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Correlation Functions and Scaling Limit of the Two-Dimensional Ising Model

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Abstract An infinite series representation for the correlation or Schwinger functions of the two-dimensional infinite lattice Ising model is obtained in a very simple and transparent way from a infinite lattice Feynman-Kac (F-K) formula in a Fermion Fock space, In the F-K formula energy-momentum and field or spin operators are defined utilizing two sets of canonical Fermion operators which are related by a proper linear canonical transformation (plct), i.e. there exists a unitary operator which implements the transformation. By exploiting the special properties of the plct we prove a generalization of Wick's theorem. Substituting the spectral representations of the energy-momentum operators in the F-K formula an infinite series representation for the Schwinger functions is obtained. The terms of the series are evaluated explicitly by a mere application of the generalized Wick's theorem. From this series representation scaling limit Schwinger functions are also obtained.

1. INTRODUCTION AND RESUCTS

In[1] two finite sets of Fermion operators $\{\xi_k\}$ and $\{\xi_\ell\}$ (the $\{k\}$ and $\{\ell\}$ are wavenumbers belonging to distinct sets) and their associated vacuum vectors are employed to determine the eigenvalues and eigenvectors of the transfer matrix of the periodic two-dimensional finite lattice Ising model. Abraham[2] uses a finite lattice Feynman-Kac (F-K) formula for the correlation functions and the fact that $\{\xi_k\}$ and $\{\xi_\ell\}$ are linearly related to develop a system of equations for the matrix elements of the spin operators occuring in the F-K formula. Generalizing these equations to an infinite lattice he obtains a system of integral equations which are solved by giving an ansatz forthesolution and an infinite series representation for the correlation functions is obtained. Each term of the series is an integral of a function given by a recursion relation reminiscent of Wick's theorem which was a surprise to the author and left unexplained, In a previous article[3] (the notation and results of [3] are used in this article) we obtained a F-K formula in a Fermion Fock space over $L^2(-\pi,\pi)$ for the infinite lattice correlation or Schwinger functions. The F-K formula is expressed in terms of energy-momentum and spin or field operators which in turn are defined in terms of the two sets of free Fermion operators $\hat{\xi}(k)$ and $\tilde{\xi}(k)$. In this way we have defined the quantum field theory of a scalar field associated with the infinite lattice Ising model. The $\hat{\xi}$ operators are the usual Fock representation and the $\tilde{\xi}$ are related to $\hat{\xi}$ by a linear canonical transformation (lct) implemented by a unitary transformation U and the sets $\{\hat{\xi}\}$, $\{\tilde{\xi}\}$ have the vacuum vectors $\hat{\psi}$ and $\tilde{\psi} = U\hat{\psi}$, respectively, belonging to the Fock space. In this way we carry over the algebraic structure of the finite lattice to the infinite lattice and consideration of the infinite lattice limits of both of the finite lattice vacuum vectors becomes unnecessary. By using the special property of U, namely $U = U^* = U^{-1}$, which follows from the fact that the reverse transformation of the lct is itself, a generalization of Wick's theorem for vacuum expectation values of products of Fermion operators is proved (see Theorem III below).

As shown in[3] the infinite lattice Schwinger functions admit a series representation by inserting the spectral representations of the energy-momentum operators in the F-K formula. We have

$$S_k = (\hat{\psi},\ \sigma_1^x\, e^{-H(n_2-n_1)}\, e^{-iP(m_2-m_1)}\, \sigma_1^x\, e^{-H(n_3-n_2)}\, e^{-iP(m_3-m_2)} \cdots e^{-H(n_k-n_{k-1})}\, e^{-iP(m_k-m_{k-1})}\, \sigma_1^x\, \hat{\psi}) \ . \tag{1.1}$$

Thus S_k has the representation (for $T>T_c$)

$$S_k = \sum_{\{\alpha_i\}\{\beta_j\}} \int dq^{\{\alpha_i\}} \int dk^{\{\beta_j\}}\, (\hat{\psi},\ \sigma_1^x\, e^{-\gamma_{\alpha_1}}\, \chi_{\alpha_1})(\chi_{\alpha_1},\ \sigma_1^x\, e^{-\gamma_{\beta_2}}\, \chi_{\beta_2}) (\chi_{\beta_2},\ \sigma_1^x\, e^{-\gamma_{\alpha_3}}\, \chi_{\alpha_3}) \cdots (\chi_{\beta_{k-2}},\sigma_1^x\, e^{-\gamma_{\alpha_{k-1}}}\chi_{\alpha_{k-1}})(\chi_{\alpha_{k-1}},\sigma_1^x\, \hat{\psi}) \tag{1.2}$$

where $\{\alpha_i\} = \{\alpha_1,\alpha_3,\ldots,\alpha_{k-1}\}$ and $\{\beta_j\} = \{\beta_2,\beta_4,\ldots,\beta_{k-2}\}$ and $\alpha_i(\beta_j)$

take on odd (even) integer values, In eq. (1.2)

$$\chi_{\alpha_i} = \frac{1}{\sqrt{\alpha_i!}} \tilde{\xi}^*(q_1)\ldots\tilde{\xi}^*(q_{\alpha_i})\tilde{\psi}$$

$$\chi_{\beta_j} = \frac{1}{\sqrt{\beta_j!}} \hat{\xi}^*(k_1)\ldots\xi^*(k_{\beta_j})\hat{\psi}$$

thus

$$(\chi_\alpha,\ \sigma_1^x\ \chi_\beta) = (\tilde{\psi},\ \tilde{\xi}(q_\alpha)\ldots\tilde{\xi}(q_1)\ \sigma_1^x\ \hat{\xi}^*(k_1)\ldots\hat{\xi}^*(k_\beta)\hat{\psi}\ )\ /\ \sqrt{\alpha!}\ \sqrt{\beta!}$$

$$= \overline{(\chi_\beta,\ \sigma_1^x\ \chi_\alpha)}\ .$$

We set

$$\sqrt{\alpha!}\ \sqrt{\beta!}\ (\chi_\alpha,\ \sigma_1^x\ \chi_\beta) \equiv M^x((e^{-ik})_{\beta,1}\ (e^{iq})_{1,\alpha})\ ,\ \text{the } e^{-ik}$$

argument occuring in order to take advantage of symmetry properties discussed below, In section 2 the explicit evaluation of the matrix elements occuring in (1.2) is reduced to an application of the generalized Wick's theorem. In this way we obtain

