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A Comment on the Null Geodesic Equations in Schwarzschild Geometry

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Abstract In this paper we present an integration of the null geodesic equations in the Schwarzschild geometry, which is valid to first order in GM/Rc^2. We compare our solution with others published in the literature and analyse their range of validity. We also clarify some misunderstandings.

In the Schwarzschild field the metric is

$$ds^2 = B(r)\, dt^2 - A(r)\, dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2 \tag{1}$$

where

$$B(r) = \left[A(r)\right]^{-1} = 1 - 2m/r \;;\; m = MG/c^2 \;;$$

$$r > 2m \;;\quad 0 < \theta < \pi;\quad 0 < \phi < 2\pi \tag{2}$$

The time-like and null geodesic equations can be written[1] as

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 + \frac{mE}{J^2} \tag{3.a}$$

$$r^2\,\frac{d\phi}{dp} = J \tag{3.b}$$

$$\theta = \pi/2 \tag{3.c}$$

$$A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = E \tag{3.d}$$

where $u = 1/r$, $dp = Bdt$, and E and J are constants. Also for $\theta = \pi/2$

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$$ds^2 = E\, dp^2 \tag{4}$$

and we see that bradyons are characterized by $E > 0$ and photons by $E=0$; when E = 0 eq. (3.a) becomes

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \tag{5}$$

and eq. (3.b) has the physical meaning that the generalized *law* of *areas* is valid also for luxons.

Observe that the constant J can be calculated from the critical point of the orbit. Indeed, if $P_0 \equiv (R_0, \pi/2)$ is the critical point of the orbit we have from eq. (3.d)

$$J^2 = R_0^2/B(R_0) \tag{6}$$

For weak fields we get, to first order in m/R_0,

$$R_0 = -m + |J| \tag{7}$$

The invertibility of eq.(6) in a weak field implies that the orbit has only one critical point (for the function *radial coordinate relative* the *center*). Recalling that $A(r) > 0$ we get frorn eq. (3.d) with $E = 0$,

$$J^2 \leq r^2/B(r) \tag{8}$$

which implies that P_0 is a point of minimum. This result is inagreement with our intuition that light must follow an orbit that looks like a hyperbola in a weak Schwarzschildfield, similar to the trajectory of a particle with speed greaterthan the escape speed. Also our intuition says that the shape of the orbit must by symrnetric with respect to the straight line through the center of the coordinate systern to F,. This is indeed true, as can be see from the following argument: eq.(5) gives us the orbit of the luxon, $r = r(\phi) = [u(\phi)]^{-1}$ once we give the initia! conditions

$$u(\pi/2) = 1/R_0 \ ; \quad u'(\pi/2) = 0 \ , \tag{9}$$

and both the initial conditions as well as eq. (5) are invariant by the substitution $\phi \rightarrow$ -$. Then $r(\phi) = r(-\phi)$.

The classical equation for the orbit of a light ray is

$$\frac{d^2u}{d\phi^2} + u = 0 \tag{10}$$

With the initial conditions given by eq. (9) we have the solution

$$u_0 = \frac{\sin\phi}{R_0} \tag{11}$$

We see that eq. (5) differs from eq. (10) by the factor $3mu^2$ and, since $u = 1/R < 1/R_0$, we can solve eq. (5) by iteration. For the first iteration,

$$\frac{d^2u_1}{d\phi^2} + u_1 = 3mu_0^2 \quad . \tag{12}$$

Ignoring terms of second and higher orders in (mu) and using eq. (9), we get

$$u_1 = \frac{1}{R_0}\left(1 - \frac{m}{R_0}\right)\sin\phi + \frac{3m}{2R_0} - \frac{m}{2R_0^2}\sin 2\phi \tag{13}$$

Also ignoring terms of second order in (mu), we have from eq. (13)

$$u_1^2 \simeq u_0^2 \tag{14}$$

Eq. (14) warrants that eq. (13) is correct to the first order in mu.

