partial r}\,(r\xi_r)\right\}-\alpha^2\left(k^2-\frac{\omega^2}{\alpha^2\Omega_t^2}\frac{n}{n_0}\right){}_r = 0 \tag{1}$$

where k, r, ξ_r are now normalized dimensionless quantities, n_0 is the plasma density at the center, $\Omega_t = V_A/a$ is the Alfvén transit frequency calculated with respect to n and B_V, $\alpha(r) \equiv B(r)/B_V \leq 1$ is related to plasma β by

$$\beta = 1 - \alpha^2(0)$$

and the shear Alfvén dispersion relation

$$\omega^2 = \alpha^2(r_s)k^2\Omega_t^2$$

is assurned to be satisfied in the second region where $1 \leq r \leq b/a$. In the first region where $r \leq 1$ the modes are described by

$$\frac{\partial}{\partial r}\left\{\frac{1}{\frac{m^2}{r^2}-\kappa^2}\,\frac{1}{r}\frac{\partial}{\partial r}\,(r\xi_r)\right\}-\xi_r = 0 \tag{2}$$

where

$$\kappa^2 = \left(\frac{\omega^2}{\alpha^2(0)\Omega_t^2}-k^2\right) > 0$$

whose solution is

$$\xi_r = A_1 \frac{\partial}{\partial r} J_m(\kappa r) \tag{3}$$

In the second region, we write the density and magnetic pressure as

$$n(r) = n_0[(b/a) - r]/[(b/a) - 1]$$

$$\begin{aligned} \alpha^2(r) &= 1 - \beta[(b/a) - r]/[(b/a) - 1] \\ &= 1 - \frac{\beta b/a}{(b/a) - 1} + \frac{\beta}{(b/a) - 1} r \end{aligned}$$

and eq. (1) becomes

$$\frac{\partial}{\partial r}\left\{\frac{(Br - A)}{k^2 + \frac{m^2}{r^2} - \frac{\omega^2}{\alpha^2\Omega_t^2}\frac{n}{n_0}} \frac{1}{r}\frac{\partial}{\partial r}(r\xi_r)\right\} - (Br-A)\xi_r = 0 \tag{4}$$

$$A = \frac{b/a}{(a/b) - 1}\frac{\omega^2}{\Omega_t^2} - \left(1 - \frac{\beta b/a}{(a/b) - 1}\right) k^2$$

$$B = \frac{1}{(a/b) - 1}\frac{\omega^2}{\Omega_t^2} + \frac{\beta}{(a/b) - 1} k^2$$

In Alfvén wave heating, we usually have $m^2/r^2 >> k^2$, so that the radial variation of eq. (4) comes mostly from the shear Alfvén branch. In the opposite case, magnetosonic cavity modes (second and higher radial modes) can be set up in the plasma[7]. The Alfvén resonant layer which is located outside the homogeneous plasma magnetosonic cutoff is near the plasma surface and the heating efficiency is low[4]. We then replace the magnetosonic branch variables by their averages as was done by Grossmann and Tataronis[1]. We rewrite eq. (4) and its solution as

$$\frac{\partial}{\partial r}\{(Br-A)\frac{\partial}{\partial r}(r\xi_r)\} - \varepsilon^2(Br-A)(r\xi_r) = 0$$

$$\varepsilon^2 = \langle k^2 + \frac{m^2}{r^2} - \frac{2}{\alpha^2\Omega_t^2}\frac{n}{n_0}\rangle \simeq \frac{m^2}{\langle r^2\rangle}$$

$$r\xi_r(r) = A_2 I_0(Z(r)) + A_3 K_0(Z(r))$$

$$Z(r) = \varepsilon(Br - A)/B \qquad (5)$$

In the third region, eq. (1) and its solution are

$$\frac{\partial}{\partial r}\left\{\frac{1}{k^2+\frac{m^2}{r^2}}\,\frac{1}{r}\,\frac{\partial}{\partial r}\,(r\xi_r)\right\} - \xi_r = 0 \qquad (6)$$

$$\xi_r(r) = A_4\,\frac{\partial}{\partial r}\,I_m(kr) + A_5\,\frac{\partial}{\partial r}\,K_m(kr) \qquad (7)$$

