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The Non-Compact CP^N Model

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Abstract The non-compact $\mathbb{C}P^N$ model is formulated as the M=1 case of the generalized non-compact Grassmannian sigma models on the coset space $SU(M,N)/[SU(M)\times SU(N)\times U(1)]$. Quantum properties obtained within an indefinite metric formalism and the $1/N$ expansion show the phenomena of dynamical generation of mass and of a composite gauge boson for certain values of the coupling constant, without any input parameter.

The non-compact $\mathbb{C}P^N$ model is the M=l case of the non- compact Grassmannian sigma model defined on the space G/H = $SU(M,N)/$ $/[SU(M)\times SU(N)\times U(1)]$[1]. The Lagrangian in the general matrix formalism ir. given by[2]

$$L = -\frac{1}{8}\,\mathrm{Tr}\ \partial_\mu q(x)\partial^\mu q(x)^{-1}, \tag{1}$$

where

$$q(x) = g^{-1}(x)g_0 g(x) \tag{2}$$

are a subset of the group elements g belonging to $SU(M,N)$, with the fixed element

$$g_0 = \begin{bmatrix} -I_M & 0 \\ 0 & +I_N \end{bmatrix} \tag{3}$$

By definition, an element of thepseudo-unitary group satisfies the condition

$$g^\dagger g_0 g = g_0 = g\ g_0 g^\dagger, \tag{4}$$

with inverse given by

$$g^{-1} = g_0 g^\dagger g_0 \quad . \tag{5}$$

The interaction in the model is provided by the constraint

$$q(x)q(x) = I_{M+N} \ . \tag{6}$$

The coset elements can be parametrized by off-diagonal $M \times N$ complex matrices X and $X^\dagger$ in the Lie algebra in the form[3]

$$q' = \exp\begin{bmatrix} 0 & X \\ X^\dagger & 0 \end{bmatrix} = \left[\begin{array}{c|c} \cosh\sqrt{XX^\dagger} & \dfrac{X\sinh\sqrt{X^\dagger X}}{\sqrt{X^\dagger X}} \\ \hline X^\dagger\dfrac{\sinh\sqrt{XX^\dagger}}{\sqrt{XX^\dagger}} & \cosh\sqrt{X^\dagger X} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \sqrt{I_M + ZZ^\dagger} & Z \\ \hline Z^\dagger & \sqrt{I_N + Z^\dagger Z} \end{array}\right] \ . \tag{7}$$

In the $M=1$ case, the block Z reduces to a single row parametrized by a set of N complex numbers $z_i (i=1,\ldots,N)$ which, with z_0 defined by the non-compact constraint $\bar{z}_0 z_0 = 1 + \bar{z}_i z_i$, become the field variables transforming as a vector z_a under $SU(1,N)$. In terms of these variables, the coset (2) can be written as

$$q = \left[\begin{array}{c|c} -1 + 2\bar{z}_0 z_0 & 2\bar{z}_0 z_i \\ \hline 2z_0\bar{z}_j & -1 - 2\bar{z}_i z_j \end{array}\right] \ . \tag{8}$$

The Lagrangian in eq. (1), after some algebra, now assumes the canonical form

$$L = -\partial_\mu \bar{z}_0 \partial^\mu z_0 + \partial_\mu \bar{z}_i \partial^\mu z_i$$

$$+ (-\bar{z}_0 \partial_\mu z_0 + \bar{z}_i \partial_\mu z_i)(-\bar{z}_0 \partial^\mu z_0 + \bar{z}_i \partial^\mu z_i)$$

or

$$L = \partial_\mu \bar{z}_a \partial^\mu z^a + \frac{1}{\bar{z}_a z^a} (\bar{z}_a \partial_\mu z^a)(\bar{z}_b \partial^\mu z^b) \quad , \quad a, b = 0, 1, \ldots, N \qquad (9)$$

in which a rescaling of field variables has been performed through the introduction of a coupling constant f in the constraint

$$\bar{z}_a z^a = -\bar{z}_0 z_0 + \bar{z}_i z_i = -\frac{N+1}{f} \ . \qquad (10)$$

Under the local transformation $z_a(x) \rightarrow z_a(x) \exp i\Lambda(x)$, the Lagrangian is seen to be invariant. Thus classically there exists acomposite field defined by $A_\mu = i\bar{z}^a \partial_\mu z^a$, which transforrns like a $U(1)$ gauge field. Al-

$$L = \overline{D_\mu z_a}\, D^\mu z^a \quad , \qquad (11)$$

with $D_\mu = \partial_\mu + iA_\mu$. The above formulation thus far is relevant for any dirnension. It is our emphasis in this note to concentrate on the quantum properties in these models, which are only renormalizable in two dimensions, and to demonstrate that the composite gauge boson can indeed be generated dynamically.