Thm. I.

$$S_k = \sum_{\{\alpha_i\}\{\beta_j\}} \int dq^{\{\alpha_i\}} \int dk^{\{\beta_j\}}\ M^x(\phi|(e^{iq})_{1,\alpha_1}) M^x((e^{-ik})_{1,\beta_2}|(e^{iq})_{1,\alpha_1})$$

$$\overline{M^x((e^{-ik})_{1,\beta_2}|(e^{iq})_{1,\alpha_3})}\ \ldots\ \overline{M^x((e^{-ik})_{1,\beta_{k-2}}|(e^{iq})_{1,\alpha_{k-1}})}$$

$$M^x(\phi|(e^{iq})_{1,\alpha_{k-1}})\ e^{-\gamma_{\alpha_1}}\ e^{-\gamma_{\beta_2}}\ldots\ e^{-\gamma_{\alpha_{k-1}}} \tag{1.3}$$

with

$$M^{x}((e^{iq})_{1,m}\ (e^{iq})_{m+1,n}) = \alpha^{m}\ \bar{\alpha}^{n-m}\ \Theta_{-}^{-1}(\infty)\ (\tilde{\psi},\hat{\psi}) .$$

$$\exp\left[i \sum_{j=1}^{n} \phi_{q_j} / (m!\,(n-m)!\,(2\pi)^{n})^{1/2}\right] .$$

$[(\det A)^{1/2} \equiv \text{Pfaffian } A]$ where A is the anti-symmetric matrix

$$A = \begin{Vmatrix}
0 & \Theta_{-1} & \cdots & \Theta_{-m} & \Theta_{-m+1} & & \Theta_{-n} \\
-\Theta_{-1} & 0 & m_{-1,2} \cdots & m_{-1,m} & m_{+1,m+1} & & m_{+1,n} \\
\vdots & & \ddots & \vdots & \vdots & & \vdots \\
\vdots & & 0 & m_{-m-1,m} & \vdots & & \vdots \\
\vdots & & & 0 & m_{+m,m+1} & \cdots & m_{+m,n} \\
\vdots & & & & 0 & m_{-m+1,m+2} \cdots & m_{-m+1,n} \\
\vdots & & & & & \ddots & \vdots \\
\vdots & & & & & 0 & m_{-n-1,n} \\
-\Theta_{-n} & & & & & & 0
\end{Vmatrix}$$

and

$$(\tilde{\psi},\hat{\psi}) = (\det\ T_1^{*}\ T_1)^{1/4}\ , \quad \Theta_{+}(z) = [(x_1-z)(x_2-z)]^{1/2}\ ,$$

$$\Theta_{-}(z) = [(x_1-z^{-1})(x_2-z^{-1})]^{-1/2}\ ,$$

$$m_\pm(i,j) \equiv m_\pm(z_i,z_j) = \frac{z_i\, z_j}{z_i\, z_j - 1}\left[\Theta_-(z_1)\Theta_+(z_2) \pm \Theta_-(z_2)\Theta_+(z_1)\right] , \; z_\ell = e^{ik_\ell}$$

$$\gamma_{\alpha_j}(\gamma_{\beta_j}) = \sum_{i=1}^{n} \varepsilon(q_i)(n_{i+1}-n_i) + \sum_{i=1}^{n} q_i(m_{i+1}-m_i),\; \alpha_j = n(\beta_j = n)$$

Remarks:

1. Recall (see [10]) that if A is an $n\times n$, n even, antisymmetric natrix with elements a_{ij} then Pfaffian A = $1/2^{n/2}$ $1/(n/2)!$.

$$\sum_{\text{Perm}} (-1)^{\text{sgn}P}\, a_{i_1i_2}\, a_{i_3i_4} \dots a_{i_{n-1}i_n} = (\det A)^{1/2}$$

Set Pf A = $(1,\dots,n)$. Under permutations $(h_1,\dots,h_n) = (-)^P\,(1,2,\dots,n)$ where $(h_1,h_2,\dots,h_n)$ is a permutation P of $(1,2,\dots,n)$ and $(-1)^P$ the sign. We have the expansion rule

$$(1,\dots,n) = \sum_{\substack{j=1\\ j\neq 1}}^{n} (-1)^{1+i+j}\,(ij)(1,\dots,i-1,i+1,\dots,j-1,j+1,\dots,n)$$

where we take $(ij) = (ji)$,

2. For the two-point function A reduces to only the upper left -hand corner and $M^x(\phi\,|(e^{iq})_1,\alpha_1)M^x(\phi\,|(e^{iq})_1,\alpha_1) \;\alpha\; [\det A]$.

3. For the four and higher point functions the singular function m_+ appears. In the integrals occuring in S_k the principal value is to be taken, as expected, since the terms of the series are limits of discrete momentum sums.

4. Note that S_k has an overall factor

$$(\tilde\psi,\hat\psi)^k = (\det T_1^* T_1)^{k/4} = ([1 - (\sinh 2K \sinh 2K)^2])^{1/8}/\cosh K^*)^k$$

5. Similar considerations hold for $T<T_c$,

An infinite series representation for the scaling limit Schwinger functions from above the critical temperature T_c is obtained from Thm. I in a straight forward way by introducing a scaling parameter

$\lambda>0$ and letting the ternperature depend on λ; the limit is then taken in such a way that

$$T(\lambda) \to T_c,\ T(\lambda) > T_c \text{ for } \lambda > 0$$
$$\lambda \to 0 \tag{1.4}$$

$$n_2-n_1 \to \infty \text{ with } (n_2-n_1)\lambda = (s_2-s_1) \text{ fixed}$$

$$m_2-m_1 \to \infty \text{ with } (m_2-m_1)\lambda = (x_2-x_1) \text{ fixed}$$

$$(2K^*-2K)/\lambda \to m > 0 ,$$

m being identified as the rnass of the single particle state. The scaling lirnit is given by