Using the linearized MHD equations[6], the continuity of the perturbed radial magnetic field and the total pressure across $r=1$ and $r = a/b$ is equivalent to the continuity of $d\log\xi_r/dr$. The condition at $r = R/a$ is $\xi_r = 0$. We join the solutions for the three regions and normalize the coefficients to As by choosing $A_5 = 1$. The eigenvalues are determined by

$$\frac{V}{U} = \{J_m'(\kappa r)_1\left[K_0(Z(b/a))I_0'(A(1)) - I_0(Z(b/a))K_0'(Z(1))\right]$$

$$+ (\kappa^2 - \frac{m^2}{r^2})_1\; J_m(\kappa r)_1\;\left[K_0(Z(b/a))I_0(Z(1)) - I_0(Z(b/a))K_0(Z(1))\right]\}$$

$$/\{J_m'(\kappa r)_1\;\left[K_0'(Z(b/a))I_0'(Z(1)) - I_0'(Z(b/a))K_0'(Z(1))\right]$$

$$+ (\kappa^2 - \frac{m^2}{r^2})_1\; J_m(\kappa r)_1\;\left[K_0(Z(b/a))I_0(Z(1)) - I_0(Z(b/a))K_0(Z(1))\right]\}$$

$$(8)$$

where $J_m'(\kappa r)$ stands for the radial derivative of $J_m(\kappa r)$ and so on, and

$$Z(1) = -\frac{\varepsilon[(b/a) - 1]W^2}{(W^2 + k^2)} < 0$$

$$Z(b/a) = \frac{\varepsilon[(b/a) - 1]k^2}{(W^2 + k^2)} > 0$$

$$W^2 \equiv (\frac{\omega^2}{\Omega_t^2} - \alpha(0)k^2) = \alpha^2(0)\kappa^2 > 0$$

$$U = \frac{b}{a}\ (k^2 + \frac{m^2}{r^2})\ \{A_4\ I_m(kr) + K_m(kr)\}_{r=b/a}$$

$$V = \frac{b}{a}\ \{A_4\ I_m'(kr) + K_m'(kr)\}_{r=b/a}$$

$$A_4 = -\{K_m'(kr)/I_m'(kr)\}_{r=R/a}$$

Eq. (8) is a complex value equation since $Z(1)<0$. Considering $b/a \cong 1$, we can expand $I_0(Z)$, $K_0(Z)$ and their derivatives by small argument expansions. We choose the branch cut of $\log Z(1) = \log(-Z(1)) - i$ where both $\log Z$ and $\log(-Z)$ are on the same sheet. Writing $W = W_R + iW_I$ and assuming $|W_R| >> |W_I|$, we can also expand $J_m(\kappa r)$ and its derivative about W_R. With all the terms considered, eq. (8) reads

$$\frac{V}{U} = (G_1 + iG_2)/(G_3 - iG_4) \tag{9}$$

$$G_1 = 2(\kappa^2 - \frac{m^2}{r^2})\ \log(\frac{W}{k})J_m(\kappa r) - \frac{(W^2 + k^2)}{(a/b) - 1}\ \frac{1}{W^2}\ J_m'(\kappa r)$$

$$G_3 = -\frac{(W^2 + k^2)}{(a/b) - 1}\ (\kappa^2 - \frac{m^2}{r^2})\ \frac{1}{k^2}\ J_m(\kappa r) + \frac{\varepsilon^2}{2}(\frac{W^2}{k^2} - \frac{k^2}{W^2})J_m'(\kappa r)$$