In the two-dimensional model, several studies[4] of the quantum theory in the positive coupling region of eq.(10) have concluded that there cannot be dynamical generation of a composite gauge boson due to the lack of a mass generation, unless a certain extra input is introduced[5]. Thus the quantum theory in such case is not too interesting. However, a previous analysis[6] of the non-compact $O(1,N)$ sigma model shows that a negative coupling does not prevent the construction of a positive definite quantum Hamiltonian in an indefinite-metric formalism[7], in spite of the fact that the corresponding classical Hamiltonian is not positive. The $1/N$ expansion study of the model further shows that in the same region there can be dynarnical mass generation, though the $O(1,N)$ model does not possess a gauge degree of freedom.

Following the ideas in the analysis of the $O(1,N)$ mode1[6],. we introduce a mass term $m^2\bar{z}_a z^a$ in eq.(9) to regulate the infrared divergente of the massless theory. The mass at this point is an extra parameter but will become related to the coupling later on. The overall Fock space of quantization in the free field theory on which a perturbative treatment of the interacting theory is based, is of course, indefinite as the fields carry both positive and negative norrns. Thus in the momentum space expansion of the complex fields, the creation and annihilation operators a_i^+, a_i satisfy the prescription

$$<0|a_i(p)a_i(p')^+|0> = \mp 2p_0\,\delta(p-p') \begin{cases} - \text{ for } i=0 \\ + \text{ for } i=\ ,\ldots,N. \end{cases} \tag{12}$$

The physical subspace, obtained by projecting states consisting of an even number of negative-metrid fields but arbitrary number of positive-metric fields, is always positive definite. As a result, all physical quantities constructed in this subspace have positive-definite eigenvalues.

We now examine several important features of the quantum theory for the non-compact CP^N model. Implementing the non-linear constraint in eq.(10) via a Lagrangian multiplier λ, we rewrite the Lagrangian as

$$L = -\overline{D_\mu z_0}\, D^\mu z_0 + \overline{D_\mu z_i}\, D^\mu z_i - m^2(-\bar z_0 z_0 + \bar z_i z_i)$$
$$- \frac{\lambda}{\sqrt{N+1}}\left(-\bar z_0 z_0 + \bar z_i z_i + \frac{N+1}{f}\right) . \tag{13}$$

The value of m can be chosen so that the vacuum expectation value

$$<\lambda> = <0|\,T\lambda \exp i\int L_{\text{int}}\ \ 0\ \ = 0 \tag{14}$$

is satisfied. This condition implies that

$$-<\bar z_0 z_0> + <\bar z_i z_i> = -\frac{N+1}{f} . \tag{15}$$

Since the field z_0 has the negative metric, its propagator carries an additional relative minus sign with respect to those of the positive-metric fields and the combined effect on its vacuurn expectation value is that it is positive! As the dynamics is invariant when z_0 is translated by a constant value, i.e. $z_0 \to z_0' = o, + a$, we may exploit this invariance. Denoting the translation of the vacuum expectation value by $<\bar z_0 z_0>' = <\bar z_0 z_0> + \sigma\sigma$, we obtain the linear part in z_0 of the shifted Lagrangian

$$L' = L + m^2(\bar{\sigma}\sigma + \bar{\sigma}z_0 + \bar{z}_0\sigma) + \frac{1}{\sqrt{N+1}}(\bar{\sigma}\sigma + \bar{\sigma}z_0 + \bar{z}_0\sigma) \qquad (16)$$
$$+ \text{ other terms,}$$

so that in the tree aproximation, the requirement that the shifted field z_0 now hsve zero vacuum expectation value further implies the condition

$$m^2\bar{\sigma}\sigma = 0 \ . \qquad (17)$$

Thus there are two distinct possibilities according to eq.(17) in the analysis of the dynamics:

Case I. If $\bar{\sigma}\sigma \neq 0$, then m^2 must be zero identically. This is the case corresponding to the absence of dynamical mass generation. Equation (15) further shows that to lowest order in the large N limit,

$$\bar{\sigma}\sigma = (N+1)\int \frac{d^2k}{(2\pi)^2}\left(\frac{i}{k^2-m^2+i}\right) + \frac{N+1}{f} \ , \qquad (18)$$

in which the integral diverges when m = 0. For positive coupling f, the quantity $\bar{\sigma}\sigma$ is therefore infinite, a result which is physically meaningless. In addition, as we shall see in later expression, the zero mass limit is not well-defined for the propagator for the composite gauge field As $p^2 \to 0$, the pole of the propagator is absent and this is why the gauge boson fails to appear dynamically.