Thm. II. Divide S_k of (1.3) by the overall factors $(\tilde{\psi},\hat{\psi})^k/\Theta_-(\infty)^k$ and take the pointwise limit in cach integrand of each term of the expansion. The result is the set of scaling limit Schwinger functions $\{S_k^L\}$ where

$$S_k^L = \sum_{\{\alpha_i\}\{\beta_j\}} \int d\mu(p)^{\{\alpha_i\}} \int d\mu(p)^{\{\beta_j\}} \overline{L^x(\phi)\,|(p)_{1,\alpha_1})\ L^x((-p)_{1,\beta_2}|(p)_{1,\alpha_1})}$$
$$\overline{L^x((-p)_{1,\beta_2}|(p)_{1,\alpha_3})\dots L^x((-p)_{1,\beta_{k-2}}|(p)_{1,\alpha_{k-1}})}$$
$$L^x(\phi|(p)_{1,\alpha_{k-1}})\ e^{-\rho_{\alpha_1}}\ e^{-\rho_{\beta_2}} \dots e^{-\rho_{\alpha_{k-1}}} \tag{1.5}$$

with

$$L^x((p)_{1,m}|(p)_{m+1,n}) = \frac{1}{\sqrt{m!}\ \sqrt{(n-m)!}\ (2\pi)^{n/2}} \left[(\det B)^{1/2} \equiv \text{Pfaffian } B\right]$$

where B is the anti-symrnetric natrix

$$B = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ -1 & 0 & \Delta_{-12} & \cdots & \Delta_{-1m} & \Delta_{+1,m+1} & \cdots & \Delta_{+1,n} \\ \vdots & & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & & & \ddots & \Delta_{-m-1,m} & \vdots & & \vdots \\ -1 & & & & 0 & \Delta_{+m,m+1} & & \Delta_{+m,n} \\ -1 & & & & & 0 & \Delta_{-m+1,m+2} & \Delta_{-m+1,n} \\ \vdots & & & & & & \ddots & \vdots \\ \vdots & & & & & & & \Delta_{-n-1,n} \\ -1 & & & & & & & 0 \end{vmatrix}$$

where

$$\Delta_{\pm}(i,j) = \frac{\omega(p_i) \pm \omega(p_j)}{p_i + p_j}\ , \quad \omega(p_i) = \sqrt{p_i^2 + m^2}, \quad d\mu(p) = dp/\omega(p)$$

and

$$\rho_{\alpha_j}(\rho_{\beta_j}) = \sum_{i=1}^{n} \omega(p_i)(s_{i+1}-s_i) + i \sum_{i=1}^{n} p_i(x_{i+1}-x_i)$$

for

$$\alpha_j = n(\beta_j=n) \quad .$$

The series (1.5) converges for all s_i such that $s_j - s_{j-1} > 0$ for all $j = 2, \ldots k$.

Remarks:

1. From lemma III.1 the overall factor that is taken out of S_k behaves ar $(T-T_c)^{k/8}$ for $T \approx T_c$. Also the integral of the singular func–

tion A_+ is to be interpreted as a principal part integral.

2. We do not concern ourselves here with the approach to the limit but only the scaling limit itself. For the approach to the limit for the 2-point function see refs.2 and 4.

3. In [5] series representations for the scaling limit functions are given; whether or not their, or our, representation satisfies the Osterwalder-Schrader axioms[6] or whether they can be used to define a Wightman field theory is an open question. In our case the point in question is whether or not the real time concinuations of our representation are tempered distributions. Abeginning analysis of the representation of ref. 5 has been given in ref. 7.

4. In[8,9] a formal expression for Heisenberg operators associated with the spin operator in the scaling limit is given. The operator is written in terms of a self-conjugate Fermi free field and eiiergy-momentum operators are not separately defined.

5. In section 3 we give the proof of Thm. II for k = 2 and sketch the proof for k > 2. A detailed proof for k > 2 will appear in a forthcoming paper.

6. Related to remark 3 is the question of whether or not a Feynman-Kac formula for the scaling limit Schwinger functions can be written in the Fermi Fock space over $L^2(-\infty,\infty)$. Formally it is obvious how to write such a formula but the analog of the kernel C is no longer Hilbert-Schmidt in the $L^2(-\infty,\infty)$ case.

Theorem I follows in a most transparent way from a generalization of Wick's theorem, which is proved in Section 2, forthevacuum expectation values of products of Fermi operators where one vacuum is the usual Fock vacuum for the $\hat{\xi}$ operators and the other vacuum is $\tilde{\psi}=U\hat{\psi}$, the vacuum for the $\tilde{\xi}$ operators, U implements the lct, i.e. $\tilde{\xi}=U\hat{\xi}U^{-1}$ and satisfies the special property $U^{-1}=U$. We have

Thm. III. Let $\hat{\xi}(f)$ be the Fock representation of the CAR in a Fock space built over the Hilbert space L and let $\tilde{\xi}(f)=\hat{\xi}(T_1f)+\hat{\xi}^*(T_2f)$ be a linear canonical transformation, with ker $T_1=0$, implemented by the unitary operator U, i.e. $\tilde{\xi}(f)=U\hat{\xi}(f)\,U^{-1}$ and $\tilde{\xi}^*(f)=U\hat{\xi}^*(f)\,U^{-1}$ for all $f\in$ L. Let $\hat{\psi}$ and $\tilde{\psi}=U\hat{\psi}$ be the normalized vacuum vectors, i.e.