$$G_2 = W_I \left\{ \left[\frac{(W^2 + k^2)}{(a/b) - 1}\ \frac{1}{W^2}\ \frac{\kappa r}{\alpha(0)}\ (1 - (\frac{\kappa r}{m})^2) + \frac{4\kappa}{\alpha(0)}\ \log(\frac{W}{k}) + 2(\kappa^2 - m^2/r^2)\frac{1}{W}\right] J_m(\kappa r) + \left[\frac{2(W^2 + k^2)}{(b/a) - 1}\ \frac{1}{W^3} - \frac{2}{(b/a) - 1}\ \frac{1}{W} + 2(\kappa^2 - m^2/r^2)\log(\frac{W}{k})\ \frac{r}{W}\right] J_m'(\kappa r) \right\} - \pi(\kappa^2 - m^2/r^2)J_m(\kappa r)$$

$$\equiv W_I . G_{x2} - P_2$$

with

$$P_2 = \pi(\kappa^2 - m^2/r^2)J_m(\kappa r)$$

and

$$G_4 = W_I\left\{\left[\frac{2(W^2+k^2)}{(b/a)-1}\ \frac{1}{k^2}\ \frac{\kappa}{\alpha(0)} + \frac{2}{(b/a)-1}\ (\kappa^2 - \frac{m^2}{r^2})\ \frac{W}{k^2}\right.\right.$$

$$\left. + \frac{\varepsilon^2}{2}\left(\frac{W^2}{k^2} - \frac{k^2}{W^2}\right)\frac{\kappa r}{\alpha(0)}\ \left(1 - \left(\frac{\kappa r}{m}\right)^2\right)\right] J_m(\kappa r)$$

$$\left. + \left[\frac{(W^2+k^2)}{(b/a)-1}\ (\kappa^2 - \frac{m^2}{r^2})\ \frac{r}{W}\ \frac{1}{k^2} - \varepsilon^2\left(\frac{W^2}{k^2} + \frac{k^2}{W^2}\right)\frac{1}{W}\right] J_m'(\kappa r)\right\}$$

$$+ \pi\,\frac{\varepsilon^2}{2}\ \frac{(b/a)-1}{(W^2+k^2)}\ k^2\ (\kappa^2 - \frac{m^2}{r^2})\ J_m(\kappa r)$$

$$\equiv W_I \cdot G_{x4} + P_4$$

with

$$P_4 = \pi\,\frac{\varepsilon^2}{2}\ \frac{(b/a)-1}{(W^2-k^2)}\ k^2(\kappa^2 - m^2/r^2)J_m(\kappa r)$$

and where, for simplicity, the subscript of W_R is suppressed and all the expressions are evaluated at $r = 1$. The dispersion relation of the magnetosonic mode is given by

$$\frac{V}{U} = \frac{G_1 \cdot G_3}{(G_3)^2 + (G_4)^2} \cong \frac{G_1}{G_3} \tag{10}$$

and the damping rate is

$$W_I = \frac{P_2 \cdot G_3 - P_4 G_1}{G_{x2} \cdot G_3 - G_{x4} \cdot G_1} \tag{11}$$

Consequently, with $\omega = \omega_r + i\omega_i$

$$\frac{\omega_r}{\omega_t} = (W_R^2 + \alpha^2(0)k^2)^{1/2} \tag{12}$$

$$\frac{\omega_i}{\Omega_t} = \frac{W_R\, W_I}{\omega_r/\Omega_t} \tag{13}$$

To evaluate the effect of compressibility, we compare with the incompressible case where, in equation (8), $J_m(\kappa r)$, $(J_m')\kappa r$ become $I_m(kr)$, $I_m'(kr)$ and κ^2 becomes $(-k^2)$. The expressions of G_1 and G_3 follow the same substitution, whereas G_2 and G_4 read

$$G_2 = W_I \left\{ \left[-2(k^2 + m^2/r^2)\, \frac{1}{W} \right] I_m(kr) + \left[\frac{2(W^2+k^2)}{(b/a)-1}\, \frac{1}{W^3} - \frac{2}{(b/a)-1}\, \frac{1}{W} \right] I_m'(kr) \right\}$$

$$- \left[-\pi(k^2 + m^2/r^2)\; I_m(kr) \right]$$

$$G_4 = W_I \left\{ \left[-\frac{2}{(b/a)-1}\, (k^2 + m^2/r^2)\frac{W}{k^2} \right] I_m(kr) + \left[-\varepsilon^2 \left(\frac{W^2}{k} + \frac{k^2}{W} \right) \frac{1}{W} \right] I_m'(kr) \right\}$$

$$+ \left[-\pi\, \frac{\varepsilon^2}{2}\, \frac{(b/a)-1}{(W^2+k^2)}\, k^2(k^2 + m^2/r^2) \right] I_m(kr)$$

