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Decay Rate of the False Vacuum at High Temperatures

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Abstract We investigate, within the semiclassical approach, the high temperature behaviour of the decay rate (Γ) of the metastable vacuum in Field Theory. We exhibit some exactly soluble (1+1) and (3+1) dimensional examples and develop a formal expression for Γ in the high temperature limit.

1. INTRODUCTION

Theories in which the symmetry is spontaneously broken might have this symmetry restored when the temperature exceeds a critical one[1] (T_c). The system described by such a theory is then supposed to undergo a phase transition if the temperature of the system reacties this critical value.

As a result of our belief in the standard model, which makes use of the spontaneous symmetry breaking mechanism, and due to the now accepted picture of a hot early universe it follows a widespread belief that the universe experienced phase transitions in the course of its expansion and cooling. These phase transitions might have an essential role in the evolution of the universe. In particular, issues like the flatness problem, the horizon problem and magnetic monopoles could be solved if the phase transition takes place with a large enough amount of supercooling. The models that satisfy this condition are called inflationary models[2].

In order to know the amount of supercooling underwent by the system one must study the decay rate of the false vacuum Γ (ie , tun-

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neling probability per unit time). Having in hand this quantity we can calculate the fraction of the universe in the new phase as a function of time and consequently the amount of supercooling,

The study of the decay of the false vacuum in field theory has been carried out by Coleman and Callan[3]. Applying the Euclidean path integral technique and the semi-classical approximation these authors developed an expression for the decay rate of the false vacuum at zero temperature.

One can very easily extend the formalism for finite temperature[4] since the quantum statistics of bosons (fermions) at finite temperature T is formally equivalent to quantum field theory in Euclidean space-time, periodic (anti-periodic) in the "complex time" direction with period T^{-1}.

In this paper we study how the "first quantum" corrections to Γ depend on T and their importance in the high temperature limit. This is done by studying (1+1) and (3+1) exactly solvable models and by the use of a formal power series.

The outline of this paper is as follows: In section 2 we review the formalism used to calculate Γ. In the following section we analyze some exaclty soluble models in (1+1) dimensions. Section 4 contains the derivation of a formal expression for Γ in the high temperature limit and also a (3+1) dimensional example. Finally, section 5 summarizes the results and gives our conclusions.

2. REVIEW OF THE FORMALISM AT FINITE TEMPERATURE

We are going to review briefly the functional integration formalism applied to quantum field theory at finite temperature. All the information about a system in thermal equilibrium at a temperature* β^{-1} is contained in the partition function which is given by

$$Z = \mathrm{tr}\left[e^{-\beta\hat{H}}\right] \tag{2.1}$$

where $\hat{H}$ is the Hamiltonian of the system. The Helmoltz free energy (A) can be obtained from Z

* Our system of units is such that $k_B = \hbar = c = 1$.

$$A = - \beta^{-1} \log Z \quad . \tag{2.2}$$

Supposing that the system under study is described by a scalar field ϕ and by a set of fields X, which can be either boson or fermion fields, one can write a path integral representation for Z [5]

$$Z = \int [D\phi DX] \; e^{-S(\phi,X)} \quad , \tag{2.3}$$

where $S(\phi,X)$ is the Euclidean action of the system and the integration is carried over periodic (anti-periodic), in the "complex time" direction, field configuration for bosonic (fermionic) fields - that is, $\phi(0,\vec{x}) = \phi(\beta,\vec{x})$ for bosonic fields and $\psi(0,\vec{x}) = -\psi(\beta,\vec{x})$ for fermionic fields.

One can integrate out the X fields in (2.3), and write[8]

$$Z = \int d\phi \; e^{-S_{eff}(\phi)} \quad , \tag{2.4}$$

where S_{eff} is the effective action of the field ϕ.

At this point one can perform the "canonical" approximation[8] and write*

$$S_{eff} = \frac{1}{2} (\partial_\mu \phi)^2 + V_{eff}(\phi) \tag{2.5}$$

- that is, S_{eff} is replaced by the first term of its low momentum[6] expansion.

The approximation (2.5) is very good in the high temperatures limit since the leading terms (in T) of S_{eff} are exactly the ones that come from the zero momentum terms of S_{eff} [7].