$\hat{\xi}(f)\hat{\psi} = 0$ and $\tilde{\xi}(f)\tilde{\psi} = 0$ for all $f \in L$, and $|\hat{\psi}| = |\tilde{\psi}| = 1$. In addition assume that $U^{-1} = U = U^*$. If A_i, $i = 1, 2, \ldots n$, n even, is an arbitrary linear combination of $\hat{\xi}$, $\hat{\xi}^*$, $\tilde{\xi}$ and $\tilde{\xi}^*$, then

$$\frac{(\tilde{\psi}, A_1 \ldots A_n \hat{\psi})}{(\tilde{\psi}, \hat{\psi})} = (\det D)^{1/2} \equiv \text{Pfaffian } D$$

where D is the $n \times n$ anti-symmetric matrix

$$D = \frac{1}{(\tilde{\psi},\hat{\psi})} \begin{vmatrix} 0 & D_{12} & \cdots & D_{1,n} \\ -D_{12} & 0 & \ddots & \vdots \\ \vdots & & 0 & D_{n-1,n} \\ -D_{1,n} & \cdots & -D_{n-1,n} & 0 \end{vmatrix}, \quad D_{ij} = (\tilde{\psi}, A_i A_j \hat{\psi}) .$$

Remark:

If $U = I$ then $\tilde{\psi} = \psi$ and we have the usual Wick's theorem, i.e. the vacuum expectation value of a product is the sum of the product, up to a sign, of the vacuum expectation values of all pair contractions. The above is a compact way of giving the correct sign to each term inthe sum.

The matrix elements occuring in (1.3) are evaluated using Thm. III in the following way. Since

$$\sigma_1^x = \frac{1}{\sqrt{2\pi}} \int \left[\alpha e^{i\phi_q} \hat{\xi}(q) + \bar{\alpha} e^{-i\phi_q} \hat{\xi}^*(q)\right] dq$$

the matrix element M^x can be reduced to the evaluation of the matrix element without a_1^x, call it M. One applies Thm, III to M reducing itto a sum of products of the generalized pair contractions

$$(\tilde{\psi},\tilde{\xi}(k_1)\tilde{\xi}(k_2)\hat{\psi}) \tag{1.6a}$$

$$(\tilde{\psi},\hat{\xi}^*(-k_1)\hat{\xi}^*(-k_2)\hat{\psi}) \quad \text{and} \quad (\tilde{\psi},\tilde{\xi}(k_1)\hat{\xi}^*(-k_2)\hat{\psi}) \ . \tag{1.6b}$$

Symmetry properties show (see lemma II.1) that (1.6a) and (1.6b) are equal so that only two distinct functions enter in the final expression for M. Using Wiener-Hopf methods an explicit formula given by a contour integral is found for these functions and their evaluation is reduced to contour integration. Another contour integration gives M^x.

2. PROOF OF THMS. I AND III

In this section we first prove the generalization of Wick's theorem (Thrn. III) and then prove Thm. I for the representation of the infinite lattice correlation or Schwinger functions. Note from ithe proof that the matrix element for the 2-point function only uses the special property of U, $U^{-1} = U$, and the usual Fock space inner product.

Prf. of Thm. III: By multi-linearity and the anti-commutation relations it is enough to show that the theorem holds for $(\tilde{\psi},\ \tilde{\xi}(f_1)\ ,\ldots \tilde{\xi}(f_n)\hat{\psi})$ and $(\tilde{\psi},\tilde{\xi}(f_1)\ldots\tilde{\xi}(f_n)\tilde{\xi}^*(f_{n+1})\ldots\tilde{\xi}^*(f_{n+m})\hat{\psi})$ for n, m arbitrary and f real.

By substituting $U\,\hat{\xi}(f)\,U^{-1}$ for $\tilde{\xi}(f)$ and using $U^* = U$ we have

$$(\tilde{\psi},\tilde{\xi}(f_1)\ldots\tilde{\xi}(f_n)\hat{\psi}) = (\hat{\psi},\hat{\xi}(f_1)\ldots\hat{\xi}(f_n)\tilde{\psi})$$

$$= (\hat{\xi}^*(f_n)\ldots\hat{\xi}^*(f_1)\hat{\psi},\tilde{\psi}) \ . \tag{2.1}$$

Recall that $\tilde{\psi} = U\hat{\psi}$ has the explicit form

$$\tilde{\psi} = U\hat{\psi} = (\tilde{\psi},\hat{\psi})\ \exp\{-\frac{1}{2}\int C(x,y)\hat{\xi}^*(x)\hat{\xi}^*(y)\ dx\ dy\}\hat{\psi}$$

where $(\tilde{\psi},\hat{\psi})$ = $(\det T_1^* \, T_1)^{1/4}$ and $C(x,y)$ is the kernel of the anti-symmetric operator $T_1'^{-1}\,T_2'$. Substituting for $\tilde{\psi}$ in (2.1) only the $n/2$ term of the expansion of the exponential contributes and we get

$$(\hat{\xi}^*(k_1)\ \ldots\ \hat{\xi}^*(k_n)\hat{\psi},\tilde{\psi})/(\tilde{\psi},\hat{\psi}) =$$

$$(-2)^{-n/2}\ (n/2)!^{-1}\ \Sigma(-1)^P\ C(k_1,k_2)\ldots C(k_{n-1},k_n)\ .\ \text{perm}\ \{k_1\ldots k_n\}$$

The right side is just $(\det F)^{1/2}$ = Pfaffian F where F is the anti-symmetric $n\times n$ matrix with elements $F_{ij} = C(k_i,k_j)$, $i<j$. But for $n = 2$

$$(\tilde{\psi},\tilde{\xi}(f_1)\tilde{\xi}(f_2)\hat{\psi}) = (\hat{\xi}^*(f_2)\hat{\xi}^*(f_1)\hat{\psi},\tilde{\psi})$$

$$= \int C(k_1,k_2)\,(\tilde{\psi},\hat{\psi})f_2(k_2)f_1(k_1)dk_1\ dk_2$$

so that the theorem follows in this case. Now by anti-commutation

$$(\tilde{\psi},\tilde{\xi}(f_1)\tilde{\xi}^*(f_2)\hat{\psi}) = (f_1,f_2)\ (\tilde{\psi},\hat{\psi})$$

since

$$(\tilde{\psi},\tilde{\xi}^*(f_2)\tilde{\xi}(f_1)\hat{\psi}) = (\tilde{\xi}(f_2)\tilde{\psi},\ \tilde{\xi}(f_1)\hat{\psi}) = 0\ .$$