In the original work on Alfvén wave heating by Grossmann and Tataronis[1], they considered the incompressible surface mode and arrived at a somewhat simpler result by taking $I_0(Z) = 1$, $I_0'(Z) = 0$. The magnetic compressibility significantly reduces the damping rate. Fig. (1) compares the damping rate for $m = 1$. The case of $\beta = 0$ is shown in fig. (2). The dispersion relation is also presented in the same figure in each case. The most interesting feature is that the $m = 1$ magnetosonic mode exists only at low frequencies. For $k \cong 0$

$$J_m(\kappa r) \cong I_m(kr)$$

$$\left(\frac{W}{\alpha^2(0)} - \frac{m^2}{r^2} \right) \cong -(k^2 + m^2/r^2) \cong -m^2/r^2$$

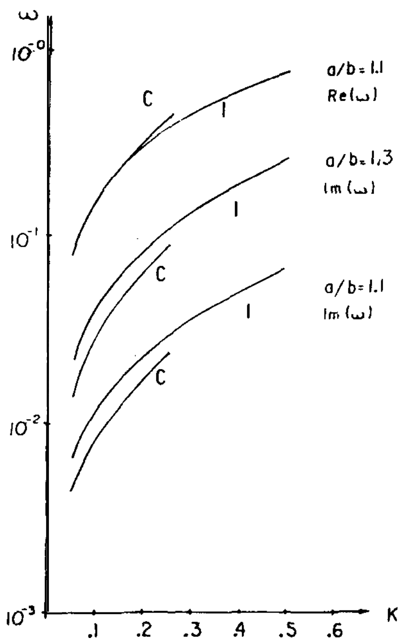

Fig. 1 - Real and imaginary part of ω versus k for different values of a/b with β=0.5 showing that the collective mode ceases at k=0.25 with compressibility (C) yet continues to k=0.5 and beyond without compressibility (I).

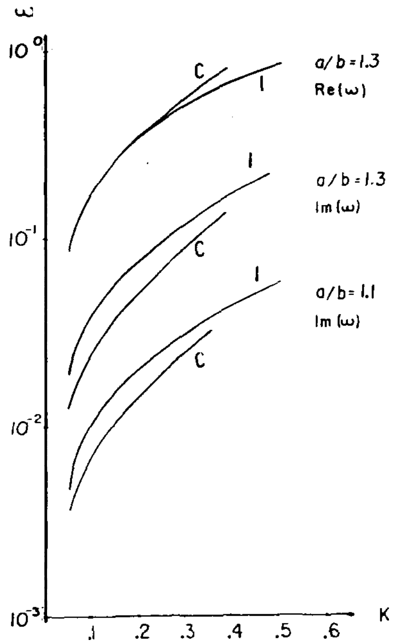

Fig.2 - Real and imaginary part of ω versus k for different values of a/b with β=0.

As k increases, magnetic compressibility becomes important and essential differences begin to appear, and for $k>k_{max}$ eq. (10) allows no solutions, which is in contrast with the surface mode that allows solutions for any k^1. This limits the collective mode to a small region of k space near zero. For m ≥ 2, k_{max} becomes much larger and there is not much difference from the surface mode.

This work was supported by the Financiadora de Estudos e Projetos (FINEP), the Comissão Nacional de Energia Nuclear (CNEN, the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), and the Laboratório de Computação Científica (LCC).

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Resumo

Considerando a compressibilidade magnética, é mostrado que, para um plasma numa configuração de θ-pinch, o modo coletivo $m=1$ poderia ser limitado em contraste com os resultados baseados no modelo do fluido incompressivel. Isto reduz o modo coletivo a uma pequena região do espaço dos k perto de zero.