Now one usually performs the semiclassical approximation in order to obtain a closed expression for Γ/V. In the semiclassical limit, the leading contributions to Z, given by (2.4) and (2.5), come from the field configurations which minimize the effective action and therefore obey the Euler-Lagrange equation

$$\sum_{i=1}^{D} \frac{\partial^2 \phi_c}{\partial x_i^2} = V'_{eff}(\phi_c) \quad . \tag{2.6}$$

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* The effective potential here contains internal lines of the X fields only since we have only performed the $[DX]$ integration of (2.3).

where ϕ_C satisfies

$$\phi_C(0,x) = \phi_C(\beta,\vec{x}) .$$

It is easy to prove that for high temperatures the relevant field configurations are those independent of the Euclidean time[4].

Now one makes a functional Taylor expansion of S_{eff} around ϕ_C and keeps only the quadratic terms in $\eta = \phi-\phi_C$

$$Z^{(1)} = e^{-S_{eff}(\phi_C)} \int [D\eta] \exp-\left\{\int d_E^D\vec{x} \left[\frac{1}{2}\sum_{i=1}^{D} (\partial_i\eta)^2 + \frac{1}{2}\eta\, V''_{eff}(\phi_C)\eta\right]\right\} . \tag{2.7}$$

The gaussian integral in (2.7) is easy to perform[3-5] and one gets formally

$$Z^{(1)} = e^{-S_{eff}(\phi_C)} \det{}^{-1/2}\left[-\sum_{i=1}^{D}\partial_i^2 + V''_{eff}(\phi_C)\right] . \tag{2.8}$$

This expression gives the contribution of just one bounce solution.

Using the dilute gas approximation one obtains

$$Z = Z^0 \exp\left(\frac{Z^1}{Z^0}\right) ,$$

where

$$Z^0 = e^{-S_{eff}(\phi_{VAC})} \det{}^{-1/2}\left[-\sum_{i=1}^{D}\partial_i^2 + V''_{eff}(\phi_{VAC})\right] , \tag{2.9}$$

and ϕ_{VAC} is the false vacuum of the theory.

Defining the transition probability[3,10] as $\Gamma = -2 \text{ Im } A$ one obtains, by treating separately the zero eigenvalues:

$$\frac{\Gamma}{V} = -2T \text{ Im} \left[\frac{S_{eff}(\phi_C)}{2\pi}\right]^{Z/2} \left[\frac{\det'(-\partial^2 + V''_{eff}(\phi_C))}{\det(-\partial^2 + V''_{eff}(\phi_{VAC}))}\right]^{-1/2} \exp - S_{eff}(\phi_C) , \tag{2.10}$$

where the prime indicates that the zero eigenvalues of $-\partial^2 + V''_{eff}(\phi_C)$ must be omitted from the determinant and Z is the number of these eigenvalues. This is essentially the result contained in ref. 4.

After some algebra (see appendix A) one can write Γ/V as

$$\frac{\Gamma}{V} = \frac{2\,T^{Z+1}}{\sin(\frac{\beta\omega}{2})}\left(\frac{S_{eff}(\phi_C)}{2\pi}\right)^{Z/2}\exp\left\{-S_{eff}(\phi_C) + \left[\frac{\beta}{2}\left(\sum_j \lambda_j^V - \sum_j{}'' \lambda_j^C\right)\right]\right.$$
$$\left. + \left[\sum_j \log\left(1 - e^{-\beta\lambda_j^V}\right) - \sum_j{}'' \log\left(1 - e^{-\beta\lambda_j^C}\right)\right]\right\} \qquad (2.11)$$

where

$$\left[\sum_{spatial} - (\partial_i)^2 + V''_{eff}(\phi_{VAC})\right]\eta_j = (\lambda_j^V)^2\,\eta_j \qquad (2.12)$$

$$\left[\sum_{spatial} - (\partial_i)^2 + V''_{eff}(\phi_C)\right]\eta_j = (\lambda_j^C)^2\,\eta_j\,, \qquad (2.13)$$

the negative eigenvalue (which is assumed to be unique) in (2.13) is written as

$$(\lambda_-^C)^2 = -\,\omega^2 \quad , \qquad (2.14)$$

and the double prime indicates that the negative and zero eigenvalues must be omitted from the summation.

3. ONE DIMENSIONAL EXAMPLES

Now we are going to analyze some specific examples in order to get the asymptotic behaviour of the decay rate at high temperatures.

3-A. An "Inverted" $\lambda\phi^4$ potential

The Lagrangian density for this first example is given by

$$L_{eff} = \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}(\partial_x\phi)^2 + V_{eff}(\phi) \quad , \qquad (3.1)$$

where

$$V_{eff}(\phi) = \frac{1}{2}m^2\phi^2 - \frac{\lambda\phi^4}{4} \quad , \qquad (3.2)$$

where m^2 and X are positive functions of the temperature. The expression (3.2) for V_{eff} looks like the usual approximation[4,9] used for V_{eff} when

ϕ_1, which is the solution to $V_{eff}(\text{false}) = V_{eff}(\phi_1)$, is much smaller than the true vacuum.