Case II. If $\bar{\sigma}\sigma = 0$, then m^2 can be finite. This is the case where there is dynamical mass generation. The evaluation of the integral in eq.(18) after a Wick's rotation leads to the following relation

$$\frac{1}{(2\pi)^2}\int \frac{d^2k_e}{k_e^2 + m^2} = -\frac{1}{f} \ , \quad k_e = k_0^2 + k_1^2 \ , \qquad (19)$$

which is valid only when the coupling f is negative. Equation (19) gives the typical asymptotic freedorn behavior of the coupling as a furictionof the cutoff A, i.e. $f^{-1}(\Lambda)$ = constant × $\log(\Lambda^2/m^2)$. The mass introduced earlier is now nonperturbatively related to the coupling and is the dynamically generated mass. The proper two-point function of the gauge field, obtained from eq. (13), is

$$\Gamma_{\mu\nu}(p) = \frac{1}{N+1}\left[2i\, g_{\mu\nu} \langle \bar{z}_a z^a \rangle + \langle : \bar{z}_a \overleftrightarrow{\partial}_\mu z^a : \, : \bar{z}_b \overleftrightarrow{\partial}_\nu z^b : \rangle\right]$$

$$= -2\, g_{\mu\nu} \int \frac{2}{(2\pi)^2} \frac{1}{(k^2-m^2)} + \int \frac{d^2k}{(2\pi)^2} \frac{(p_\mu+2k_\mu)(p_\nu+2k_\nu)}{(k^2-m^2)\left[(p+k)^2-m^2\right]},$$

i.e.

$$\Gamma_{\mu\nu}(p) = \text{constant} \times \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) \int_0^1 d\alpha \; \frac{(1-2\alpha)^2 \, p^2}{\alpha(1-\alpha)p^2-m^2} \quad , \qquad (20)$$

which is the same as the corresponding function in the compact model[8]. In this massive case, the expression in eq. (20) shows that the gauge boson propagator develops a pole at $p^2 = 0$, indicating the presence of a physical gauge boson in the quantum theory. The same expression shows that the pole at $p^2 = 0$ variishes when $m = 0$. In fact, the integral in eq.(20) is infrared divergent if $m = 0$. This is the reason why dynamical generation of a gauge boson is dependent on mass generation in the model. In general, the dynamics of the compact and the nsn-compact CP^N models are very similar. In addition to the negative sign carried by the propagator of the negative-metric field, the Green's functions involving n negative-metric fields in the non-compact model are related to the corresponding Green's functions in the compact model by the factor $(-1)^{n/2}$ (n is always even).

The phenomenon of mass generation with asymptotic freedom behavior in the negative coupling region is a common feature of the $SO(1,N)/SO(N)$ and the $SU(1,N)/SU(N)$ $U(1)$ sigma models. One important difference between these two types of theories is that in the nun- compact CP^N model there are no single-particle asymptotic states corresponding to the scaiar fields of the Lagrangian. Unlike the $SO(1,N)/SO(N)$ model in which the physical asymptotic states contain an even number of negative-metric particles and the S-matrix structure for two-body scattering is well defined, it is no longer possible to discuss the scattering of fundamental particles in the non-compact CP^N model. Thescalar fields are permanently confined by the Coulomb potential which is linear in two dimensions[8]. However, if the non-compact CP^N model is coupled minimally with a fermion then the gauge boson pole at $p^2 = 0$ will disappear . In such a situation, the scalar fields are no longer confined and scattering between the scalar particles becomes possible. One can then discuss the features os particle scattering in this non- compact model as has been done in the O(1,N) sigma model[9].

We finally remark on the equivalence between the non-compact $SU(1,1)/U(1)$ sigma model and thecon-compact $SO(2,1)$ sigmarnodel, the latter being defined by the Lagrangian

$$L = \sum_{i=1}^{3} \frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi_i , \quad \sum_{i=}^{3} \phi^i \phi_i = \text{constant} . \tag{21}$$

In the spinor representation, the generators of $SO(2,1)$ can be given in terms of the Pauli matrices σ_i as: $\Gamma_1 = i\sigma_1$, $\Gamma_2 = i\sigma_2$, $\Gamma_3 = \sigma_3$. Defining the composite fields to be

$$\phi_i = \bar{z}^a (\Gamma_i)_a{}^b z_b , \quad a, b = 0,1 , \tag{22}$$

which are real, we find that $\phi_1^2 + \phi_2^2 - \phi_3^2 = -\bar{z}_0 z_0 + \bar{z}_1 z_1$. Similarly, a simple calculation using eq. (22) shows that the Lagrangian resulting from eq. (21) is exactly of the form of the non-compact CP^1 model.

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Resumo

O modelo CP^N não compacto é formulado como o caso $M=1$ dos modelos sigma generalizados def'inidos no espaco coset $SU(M,N)/SU(M)\times SU(N)\times U(1)$. As propriedades quânticas obtidas dentro do formalismo de métrica indefinida e expansão $1/N$, exibem o fenômeno de geração dinâmica de massa e de um boson de gauge composto, numa certa região da constante de acoplamento. Isto resulta sem uso de nenhum parâmetro auxiliar.