Again by anti-commutation

$$(\tilde{\psi},\tilde{\xi}(f_1)\ldots\tilde{\xi}(f_n)\tilde{\xi}^*(f_{n+1})\hat{\psi}) = \sum_{j=1}^{n}\ (-1)^{j-n}\ (f_j,\ f_{n+1})\ .$$

$$(\tilde{\psi},\tilde{\xi}(f_1)\ldots\tilde{\xi}(f_{j-1})\tilde{\xi}(f_{j+1})\ldots\tilde{\xi}(f_n)\hat{\psi})$$

which upon substituting

$$(\tilde{\psi},\tilde{\xi}(f_j)\tilde{\xi}^*(f_{n+1})\hat{\psi})/(\tilde{\psi},\hat{\psi}) = (f_j,f_{n+1})$$

gives

$$(\tilde{\psi},\hat{\psi})^{-1}(\tilde{\psi},\tilde{\xi}(f_1)\ldots\tilde{\xi}(f_n)\tilde{\xi}^*(f_{n+1})\hat{\psi}) = \sum_{j=1}^{n}\ (-1)^{j-n}(\tilde{\psi},\tilde{\xi}(f_j)\tilde{\xi}^*(f_n)\hat{\psi})\ .$$

$$(\tilde{\psi},\hat{\psi})^{-2}\ (\tilde{\psi},\tilde{\xi}(f_1)\ldots\ \tilde{\xi}(f_{j-1})\tilde{\xi}(f_{j+1})\ldots\ \tilde{\xi}(f_n)\hat{\psi})\ . \tag{2.2}$$

The theorem holds in this case since the right side of (2.2) coincides with the theorem by the rule for the expansion of a Pfaffian. Finally, for n fixed, an induction argument on m shows that the theorem tiolds for $(\tilde{\psi},\xi(f_1)\ldots\xi(f_n)\xi^*(f_{n+1})\ldots\xi^*(f_{n+m})\hat{\psi})$. We now have

Lemma II.1. Let

$$m_\pm(z_1,z_2) = \frac{z_1 z_2}{z_1 z_2-1}\left[\Theta_-(z_1)\Theta_+(z_2) \pm \Theta_-(z_2)\Theta_+(z_1)\right] ,$$

$$c_+ = -\frac{1}{2\pi} e^{i(\phi_1+\phi_2)} (\tilde{\psi},\hat{\psi}) ,$$

$$c_- = \frac{i}{2\pi} e^{i(\phi_1+\phi_2)} (\tilde{\psi},\hat{\psi})$$

where

$$\Theta_+(z) = \left[(x_1-z)(x_2-z)\right]^{1/2} ,$$

$$\Theta_-(2) = \left[(x_1-z^{-1})(2,-2^{-1})\right]^{-1/2}$$

and $\Theta = \Theta_+\Theta_-$, then

$$a)\quad (\tilde{\psi},\tilde{\xi}(k_2)\tilde{\xi}(k_1)\hat{\psi}) = c_- m_-(z_1,z_2) \quad , z_i = e^{ik_i}$$

$$b)\quad (\tilde{\psi},\hat{\xi}^*(-k_1)\hat{\xi}^*(-k_2)\hat{\psi}) = (\tilde{\psi},\tilde{\xi}(k_1)\tilde{\xi}(k_2)\hat{\psi})$$

$$c)\quad (\tilde{\psi},\tilde{\xi}(k_1)\hat{\xi}^*(k_2)\hat{\psi}) = \overline{(\tilde{\psi},\tilde{\xi}(k_2)\hat{\xi}^*(k_1)\hat{\psi})} \quad \text{and}$$

$$(\tilde{\psi},\tilde{\xi}(k_2)\hat{\xi}^*(-k_1)\hat{\psi}) = c_+ m_+(z_1,z_2) , \quad z_i = e^{ik_i}$$

Remark: The singular function m_+ is to be interpreted as the inverse Fourier series transform of the Fourier series of m_+ calculated by using the principal value integral.

Proof of lemma II.1:

a) From the proof of Thm. II1 $(\tilde{\psi},\tilde{\xi}(k_2)\tilde{\xi}(k_1)\hat{\psi})/(\tilde{\psi},\hat{\psi}) = -C(k_1,k_2)$ where $(T_1'(k,.)C(.,k'))(k,k') = T_2'(k,k')$. T_1 and Y of ref.3 are related by $Y = -\bar{H} - e^{-2i\phi}\bar{H}e^{2i\phi} = -2e^{-i\phi}\bar{T}_1 e^{i\phi}$. Furthermore in ref. 2 it is shown that $Y_+ \ni e^{2i\phi} Y e^{-2i\phi}$ has kernel zero thus Y has kernel 0. By contour integration

$$(Y_+ f_-(.,z_2))(z_1,z_2) = -\frac{2\ z_1\ z_2}{z_1\ z_2 - 1}\ .$$

$$(\Theta(z_1)^{-1}\Theta(z_1)^{-1} - 1) = \Theta(z_1)^{-1}\ \Theta(z_2)^{-1}\ h(z_1,z_2)\ , \tag{2.3}$$

where $f_-(z_1,z_2) = \Theta(z_1)^{-1}\ \Theta(z_2)^{-1}\ m_-(z_1,z_2)$, the integration conveniently being carried out for $1 < |z_2| < x_2$ and then passing to the limit $|z_2| = 1$ which agrees with the principal value integral since f_- is analytic in z in an annulus about $|z_2| = 1$ for $|z_1| = 1$. Thus $Ym_- = h$, which is equivalent to $T_1^! C = T_2^!$, since from[3]

$$m_- = G(k_1,k_2) = -e^{-i(\phi_1+\phi_2)}\ 2\pi\ e^{-2i\pi/4}\ C(k_1,k_2) \quad \text{so}$$

$$-C(k_1,k_2) = \frac{i}{2\pi}\ e^{i(\phi_1+\phi_2)}\ m_-(k_1,k_2)\ .$$

C is unique since dim ker $T_1^!$ = dim ker Y = dim ker $Y_\pm = 0$.