Clearly, the state $\phi=0$ is metastable, We shall calculate its decay rate per unit volume at high temperatures*, First of all, we have to obtain a static solution to

$$\partial_{xx}\phi - m^2\phi + \lambda\phi^3 = 0 \ . \tag{3.3}$$

A solution to (3.3) is

$$\phi_C = \sqrt{2/\lambda}\ m\ \text{sech}(mx) \ . \tag{3.4}$$

The Euclidean action of this solution is given by

$$S_{\text{eff}}(\phi_C) = \frac{4m^3}{3\lambda}\frac{1}{T} \ . \tag{3.5}$$

The eigenvalues of the operator $-\Box + V''_{\text{eff}}(\phi_C)$

$$\left[-\Box + V''_{\text{eff}}(\phi_C)\right]\eta = \left[-\partial_{\tau\tau} - \partial_{xx} + m^2 - 6m^2\ \text{sech}^2(mx)\right]\eta = \varepsilon\eta \tag{3.6}$$

are well known[16]

$$\varepsilon = \left(\frac{2\pi n}{\beta}\right) + \begin{cases} -3m^2 \\ 0 \\ k'^2 + m^2 \end{cases} \tag{3.7}$$

where n is an integer.

For periodic boundary conditions in a box of side L

$$k'L + \delta(k') = 2\pi n'$$

where n' is an integer and $\delta(k')$ is the phase-shift

$$\delta(k) = -\frac{2}{m}\arctan\left[\frac{3km}{2m^2 - k^2}\right] \ . \tag{3.8}$$

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* In order to the semiclassical approximation be applicable, one must have the following necessary condition satisfied: $\lambda T\to 0$, $\lambda \gg 1$.

In order to evaluate the imaginary part of the determinant ratio ($\equiv R$), which is given by (A.6) we recall that $\phi_{VAC} = 0$. Then we have:

$$\text{Im } R = \frac{T}{\sin\left(\frac{\sqrt{3}\,\beta m}{2}\right)} \exp\left\{\frac{\beta}{2}\left[\sum_k \sqrt{k^2+m^2} - \sum_{k'} \sqrt{k'^2+m^2}\right]\right.$$

$$\left. + \left[\sum_k \log\left(1 - e^{-\beta\sqrt{k^2+m^2}}\right) - \sum_{k'} \log\left(1 - e^{-\beta\sqrt{k'^2+m^2}}\right)\right]\right\} . \quad (3.9)$$

The above expression contains a divergent part given by

$$E1 = \frac{1}{2}\sum_k \sqrt{k^2+m^2} - \frac{1}{2}\sum_{k'} \sqrt{k'^2+m^2} . \quad (3.10)$$

For large L this expression becomes[11]

$$E1 = \frac{3\sqrt{2m^2}}{2\pi} + \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)} \frac{d\delta(k)}{dk} \sqrt{k^2+m^2} \quad (3.11)$$

$E1$ can made finite by adding the counterterm to the effective action

$$CT = \frac{1}{2}\int_{-\infty}^{+\infty} dx \left[\phi_C^2 - \frac{m^2}{\lambda}\right] \times 3\lambda \int_{-\infty}^{+\infty} \frac{dk}{(2\pi)} \frac{1}{\sqrt{k^2+m^2}} . \quad (3.12)$$

Writting

$$\text{Im } R = \frac{T}{\sin\left(\frac{\beta\sqrt{3}m}{2}\right)} \exp\{\beta E_1 + \beta CT + E_2\} , \quad (3.13)$$

we have, for L going to infinity,

$$E2 = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{d\delta}{dk} \log\left(1 - e^{-\beta\sqrt{k^2+M^2}}\right) \quad (3.14)$$

Defining a new variable $\mu = \beta k$ we can write

$$E2 = \beta m \int_{-\infty}^{+\infty} \frac{d\mu}{2\pi} \log\left(1 - e^{-\sqrt{\mu^2+\beta^2 m^2}}\right)\left[\frac{4}{\mu^2+4\beta^2 m^2} + \frac{2}{\mu^2+\beta^2 m^2}\right] \quad (3.15)$$

The behaviour of $E2$ in the high temperature limit can be found in the literature[12] and is

$$E2 \simeq \text{constant} \times \beta \times \log(\beta m) \quad . \tag{3.16}$$

Thus, for $\beta \to 0$ we have

$$\frac{\Gamma}{L} = \frac{2T^2}{\sin\left(\frac{\sqrt{3}m}{2}\right)} \left(\frac{2m^3}{3\pi\lambda T}\right)^{1/2} \exp\left[m\beta\left(\text{constant } \frac{m^2}{\lambda} + \text{constant' } \log \beta m\right)\right] \quad . \tag{3.17}$$

In this case we see explicitly that in the high temperature limit the contribution from the determinant ratio (constant' log (βm)) may give a non negligible correction to the exponential factor (β constant) depending on the values of β, m, and λ.

