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Density Pertubations in a Brans-Dicke Cosmological Model

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Abstract In this paper the calculation of time evolution of density perturbation, using the Raychaudhuri equation in Brâns-Dicke theory is re-examined, We give here the correct form of the perturbed differential equation and show the correct analytical time behavior of density perturbation, We also remark that the procedure of calculation of density perturbation via Raychaudhuri equation is a particular case of a general and more rigorous method proposed in the literature.

1. INTRODUCTION

Density perturbations in relativistic cosmological models remain the best known mechanism describing the galaxy formation in the linear phase. However, many technical difficulties arise in this picture. To obtain inhomogeneties emerging at the non-linear phase with a mass of the order of the typical galactic mass, we must assume perturbations originated at early epochs which may be larger than the particle horizon. On the other hand, we must take care of the perturbed quantities which we obtain by perturbing the General Relativity equations because it happens that these quantities may be merely a gauge effect, i.e., unphysical perturbations which can be eliminated by an appropriated choice of coordinates.

In order to get a meaningful conclusion from the analysis of the physical properties of perturbations larger than the particle horizon, we must adopt a procedure of calculation which ensures both invariance of mathematical representation of the perturbed quantities and, of course, their gauge independence, Many authors[1,2,3] have proposed such a procedure which has been improved by Novello and *al.*[4].

In what follows we re-examine the calculation of density perturbations in Brans-Dicke cosmology given by N. Bandyopadhyay[5] because, as pointed elsewhere[6], his differential equation for density contrast is not correct and its corresponding solution was found under inappropriate conditions. We propose here to derive the correct equation and to show a more general solution to it.

Our motivations to re-examine this calculation are the following: i) the use of a theory of gravitation with a varying G, in particular the Brans-Dicke theory, allows new estimations of observable astrophysical parameters[7]; ii) we intend to compare the behavior of a newtonian perturbation in an "approximate newtonian cosmological model"[6], with the behavior of the density perturbation correctly calculated in the Brans-Dicke cosmology; iii) the method proposed by Bandyopadhyay is a very interesting one because, as we shall see, it may be considered as a particular case of scalar perturbations (see ref. 4 eqs. (29), (35) and (36) and ref, 5). In fact, the correct density perturbation given by the Bandyopadhyay approach has all the "good" properties ensured by the method proposed by Hawking-Olson-Novello *et al.*; iv) in addition, we remark that in Brans-Dicke cosmology it is possible to solve the horizon problem by making a fine tunning of some parameters[8].

2. SCALAR PERTURBATIONS IN BRANS-DICKE COSMOLOGY

The gravitational equations in Brans-Dicke theory are

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R = (8\pi/\phi)T_{\mu\nu} + (\omega/\phi^2)\{\phi_{,\mu}\phi_{,\nu} - (1/2)g_{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}\} +$$

$$+ (1/\phi)\{(\phi_{,\mu;\nu} - g_{\mu\nu}\phi_{,\alpha}{}^{;\alpha})\} \qquad (1)$$

$$\phi_{,\alpha}{}^{;\alpha} = \{8\pi/(3+2\pi)\}T \ , \qquad (2)$$

where the semicolons represent covariant derivatives, $T \equiv T^{\mu}{}_{\mu}$ is the trace of the energy-momentum tensor $T_{\mu\nu}$ and ω is a positive real number, In the limit $\omega\to\infty$, $\phi^{-1}\to G$, the gravitational constant, the theory reduces to the conventional Einstein theory of gravitation. As in Einstein's theory we have here

$$\cdot \quad {}_{;\nu} = 0 \ . \qquad (3)$$

Assuming that the Universe is spatially homogeneous, and

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} - pg^{\mu\nu} , \tag{4}$$

we will make use of the Raychaudhuri type equation obtained by Banerji[9], which, in our case, may be written as

$$\dot{\theta} + (1/3)\Theta^2 - \dot{u}^{\mu}{}_{;\mu} + 2(\sigma^2 - \Omega^2) = - (8\pi/\phi)\{(2 + \omega)\rho/(3 + 2\omega) +$$

$$+ 3(1+\omega)p/(3+2\omega)\} - (\omega/\phi^2)\dot{\phi}^2 - \ddot{\phi}/\phi + \phi_{,\mu}\dot{u}^{\mu}/\phi \tag{5}$$

where a dot means time derivative, 0, a and R are the expansion, the shear and the rotation respectively.