b) By substituting $\tilde{\xi}(k) = U\hat{\xi}(k)\ U^{-1}$ in $(\tilde{\psi},\tilde{\xi}(-k_2)\tilde{\xi}(-k_1)\hat{\psi})$ weget

$$\frac{(\tilde{\psi},\tilde{\xi}(-k_2)\tilde{\xi}(-k_1)\hat{\psi})}{(\tilde{\psi},\hat{\psi})} = \frac{(\tilde{\psi},\hat{\xi}^*(-k_1)\hat{\xi}^*(-k_2)\hat{\psi})}{(\tilde{\psi}\ ,\ \hat{\psi})} = -\bar{C}(-k_1,-k_2)\ .$$

Taking thecomplexconjugateof $(T_1^!\ C(.,k_2))(k_1,k_2) = T_2^!(k_1,k_2)$ and changing k,, k_2 to $-k_1$, $-k_2$ we get

$$(T_1^!\ \bar{C}(-.,-k_2))(k_1,k_2) = -\ T_2^!(k_1,k_2)$$

so by the uniqueness of the solution (ker $T_1^! = 0$)

$$\bar{C}(-k_1,-k_2) = -C(k_1,k_2) = (\tilde{\psi},\tilde{\xi}(k_2)\tilde{\xi}(k_1)\hat{\psi})/(\tilde{\psi},\hat{\psi})\ .$$

c) The first line follows by substituting $U\hat{\xi}(k_1)U^{-1}$ for $\tilde{\xi}(k_1)$ and using $U^{-1}\ \hat{\xi}^*(k_2)U = \tilde{\xi}^*(k_2)$. Substituting

$$\hat{\xi}^*(-k_1) = \int \tilde{\xi}^*(q)\bar{T}_1(q,-k_1)\ dq + \int \tilde{\xi}(q)\bar{T}_2(q,-k_1)\ dq$$

gives

$$(\tilde{\psi},\tilde{\xi}(k_2)\xi^*(-k_1)\hat{\psi}) = (\tilde{\psi},\hat{\psi})\bar{T}_1(k_2,-k_1) + \int \bar{T}_2(q,-k_1)(\tilde{\psi},\tilde{\xi}(k_2)\tilde{\xi}(q)\hat{\psi})dq \ . \tag{2.4}$$

Define $W = -\bar{H} + e^{-2i\phi}\ \bar{H}\ e^{2i\phi}$. Then

$$\bar{T}_2(q,-k_1) = \bar{T}_2(-k_1,q) = \frac{1}{2i}\ (e^{i\phi}\ W\ e^{-i\phi})(k_1,q)$$

so that the second term on the right-hand side of (2.4) becomes

$$\int \frac{1}{i}\left(\frac{e^{i\phi}\quad W\ e^{-i\phi}}{2}\right)(k_1,q)(\tilde{\psi},\tilde{\xi}(k_2)\tilde{\xi}(q)\hat{\psi})dq = \frac{(\tilde{\psi},\hat{\psi})}{4\pi}\ e^{i(\phi_1+\phi_2)}\ (Wm_-)(k_1,k_2). \tag{2.5}$$

By contour integration, carried out as in a), setting $Y_- \equiv -e^{2i\phi}We^{-2i\phi}$

$$Y_-f_-(\cdot,z_2))(z_1,z_2) = 2f_+(z_1,z_2) + \frac{z_1\ z_2}{z_1z_2-1}\left(1+\frac{1}{\Theta(z_1)\Theta(z_2)}\right)$$

where

$$f_+ = \Theta(z_1)^{-1}\ \Theta(z_2)^{-1}m_+(z_1,z_2)$$

so that

$$-(Wm_-)(z_1,z_2) = 2m_+(z_1,z_2) + \frac{2\ z_1\ z_2}{z_1\ z_2-1}\ (\Theta(z_1)\Theta(z_2)+1\ ).$$

Thus eq. (2.5) becomes

$$(-\frac{1}{2\pi}\ e^{i(\phi_1+\phi_2)}\ m_+(z_1,z_2) - \frac{1}{2\pi}\ \frac{z_1z_2}{z_1z_2-1}\ (e^{-i(\phi_1+\phi_2)} + e^{i(\phi_1+\phi_2)}))(\tilde{\psi},\hat{\psi}) \tag{2.6}$$

and the 2nd term of (2.6) cancels the 1st term in (2.4) giving the result.

Remarks:

1. The function f_- is not simply an ansatz for the solution of eq. (2.3). In ref.2 it is shown that, by passing to Fourier coefficients, eq. (2.3) becomes two Toeplitz matrix equations which can be solved

using Wiener-Hopf factorization, $\Theta(z)$ admitting the factorization $\Theta = \Theta_+ \Theta_-$ where $\Theta_+(\Theta_-)$ is analyticand non-zero in $|z| \leq 1$ ($|z| \geq 1$). By the theory of these equations (see ref, 11) the unique (follows from ker $Y_+ = 0$) solution can be explicited as a contour integral involving h and the known functions in Y_+ as on page 230, eq, (13.8) of ref. 11.

Proof of Thm. I: The proof of the theorem reduces to showing that the matrix element

$$M^x((e^{ik})_{1,m}|(e^{ik})_{m+1,n}) = (\tilde{\psi},\tilde{\xi}(k_n)\ldots\tilde{\xi}(k_{m+1})\;\sigma_1^x\;\hat{\xi}^*(-k_m)\ldots\hat{\xi}^*(-k_1)\hat{\psi})$$

has the representation given by the theorem, Substituting for σ_1^x in terms of $\hat{\xi}$ and $\hat{\xi}^*$ in M^x gives

$$M^x((e^{ik})_{1,m}|(e^{ik})_{m+1,n}) = \frac{1}{\sqrt{2\pi}}\,\alpha\int e^{i\phi_q}\,(\tilde{\psi},\tilde{\xi}(k_n)\ldots\tilde{\xi}(k_{m+1})\hat{\xi}(q)\hat{\xi}^*(-k_m)\cdot$$
$$\ldots\hat{\xi}^*(-k_1)\hat{\psi})dq + \frac{1}{\sqrt{2\pi}}\,\bar{\alpha}\int e^{-i\phi_q}\,(\tilde{\psi},\tilde{\xi}(k_n)\ldots$$
$$\tilde{\xi}(k_{m+1})\hat{\xi}^*(q)\hat{\xi}^*(-k_m)\ldots\;\hat{\xi}^*(-k_1)\hat{\psi})\,dq\quad. \tag{2.7}$$