3-B. Spontaneously broken $\lambda\phi^4$ with a source term

Now we are going to consider $V_{eff}(\phi)$, which appears in (3.1), of the form

$$V_{eff}(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{\lambda\phi^4}{4} + \varepsilon\phi \tag{3.18}$$

where m^2, λ, and ε are positive and temperature dependent. (3.18) resembles the effective potential of the (1+1) dirnensional scalar electrodynamics in the high temperatures limit[8].

We will consider the case $\varepsilon \ll 1$ - that is, we will perform a thin wali approximation.

The relative minima ϕ_- ($\simeq \frac{m}{\sqrt{\lambda}} + \frac{\varepsilon}{m^2}$) is metastable and it decays to ϕ_+ ($= -\frac{m}{\sqrt{\lambda}} + \frac{\varepsilon}{m^2}$) with a decay rate per unit length Γ/L. We will obtain Γ/L to the leading order in ε.

For high temperatures[3] the static classical solution must satisfy

$$\partial_{xx}\,\phi_C = \left.\frac{\delta V}{\delta\phi}\right|_{\phi=\phi_C} = -\mu^2\phi_C + \lambda\phi_C^2 + \varepsilon \quad . \tag{3.19}$$

We can expand ϕ_C in powers of ε as follows

$$\phi_C = \sum_{n=0}^{\infty} \varepsilon^n \phi_n \quad . \tag{3.20}$$

Plugging (3.20) into (3.19) and solving the resulting equation for ϕ_0 we get

$$\phi_0 = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx}{\sqrt{2}}\right) \quad . \tag{3.21}$$

Next we need to solve the following eigenvalue problem

$$\left[-\partial^2 + V''_{\text{eff}}(\phi_C)\right]\eta_j = \alpha_j \eta_j \quad . \tag{3.22}$$

Again we also expand η_j and α_j in powers of ε

$$\eta_j = \sum_{n=0}^{\infty} \varepsilon^n \eta_{j,n} \quad , \tag{3.23}$$

$$\alpha_j = \sum_{n=0}^{\infty} \varepsilon^n \alpha_{j,n} \quad . \tag{3.24}$$

We can calculate explicitly the eigenvalues to zero order in ε and the result is

$$\alpha_0 = \left(\frac{2\pi n}{\beta}\right)^2 + \begin{cases} 0 \\ \frac{3}{2} m^2 \\ k^2 + 2m^2 \text{ (for the continuous spectrum)} \end{cases} , \tag{3.25}$$

We are going to assume the existence of just one negative eigenvalue and that it is at least of order $\varepsilon(\alpha_{neg} = -\gamma^2)$. Having the eigenvalues, we can calculate the pre-exponential factor, given by (A.6), to the lowest order in ε

$$\text{Im } R = \frac{T}{\sin\left(\frac{\gamma}{2T}\right)} \exp\left\{\frac{\beta}{2}\left[\frac{L}{2\pi}\int dk \sqrt{k^2+2m^2} - \sum_{k'} \sqrt{k'^2+2m^2} - \sqrt{3/2}\, m\right]\right.$$
$$\left. + \left[\frac{L}{2\pi}\int dk \log(1-e^{-\beta\sqrt{k^2+2m^2}}) - \sum_{k'} \log(1-e^{-\beta\sqrt{k'^2+2m^2}}) - \log(1-e^{-\beta\sqrt{\frac{3}{2}}\, m^2})\right]\right\} . \tag{3.26}$$

The two first factors appearing in the above exponential are divergent. In order to render these contributions finite we must renormalize them by adding the counterterm

$$CT = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left[\phi_C^2 - \frac{m^2}{\lambda} \right] \times 3\lambda \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{1}{\sqrt{k^2+2m^2}} \quad . \tag{3.27}$$