As is well known[10] the gravitational perturbations may be classified in three classes: scalar perturbations, vectorial perturbations and tensorial. The first one is the most important for our purpose, i.e., it concerns density perturbations directly. In this case, we can write expressions for the perturbations on the gravitational field, on the four-velocity and on the density, with the help of a unique scalar function Q. In the particular case of a flat Friedmann non-perturbed Universe and for the value $K = 0$ in equation (35) of Novello *et al.* (ref. 4), Q satisfies a Laplacian equation.

Bandyopadhyay proposes a procedure which gives a straightforward way to obtain the differential equation for density perturbations by perturbing the Raychaudhuri equation and the field equation for the scalar field ϕ, equations (5) and (2), respectively. We assume then that the unperturbed Universe is isotropicandhomogenous with the Robertson-Walker metric,

$$ds^2 = dt^2 - R(t)^2(dr^2 + r^2d\theta^2 + r^2\text{sen}^2\theta d\phi^2) , \tag{6}$$

where R is the scale factor.

Let 0, $g_{\alpha\beta}$, ρ, p, u^{α} and ϕ be solutions for the unperturbed Universe. Introducing the small perturbations 60, $h_{\alpha\beta}$, $\delta\rho$, δp , δu^{α} and $\delta\phi$, we assume that

$$\theta + \delta\Theta,\ g_{\alpha\beta} + h_{\alpha\beta},\ \rho + \delta\rho\ ,\ u^{\alpha} + \delta u^{\alpha} \text{ and } \phi + \delta\phi,\ p + \delta p \tag{7}$$

are also solutions of the equation describing a Brans-Dicke cosmological model. We take the equation of state in the form

$$p = \alpha\rho \quad , \quad \alpha = \text{constant} \quad . \tag{8}$$

Using that equation of state, we have from equations (5) and (2), taking into account equation (7), we obtain

$$\delta\dot{\Theta} + (2/3)\Theta\delta\Theta - \delta\dot{u}^{\alpha}{}_{;\alpha} + (8\pi/\phi)\{(2+\omega)/(3+2\omega) + 3(1+\omega)\alpha/(3+2\omega)\}\delta\rho -$$

$$- (8\pi/\phi^2)\{(2+\omega)/(3+2\omega) + 3(1+\omega)\alpha/(3+2\omega)\}\rho\delta\phi =$$

$$- (2\omega/\phi^2)\dot{\phi}(\delta\phi)^{\cdot} + (2\omega/\phi^2)\dot{\phi}^2\delta\phi - (\delta\phi)^{\cdot\cdot}/\phi + \delta\phi\ddot{\phi}/\phi^2 \quad , \tag{9}$$

and

$$(\delta\phi)^{\cdot\cdot} + \nabla^2\delta\phi + (3\dot{R}/R)(\delta\phi)^{\cdot} + \delta\Theta\dot{\phi}/(1+\alpha) = 8\pi(1-3\alpha)\delta\rho/(3+2\omega) \quad , \tag{10}$$

where ∇^2 is the Laplacian for the three space metric[11]. We remark also that in the assumed unperturbed Universe $\sigma = \dot{R} = 0$, so that σ^2 and $\dot{R}^2$ are neglected second order contributions.

We will consider now scalar perturbations. In this case all perturbed quantities will be constructed with the help of a scalar function which, for a flat cosmological model, satisfies a Laplacian equation. We can write (ref.4, eqs. (37) and (38)).

$$\delta\Theta = \varepsilon(t)Q \quad ,$$

$$\delta\rho = \rho\mu(t)Q \quad , \tag{11}$$

$$\delta\phi = \psi(t)Q \quad ,$$

From the conservation law (3), equations (11) and the equation of state, we can write,

$$\varepsilon(t) = -\dot{\mu}(t)/(1+\alpha) \quad . \tag{12}$$

Taking into account equations (11) and (12) and again the conservation equation $T^{\mu\nu}{}_{;\nu} = 0$, we can write the perturbation equations. From eq. (9) and eq. (10) we have

$$\ddot{\mu}/(1+\alpha) + 2(\dot{R}/R)\dot{\mu}/(1+\alpha) - (8\pi/\phi)\{(2+\omega)/(3+2\omega) + 3(1+\omega)\alpha/(3+2\omega)\}\rho\mu =$$