Letting

$$M((e^{ik})_{1,m}|(e^{ik})_{m+1,n}) = (\tilde{\psi},\tilde{\xi}(k_n)\ldots\tilde{\xi}(k_{m+1})\hat{\xi}^*(-k_m)\ldots\hat{\xi}^*(-k_1)\hat{\psi})$$

and using anti-commutation we can write (2.7) as $(\alpha = e^{-i\pi/4})$

$$M^x = \frac{1}{\sqrt{2\pi}}\;\alpha\sum_{j=1}^{m}(-1)^{j-m}\,e^{i\phi_{-k_j}}\,M(\Delta_j(e^{ik})_{1,m}|(e^{ik})_{m+1,n}) +$$
$$\frac{1}{\sqrt{2\pi}}\,\bar{\alpha}\int e^{-i\phi_q}\,M((e^{ik})_{1,m},e^{-iq}|(e^{ik})_{m+1,n})\;dq\;. \tag{2.8}$$

Now from Thm. III using the rules for the expansion of a Pfaffian we have

$$\frac{M}{(\tilde{\psi},\hat{\psi})} = \sum_{j=1}^{m-1} (-1)^{j-m-1} (\tilde{\psi},\hat{\xi}^*(-k_m)\hat{\xi}^*(-k_j)\hat{\psi})M(\Delta_j(e^{ik})_{1,m}|(e^{ik})_{m+1,n})(\tilde{\psi},\hat{\psi})^{-2}$$

$$+ \sum_{j=m+1}^{n} (-1)^{j-m-1}(\tilde{\psi},\tilde{\xi}(k_j)\hat{\xi}^*(-k_m)\hat{\psi})M((e^{ik})_{1,m-1}|\Delta_j(e^{ik})_{m+1,n})(\tilde{\psi},\hat{\psi})^{-2}$$

$$= (\tilde{\psi},\hat{\psi})^{-2}\Big[\sum_{j=1}^{m-1} (-1)^{j-m-1} M(e^{ik_j}, e^{ik_m}|\phi)M(\Delta_j(e^{ik})_{1,m-1}|(e^{ik})_{m+1,n})$$

$$+ \sum_{j=m+1}^{n} (-1)^{j-(m+1)}M(e^{ik_m}|e^{ik_j})M((e^{ik})_{1,m-1}|\Delta_j(e^{ik})_{m+1,n})\Big] \qquad (2.9)$$

Substituting (2.9) in *(2.8)* gives

$$M^x = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{m} (-1)^{j-m}\, e^{i\phi-k_j}\, M(\Delta_j(e^{ik})_{1,m}|(e^{ik})_{m+1,n})$$

$$+ \frac{(\tilde{\psi},\hat{\psi})^{-1}}{\sqrt{2\pi}}\,\bar{\alpha}\int e^{-i\phi_q}\Big\{\sum_{j=1}^{m} (-1)^{j-m} M(e^{ik_j},e^{-iq}|\phi)M(\Delta_j(e^{ik})_{1,m}|$$

$$(e^{ik})_{m+1,n}) + \sum_{j=m+1}^{n} (-1)^{j-(m+1)}M(e^{-iq}|\, e^{ik_j})\,.$$

$$M((e^{ik})_{1,m}|\Delta_j(e^{ik})_{m+1,n})\Big\}dq\,. \qquad (2.10)$$

Substituting from lema II.1 for $M(e^{ik_j}, e^{-iq}|\phi)$ and $M(e^{-iq}\,|e^{ik_j})$ we find for the integrals in *(2.10)*

$$\int e^{-i\phi_q} M(e^{ik_j}, e^{-iq}|\phi)dq = \left(+i\, e^{-i\phi k_j} - i\, e^{i\phi k_j}\,\frac{\Theta_-(z_j)}{\Theta_-(\infty)}\right)(\tilde{\psi},\hat{\psi})$$

and

$$\int e^{-i\phi_q} M(e^{-iq}|e^{ik_j})dq = (-e^{i\phi k_j}\,\Theta_-(z_j)/\Theta_-(\infty))(\tilde{\psi},\hat{\psi})$$

so that (2.10) becomes

$$M^x = +\frac{\alpha}{\sqrt{2\pi}}\sum_{j=1}^{m}(-1)^{j-m}e^{i\phi_{k_j}}\Theta_-(z_j)/\Theta_-(\infty)\,.\,M(\Delta_j(e^{ik})_{1,m}|(e^{ik})_{m+1,n})$$

$$-\frac{\bar{\alpha}}{\sqrt{2\pi}}\sum_{j=m+1}^{n}(-1)^{j-m-1}e^{i\phi_{k_j}}(\Theta_-(z_j)/\Theta_-(\infty)\,.\,M((e^{ik})_{1,m}|\Delta_j(e^{ik})_{m+1,n})$$

which agrees with (1.3) by the expansion rules for the Pfaffian recalling that the matrix elements $M(\Delta_j(e^{ik})_{1,n}|(e^{ik})_{m+1,n})$ and $M((e^{ik})_{1,m}|\Delta_j(e^{ik})_{m+1,n})$ have a Pfaffian expansion by Thm. III, the terms being sums of products of the functions $m_\pm$ by lemma 11.1. Also by lemma 11.1 an overall factor of

$$(2\pi)^{-\frac{(n-1)}{2}}e^{i\sum_{\substack{\ell=1\\ \ell\neq j}}^{n}\phi_{k_\ell}}$$

is present.