In the limit of L going to infinity, weobtain

$$\text{Im } R = \frac{T}{\sin\left(\frac{\beta\gamma}{2}\right)} \exp\left\{ \frac{\beta}{2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk\ \delta(k) \frac{d}{dk} \sqrt{k^2 + 2m^2} + \beta CT \right.$$
$$\left. - \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{d\delta}{dk} \log\left(1 - e^{-\beta\sqrt{k^2+2m^2}}\right) - \sqrt{\frac{3}{2}}\, m \frac{\beta}{2} - \log\left(1 - e^{-\beta\sqrt{\frac{3}{2} m^2}}\right) \right\} . \tag{3.28}$$

In the high temperature limit, the behaviour of Im R is given by

$$\text{Im } R = \frac{T}{\sin\left(\frac{\beta\gamma}{2}\right)} \exp\{ C_1 m\beta + C_2 \log \beta m) \} \ ,$$

where C_1 and C_2 are numerical constants. Therefore, we have

$$\frac{\Gamma}{L} = \frac{2T^2}{\sin\left(\frac{\beta\gamma}{2}\right)} \left[\frac{(2m^2)^{3/2}}{6\pi\lambda T} \right]^{1/2} \exp\left\{ - \frac{(2m^2)^{3/2}}{3\lambda T} + C_1 \frac{m}{T} + C_2 \log(\beta m) \right\} . \tag{3.29}$$

From (3.29) we can see again that the pre-exponential factor may give a non negligible correction to Γ, in the high temperature limit, depending on the values of β, m and λ.

3-C, The Birula Mycielski model[13,14]

Now we are going to repeat the calculation of **3-A-8** for the Lagrangian density

$$L_{eff} = \frac{1}{2} (\partial_\tau \phi)^2 + \frac{1}{2} (\partial_x \phi) + V_{eff}(\phi) \tag{3.30}$$

where

$$V_{eff}(\phi) = \frac{m^2\phi^2}{2}\left(1 - \log\frac{\phi^2}{C^2}\right) . \tag{3.31}$$

For high temperatures ($T > m/\pi\sqrt{2}$) the relevant static classical solution is

$$\phi_C = C\sqrt{e}\,\exp\left(-\frac{m^2x^2}{2}\right) . \tag{3.32}$$

The effective action associated to this field configuration is

$$S_{eff}(\phi_C) = \frac{e\sqrt{\pi}}{2T}\, C^2 m \quad . \tag{3.33}$$

In order to calculate the determinant of the fluctuati'ons we need to know the eigenvalues of $-\partial^2 + V''(\phi_C)$

$$[-\partial^2 + V''_{eff}(\phi_C)]\eta = \varepsilon\eta = (-\partial_{\tau\tau} - \partial_{xx} + m^4x^2 - 3m^2)\eta . \tag{3.34}$$

It is easy to check that $\varepsilon_{n,\ell}$ is given by

$$\varepsilon_{n,\ell} = \left(\frac{2\pi n}{\beta}\right)^2 + 2m^2(\ell-1) \quad , \tag{3.35}$$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$
$\ell = 0, 1, 2, \ldots$

We are going to assume that $V''_{eff}(\phi_{VAC}) = m^2$. So the pre-exponential factor for this system is given by

$$\text{Im}\, R = \frac{T}{\sin\left(\frac{m}{\sqrt{2}\,T}\right)} \times \exp\left\{\frac{\beta}{2}\left[\sum_k \sqrt{k^2+m_0^2} - \sum_{\ell=2}^{\infty}\sqrt{2m^2(\ell-1)}\right]\right.$$

$$\left. + \left[\sum_k \log\left(1 - e^{-\beta\sqrt{k^2+m_0^2}}\right) - \sum_{\ell=2}^{\infty}\log\left(1 - e^{-\beta\sqrt{2m^2(\ell-1)}}\right)\right]\right\} . \tag{3.36}$$

The first term between bracketsin the exponential corresponds to the zero-point energy and due to that it will be neglected. Then

$$\text{Im}\,R = \frac{T}{\sin\left(\frac{m}{\sqrt{2}\,T}\right)}\exp\left\{\int_{-\infty}^{+\infty} dk\,\log\left(1-e^{-\beta\sqrt{k^2+m_0^2}}\right) - \sum_{\ell=2}^{\infty}\log\left(1-e^{-\beta\sqrt{2m^2(\ell-1)}}\right)\right\}.$$

In the high temperature limit, we have

$$\int_{-\infty}^{+\infty} dk\,\log\left(1-e^{-\beta\sqrt{k^2+m_0^2}}\right) \cong 2\beta^{-1}\int_0^{\infty} d\mu\,\log(1-e^{-\mu})$$

and

$$\sum_{\ell=2}^{\infty}\log\left(1-e^{-\beta\sqrt{2m^2(\ell-1)}}\right) \cong \frac{\beta^{-2}}{m^2}\int_0^{\infty} d\mu\,\mu\,\log(1-e^{-\mu}).$$