$$= -(8\pi/\phi^2)\{(2+\omega)/(3+2\omega) + 3(1+\omega)\alpha/(3+2\omega)\}\rho\psi + 2\omega\dot{\phi}\dot{\psi}/\phi^2 -$$

$$- 2\omega\dot{\phi}^2\psi/\phi^3 + \ddot{\psi}/\phi - \ddot{\phi}\psi/\phi \quad , \tag{13}$$

$$\ddot{\psi} + (3\dot{R}/R)\dot{\psi} - \dot{\mu}\dot{\phi}/(1+\alpha) = 8\pi(1-3\alpha)\rho\mu/(3+2\omega) \quad , \tag{14}$$

where ρ, R and ϕ are solutions of the unperturbed equations (1) and (2), μ is the density contrast and ψ is the perturbations in ϕ.

Equations (13) and (14) are the differential equations for the perturbations $\mu(t)$ and $\psi(t)$ in Brans-Dicke theory. We can easily see that for $\omega\to\infty$, equations (13) and (14) give us the corresponding equation of General Relativity[12],

$$\ddot{\mu} + 2(\dot{R}/R)\dot{\mu} - 4\pi G\rho(1+3\alpha)(1+\alpha)\mu = 0 \ . \tag{15}$$

3. THE BEHAVIOUR OF DENSITY PERTURBATION IN BRANS-DICKE COSMOLOGY

As it is well known, the analytical solutions for an unperturbed homogeneous and isotropic Universe in the case of dust matter (α=0) are[3]

$$R = R_0 t^{2(\omega+1)/(3\omega+4)} \ ;$$

$$\rho = \rho_0 t^{-6(1+\omega)/(3\omega+4)} \ ; \tag{16}$$

$$\dot{\phi} = At^{-(3\omega+2)/(3\omega+4)} \ .$$

We can easily solve the system of equations (13) and (14) for this case. We can see that, taking into account the field equation (2), equation (14) gives us the first integral

$$\dot{\psi}R^3 = \mu\dot{\phi}R^3 + A \quad , \tag{17}$$

where A equals zero, according the initial condition.

Eliminating ψ from equations (17) and (13) and taking into account (16), we have

$$\dddot{\mu}t^3 + \{(12+10\omega)/(3\omega+4)\}\ddot{\mu}t^2 + \{2\omega/(3\omega+4)\}\dot{\mu}t = 0 \ . \tag{18}$$

The solutions of eq. (18) are

$$\mu_1(t) = \mu_{01}t^{-1} \quad , \tag{19}$$

$$\mu_2(t) = \mu_{02}t^{(4+2\omega)/(4+3\omega)} \quad , \tag{20}$$

$$\mu_3(t) = \mu_{03} = \text{const.} \quad . \tag{21}$$

4. CONCLUSIONS

We have shown that the perturbations of the Rachaydhuri equation in Brans-Dicke theory is indeed a useful and straightforward way to obtain the exact solutions for scalar perturbations. The mathematical structure of the Brans-Dicke theory and the further fact that the scalar field ϕ may be considered as a source term for gravitational field allow the application of a general and rigorous procedure for calculating the perturbation as given by Hawking-Olson-Novello et *al*. This fact ensures that the perturbations given by Bandyopadhyay's approach have all the good properties suggested by this general method.

The expressions given by equations (19), (20) and (21) are the correct solutions for the evolution of density perturbation obtained by the calculation proposed by Bandyopadhyay. Equation (21) represents a trivial solution and equation (19) gives the damped mode for density perturbation. The physically interesting behavior is the growing perturbation. We are interested in the growing perturbation, given by equation (20), which represents a time evolution slightly faster than the corresponding growing mode given by General Relativity theory.

We remark that our result obtained for an "approximate model"[6] is recovered in equation (20) for $\omega = 6$, i.e., $\mu \sim t^{0,72}$.

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Resumo

Neste artigo nós re-examinamos os cálculos da evoluçao temporal das perturbações de densidade obtida através da perturbação da equação de Raychaudhuri em teoria de Brans-Dicke. Nós deduzimos a forma correta da equação para as perturbações de densidade e damos a expressão analítica correta para a evolução temporal das mesmas. Ressaltemos também que este método de cálculo é, em verdade, um caso particular do método mais geral e mais rigoroso já apresentado na literatura.