We now analyse a possible misunderstanding which appears in some text-books[2]. There eqs. (3) are written as

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{h^2} + 3mu^2 \tag{15.a}$$

$$r^2\frac{d\phi}{ds} = h \quad ; \quad h = \text{constant} \tag{15.b}$$

$$\theta = \pi/2$$

We first observe that the use of s as the parameter in eq.(15.a) is irnproper since for luxons it is $ds^2 = 0$. Also eqs.(3) depend on two integration constants E and J while eqs. (15) depend only on the constant h, and the only way to make them compatible is to write (using eq.(4))

$$ds^2 = Edp^2 \; ; \quad h = J/\sqrt{E} \tag{16}$$

For the case of luxons E = 0 and it is necessary to put h = ∞, which is indeed what Rindler did when he calculated the bending of light by the sun. With h = ∞ eq.(15.a) becomes

$$\frac{d^2u}{d\phi^2} + u = 3mu^2 \tag{17}$$

and eq. (15.b) makes no sense. Also, in Rindler's book[2], for example, instead of eq. (13) he found the solution

$$u_1^r \simeq \frac{\sin\phi}{R_0} + \frac{3m}{2R_0^2} - \frac{m}{2R_0^2}\cos 2\phi \tag{18}$$

Although eq. (18) gives the same result as eq. (13) for the particular problem of the bending of light in the sun's field, we observe that

$$u_1^R(\pi/2) = \frac{1}{R_0}\left(1 + \frac{2m}{R_0}\right) \tag{19}$$

and Rindler's formula, contrary to eq.(13), leads to an error of first order in (m/R_0) when we calculate the critica1 point of the orbit.

Wewould like to comment also in the so-called *radar echo delay formula*[1]. Observe that using $E = 0$ and $dp^2 = Bdt^2$ in eq. (3.d) we get

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = 0 \tag{20}$$

and using the value of J in eq.(7) we have

$$\frac{dt}{dr} = \left[\frac{A(r)/B(r)}{\left[1 - \frac{B(r)}{B(r_0)}\;\frac{R_0^2}{r^2}\right]}\right]^{1/2} \tag{21}$$

which gives, to first order in m/R_0,

$$\frac{dt}{dr} = \left[1 - \left(\frac{R_0}{r}\right)^2\right]^{-1/2} \left[1 + \frac{mR_0}{r(R_0)r} + \frac{2m}{r}\right] . \tag{22}$$

Integration of eq. (22) in the (R_0,r) interval gives

$$t(r,R_0) = \sqrt{r^2 - R_0^2} + 2m \log\left(\frac{r + \sqrt{r^2-R_0^2}}{R_0}\right) + \left(\frac{r-R_0}{r+R_0}\right)^{1/2} \tag{23}$$

Eq. (23), the ***radar-echo*** *delay formula* can be used to calculate the time for a luxon to transit from the critica1 point of the orbit to the point with coordinate r on the orbit. Observe that for **points near** R, eq. (23) cannot be used. Indeed $(dt/dr) \to \infty$ when $r \to R_0$.

This point is important when we want to calculate, for example, the time of flight of a laser beam in a Michelson-Morley interferometer, with one of the arms in the vertical direction, in the **gravitational** field of the **Earth**[3].

We end this paper with the observation that an analogous error can happen in the case of the calculation of phases. Indeed, if we divide **eq.(1)** with $ds^2 = 0$ (and $\theta = \pi/2$) by $(d\phi)^2$ and use **eq.(3.b)** we get

$$\left(\frac{d\phi}{dr}\right) = \frac{A^{1/2}(r)}{r\left[\left(\frac{r}{R_0}\right)^2 \frac{B(R_0)}{B(r)} - 1\right]} \tag{24}$$

and to first order in m/R_0 we get

$$\frac{d\phi}{dr} = \left[\frac{1}{r}\left(\frac{r}{R_0}\right)^2 - 1\right]^{1/2} \left[1 + \frac{m}{r} + \frac{mr}{R_0(r+R_0)}\right] \tag{25}$$

Integration of eq. **(25)** in the $(R\ ,r)$ **interval gives**

$$\phi(r) - \phi(R_0) = \cos^{-1}\left(\frac{R_0}{r}\right) + \frac{m}{R_0}\sqrt{1 - \left(\frac{R_0}{r}\right)^2} + \frac{m}{R_0}\left(\frac{r-R_0}{r+R_0}\right)^{1/2} \tag{26}$$

This equation, like **eq.(22)**, can only be used for values of $r >> R_0$

Eq.(25) differs from eq. (8.5.7) of Weinberg's book[1], where there is a sen^{-1} term instead of the $\cos^{-1}$ term. Weinberg's integration is obviously wrong, although, for the specific problem where he used eq. (8.5.7), the final result is correct.

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Resumo

Neste trabalho apresentamos uma integração das equações de geodésica nula na geometria de Schwarzschild que é válida em primeira ordem de GM/Rc^2. Comparamos nossa solução com outras publicadas na literatura e analisamos o seu limite de aplicabilidade. Também esclarecemos alguns mal-entendidos.