Finally we obtain:

$$\frac{\Gamma}{L} \cong \left(\frac{e}{4\sqrt{\pi}\,T}\,C^2 m\right)^{1/2}\frac{2T^2}{\sin\left(\frac{m}{\sqrt{2}\,T}\right)}\exp\left\{-\frac{e\sqrt{\pi}}{2T}\,C^2 m + \frac{T^2}{m^2}\int_0^{\infty} d\mu\,\mu\,\log(1-e^{-\mu}) + 2T\int_0^{\infty} d\mu'\,\log(1-e^{-\mu'})\right\}. \quad (3.37)$$

This example shows again that the preexponential factor might give non negligible contributions in the high temperature limit.

4. FORMAL HIGH-TEMPERATURE EXPANSION OF Γ/V

We shall develop a formal expansion for the ratio of determinants (R) which appears in (2.10) that will be useful in order to extract the dependence of H on T at high temperatures. R can be written as

$$R = \exp -\frac{1}{2}\left\{\text{tr}\,\log\left(-\Box_E + V''_{\text{eff}}(\phi_C)\right) - \text{tr}\,\log\left(-\Box_E + V''_{\text{eff}}(\phi_{\text{VAC}})\right)\right\}. \quad (4.1)$$

Then, from (4.1), it is easy to see that R canbe written under the forrn

$$R = \exp - \frac{1}{2}\left\{ \mathrm{tr}\ \log\left[1 + \frac{1}{-\Box_E + V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})}\,(V''_{\mathrm{eff}}(\phi_C) - V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}}))\right]\right\}$$

where

$$\frac{1}{-\Box_E + V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})} = G_\beta$$

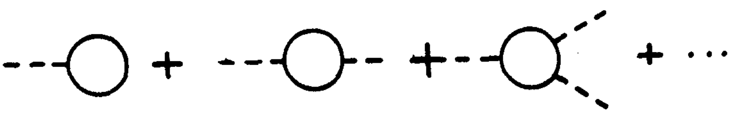
is just the free propagator at finite temperafure, with mass $\sqrt{V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})}$.

If we expand the log above in powers of

$$\frac{1}{-\Box_E + V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})}\left[V''_{\mathrm{eff}}(\phi_C) - V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})\right] ,$$

we get formally

$$\mathrm{tr}\ \log\left\{1 + \frac{1}{-\Box + V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})}\left[V''_{\mathrm{eff}}(\phi_C) - V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})\right]\right\} \equiv$$



(4.2)

where the dashed lines correspondto the "background field" $(V''_{\mathrm{eff}}(\phi_C) - V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}}))$, and the interna1 lines denote propagators G_β.

It is shown in appendix B that the first term of this series gives the leading contribution for β going to zero when the space-time dirnension is four. Then, we have:

$$R = \exp - \frac{1}{2}\ \mathrm{tr}\ \frac{1}{-\Box_E + V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})}\left[V''_{\mathrm{eff}}(\phi_C) - V''_{\mathrm{eff}}(\phi_{\mathrm{VAC}})\right] \qquad (4.3)$$

for $\beta \to 0$ $(T \to \infty)$.

We need to be careful when using (4.3). The formal manipulations that we made in order to get (4.3) work just for the eingenvalues belonging to the continuurn, Then, negative and zero eigenvalues can be treated as we did in appendix A and the result for Im R is

$$\text{Im } R = \frac{T^2}{\sin \frac{\beta\omega}{2}} \exp - \frac{1}{2} \operatorname{tr}\{G_\beta [V''_{eff}(\phi_C) - V''_{eff}(\phi_{VAC})]\} \tag{4.4}$$

We expect this expression to hold for high temperatures. Lets find out the dependence on T of the exponent in (4.4) for the usual four dimensional space in this limit. We denote this exponent by σ - that is,

$$\sigma \equiv -\frac{1}{2} \operatorname{tr}\{G_\beta [V''_{eff}(\phi_C) - V''_{eff}(\phi_{VAC})]\} \tag{4.5}$$

The reason why a does not control the high temperature behaviour of the pre-exponential factor for 1 and two spatial dimensions is given in appendix B.

4-A. (3+1) dimensional space formal expression

For (3+1) dimensional space we have, from (4.5) :

$$\sigma = -\frac{1}{2} \sum_{n,k} \frac{1}{\left(\frac{2\pi n}{\beta}\right)^2 + k^2 + m^2} \frac{1}{\beta L} \int d^2x_E \, [V''_{eff}(\phi_C) - V''_{eff}(\phi_{VAC})] \tag{4.6}$$

where $m^2 = V''_{eff}(\phi_{VAC})$.