3. SCALING LIMIT

We now take the limit from above the critical temperature in the way described in the introduction. First recall the definitions: $\tanh K^* = e^{-2K}$, $x_1 = \mathrm{ctnh}\,K^*\,\mathrm{cthn}\,K$, $x_2 = \mathrm{ctnh}\,K/\mathrm{ctnh}\,K^*$, $K = JT^{-1}$, and the derived identities $\tanh K = e^{-2K^*}$ and $\sinh 2K.\sinh 2K^* = 1$. We have $x_1 = e^{2(K+K^*)} > 1$ for all T and $x_2 = e^{2K^*}e^{-2K} > 1$, $x_1 > x_2$ for $T > T_c$ since $K^* > K$. For $T = T_c$ $x_2 = 1$, $x_1 = e^{4K_c}$ and $e^{2K_c} = 1+\sqrt{2}$ follows from $\sinh^2 2K_c = 1$.

We consider directly the series representation (1.3) of Thm. I. Make the change of variables $q = \lambda p$, $k = \lambda p$ in all integrais. Thus the integrals will be over the interval $-\pi/\lambda$ to π/λ and if M^x has n variables we can group the λ factcrs such that M^x has a factor $\lambda^{n/2}$. Thus in addition to the energy factors and overall factors in M^x we consider the factor $\lambda^{n/2}$ A. We will need lirnits of the various functions that appear in (1.3) and (1.4). We have

Lemma III.I Let lim mean the $\lambda\to 0$ limit as described in (1.4).

Then

a) $\lim x_2 = 1$, $\lim x_1 = e^{4K_c} = (1+\sqrt{2})^2$

b) $\lim E\frac{(\lambda p)}{\lambda} = (p^2+m^2)^{1/2} \equiv \omega(p)$

c) Let $\Theta_+(z) = (x_1-z)^{1/2}(x_2-z)^{1/2}$, $\Theta_-(z) = 1/(x_1-z^{-1})^{1/2}$.

$(x_2-z^{-1})^{1/2}$ and $\Theta(z) = \Theta_+(z)\Theta_-(z)$. Then

$$\lim \Theta_+(e^{i\lambda p})/\lambda^{1/2} = (-i(p+im))^{1/2}\sqrt{2}\ (1+\sqrt{2})$$

$$\lim \lambda^{1/2}\Theta_-(e^{i\lambda p}) = 1/((i(p-im))^{1/2}\sqrt{2}\ (1+\sqrt{2})\ )$$

$$\lim \Theta(e^{i\lambda p}) = \left(\frac{p+im}{p-im}\right)^{1/2}, \quad \lim(\Theta_-(\infty)^{-1} = \sqrt{x_1x_2}) = 1+\sqrt{2}$$

d) $(\tilde{\psi},\hat{\psi}) = (\det T_1^*T_1)^{1/4} = \hat{m}/\cosh K^*$ where

$\hat{m} = [1 - (\sinh 2K \sinh 2K)^2]^{1/8}$ and

$$\lim(\hat{m}/\cosh K^*)/|K - C|^{1/8} = (4\cosh 2K_c)^{1/8}/\cosh K_c$$

e) $$\lim \lambda m_{\pm}(e^{i\lambda p}, e^{i\lambda q}) = -\frac{1}{(p-im)^{1/2}(q-im)^{1/2}}\left(\frac{\omega(p)\pm\omega(q)}{p+q}\right)$$

Proof of lemma III.1: a,b,c and e are immrdiate. d) The identity $(\tilde{\psi},\hat{\psi}) = \hat{m}/\cosh K^*$ follows from refs. 3 and 2,

Proof of Thm. II. Theoverall factor $(\Theta_-(\infty)(\tilde{\psi},\hat{\psi}))^k$ that is dividedout behaves as $(T-T_c)^{k/8}$. Now the factor $e^{i\sum_{j=1}^{n}\phi_{q_j}}$ that occurs in (1.3) will also appear with its complex conjugate to give 1. In M^x as seen from c and e there will be an overall factor of $\prod_{j=1}^{n}(p_j-im)^{-1/2}$ which also appears with its complex conjuagate to give the overall factor $\prod_{j=1}^{n}\omega(p_j)^{-1}$ accounting for the weighted integrals in (1.5) and the formof the matrix

B in the theorem. By b) of the lemma the exponential energy-momentum factors of (1.3) converge to those of (1.5).

Concerning the convergence of the 2-point function recall Hadamard's inequality: If N is an $n\times n$ matrix with matrix elements n_{ij}, then

$$|\det N| \leqslant n^{n/2} \{\max_{i,j} |n_{ij}|\}^n \quad .$$

Applying this inequality to the n th term (n odd) of S_2^L and using the fact that $\sup_{p,q} |\Delta_-(p,q)| \leqslant 1$ we obtain the bound

$$(n!(2\ )^n)^{-1} \left(\int_{-\infty}^{\infty} \frac{e^{-\omega(p)(s_2-s_1)}}{\omega(p)} dp \right) . \ (n+1)^{(n+1)/2}$$

which implies convergence of the series by the ratio test.

For $k>2$ Hadamard's inequality is not directly applicable to L^x or L since the singular functions Δ_+ appear. A bound on L is obtained by expanding the Pfaffian of B as a sum of terms where each term is a product of a fixed number of Δ_+'s times the product of two Pfaffians which only involve the Δ 's. Each one of these terms is bounded by using Hadamard's inequality on the Pfaffian of the Δ_-'s and using the fact that with respect to Lebesque measure the denominator of A_+/π is the kernel of the Hilbert transform of norm 1.

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Resumo

Uma representação em série das funções de Schwinger (correlação) do modelo bidimensional de Ising na rêde infinita é obtida de um modo muito simples e transparente por uma fórmula de Feynman-Kac (F-K) num espaço de Fock fermiônico. Operadores de energia-momentum e campo (ou spin) são definidas na fórmula de F-K utilizando dois conjuntos canônicos de operadores fermiônicos relacionados entre si por uma transformação linear canônica própria (plct), i.e., implementada por um operador unitário. Explorando propriedades especiais da plct, estabelecemos uma generalização do teorema de Wick. Substituindo as representações espectrais dos operadores de energia-momentum na fórmula de F-K, obtemos a representação em série das funções de Schwinger. Os termos da série são calculados explicitamene por uma simples aplicação do teorema generalizado de Wick. Usando esta representação em série, obtemos também as funções de Schwinger no limite de escala.