Performing the n summation and taking into account that for high temperatures the relevant classical solution in independent of the Euclidean time, we can further simplify (4.6)

$$\sigma = -\frac{1}{2} \int d^3x \, [V''_{eff}(\phi_C(x)) - m^2] \int \frac{d^3k}{(2\pi)^3} \left\{ \frac{\beta}{2\sqrt{k^2+m^2}} + \frac{\beta}{\sqrt{k^2+m^2}\left(e^{\beta\sqrt{k^2+m^2}} - 1\right)} \right\} \tag{4.7}$$

The first integral in d^3k is infinity and must be renormalized. For a renormalizable theory, they way we get rid of these divergences is very simple. We just add to the Lagrangian the usual counterterrns defined in perturbation theory[1,3]. These counterterms, for renorinalizable theories, cancels the divergences which appear in the formal expansions and, in particular, cancels the divergent piece in (4.7).

Therefore, in the high temperature limit, the main contribution to σ is given by

$$\sigma = -\frac{1}{2}\int d^3x\ \left[V''_{eff}(\phi_C)-m^2\right]\int\frac{d^3k}{(2\pi)^3}\ \frac{\beta}{\sqrt{k^2+m^2}\left(e^{\beta\sqrt{k^2+m^2}}-1\right)}\ . \qquad (4.8)$$

Then, in the high temperature limit, σ behaves as

$$\sigma = A\times T \qquad (4.9)$$

where

$$A = -\frac{1}{2}\int d^3x\ \left(V''_{eff}(\phi_C)-m^2\right)\int\frac{d^3\vec{k}}{(2\pi)^3}\ \frac{1}{k(e^k-1)}\ . \qquad (4.10)$$

Therefore

$$\frac{\Gamma}{V} = \frac{2T^{Z+1}}{\sin\left(\frac{\beta\omega}{2}\right)}\left[\frac{S_{eff}(\phi_C)}{2\pi}\right]^{Z/2}\exp\left\{-\frac{B}{T}+AT\right\} \qquad (4.11)$$

where $\frac{B}{T} = S_{eff}(\phi_C)$.

Let us make some comments on (4.11). We have to keep in mind that for obtaining (4.11) we have used the semiclassical approxirnation and the high temperature limit, So, when employing (4.11), we have to verify if the temperature we are working allows us to make use of these approximations.

4-B. Soluble (3+1) dimensional example

The (3+1) dimensional system, that we are going to consíder, is described by the effective Lagrangian density

$$L_{eff} = \frac{1}{2}\sum_{i=1}^{4}(\partial_i\phi)^2 + \varepsilon\phi - \frac{m^2}{2}\phi^2 + \frac{\lambda\phi^4}{4} \qquad (4.12)$$

where ϵ, m^2, and λ are positive and ϵ is much less than 1.

We will proceed like we did in another example - that is, we are going to calculate Γ/V to the lowest order in ϵ.

Expanding ϕ_C in powers of ϵ like in (3.20) and substituting into the classical equations of motion we obtain

$$\phi_0 = \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx}{\sqrt{2}} \right) . \qquad (4.13)$$

ϕ_0 describes a domain wall (Bloch wall) in three spatial dimensions[15].

Although ϕ_0, given by (4.13) depends on just one spatial variable, one can show that it describes some important features of the bounce solution[10]. Although this bounce solution will give Γ identically zero in the thermodynamic limit, it is useful for one can check in this example whether (4.11) works well.

The eigenvalues of $-\partial^2 + V''(\phi_C)$ to the lowest order in ϵ are given by

$$\alpha_0 = \left(\frac{2\pi n}{\beta}\right)^2 + kz^2 + ky^2 + \begin{cases} 0 \\ \frac{3}{2} m^2 \\ k'^2 + 2m^2 \end{cases} \qquad (4.14)$$

It is possible to prove the existence[3] of a negative eigenvalue which we will denote by $-w^2$ and assume that is unique.

After using (A.6) and renormalizing, the result we obtain in the high temperature limit[15] is

$$\frac{\Gamma}{V} = 2T^4 \left(\frac{\sqrt{2}\, m^3}{2\pi 3\lambda T}\right)^{3/2} \frac{1}{\sin\left(\frac{\beta\omega}{2}\right)} \exp\left\{ - \frac{2\sqrt{2}\, m^3 A}{3\lambda T} + \frac{T A m}{2\sqrt{2}} \right\}, \qquad (4.15)$$

where $A = V^{2/3}$, and V is the volume of the space.

This example shows that our formal expression (4.11) works, as it should, in four dimensional problems.

5. CONCLUSIONS

As pointed out in the introduction, it has been proposed that some aspects on the evolution of the early universe should be strongly dependent on the decay rate of the false vacuum. Phenomenological implications such as monopole density in the early universe and the Great Supercooling that the universe underwent are among the consequences of the vacuum decay process.

Some conclusions were drawn based on a fairly simple parametrization for the decay rate[2,4,7], namely

$$\frac{\Gamma}{V} = T^4 \; e^{-S_C/T} \left(\frac{S_C}{2\pi T}\right)^{3/2} . \tag{5.1}$$

The expression (5.1) obviously does not take into account, in a proper way, the contribution coming from the determinant ratios in (2.10) since (5.1) is basically given by zero loop contributions.

We have devised a method which allows us to infer the high temperature behaviour of the determinant ratios in (2.10) without solving the complete eigenvalue problem. The method relies on a simple graphical expansion which allows us to get the proper asymptotic behaviour as well as to perform the renormalization of the determinant in a straightforward way.

One can see from (4.11) that the contribution from the determinant ratio makes the decay rate Γ be bigger than the value given by (5.1). This (small) increase in Γ might have some consequences in the inflationary scenario, and we will be concerned with these consequences in a future publication.

We also have studied some exactly soluble toy models in (1 + 1) dimensions. From the solution of these problems we could extract an idea of the order of magnitude of the pre-exponential contribution for Γ/V. For instance, from (3.17) or (3.29), with a suitable choice for the parameters and the temperature one can get an increase of Γ/V by a few percent.

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APPENDIX A

Here we are going to obtain an expression for the imaginary part of the determinant ratio that appears in (2.10). We assume that $-\Delta + V''_{eff}(\phi_C)$ has only one negative $(-\omega^2)$ eigenvalue and that there are z zero eigenvalues $[(-\Delta + V''(\phi_C))\eta_j = \lambda_j^2 \eta_j]$

$$R = \left[\frac{\prod'_{n,j} (\omega_n^2 + \lambda_j^{S^2})}{\prod_{n,j} (\omega_n^2 + \lambda_j^{V^2})}\right]^{-1/2} \tag{A.1}$$

where $\omega_n^2 = (2\pi n/\beta)^2$.

It is easy to see that

$$R = \frac{\prod_j \lambda_j^V \prod_j \dfrac{\sinh \beta \lambda_j^V/2}{\beta \lambda_j^V/2}}{\prod'_j \lambda_j^S \prod_j \dfrac{\sinh \beta \lambda_j^S/2}{\beta \lambda_j^S/2}} \tag{A.2}$$

where we have used the identity

$$\prod_{n=1}^{\infty} (1 + z^2/n^2) = \frac{\sinh \pi z}{\pi z} \tag{A.3}$$

We notice that the negative eigenvalue makes R pure imaginary. Analyzing (A.2) with care we get that

$$\text{Im } R = \left(\frac{2}{\beta}\right)^z \frac{\prod_j \sinh(\beta \lambda_j^V/2)}{\sin \frac{\beta\omega}{2} \prod''_j \sinh(\beta \lambda_j^S/2)} \tag{A.4}$$

where the double prime indicates that the negative and zero eigenvalue are excluded from the product.

We can further transform (A.4) using that

$$\log \sinh(\beta z/2) = \frac{\beta z}{2} + \log(1 - e^{-\beta z}) - \log 2 \tag{A.5}$$

in order to get

$$\text{Im } R = \frac{T^Z}{\sin(\frac{\beta\omega}{2})} \exp\left\{\frac{\beta}{2}\left[\sum_j \lambda_j^V - \sum_j{}'' \lambda_j^S\right]\right. \tag{A.6}$$

$$\left. + \left[\sum_j \log(1 - e^{-\beta\lambda_j^V}) - \sum_j{}'' \log(1 - e^{-\beta\lambda_j^S})\right]\right\}$$

APPENDIX B

In this appendix we will analyze the temperature dependence of each term appearing in *(4.2)*. First of all, we would like to point out that each graphic appearing in *(4.2)* have zero external momentum[7] - that is, for high temperatures:

$$(3.2) = \sum_{n=1}^{\infty} \frac{(-1)^j}{j} \int d^4xE \left[V''_{\text{eff}}(\phi_C) - V''_{\text{eff}}(\phi_{\text{VAC}})\right]^j \frac{1}{\beta} \sum_{n=-\infty}^{+\infty}$$

$$\times \int \frac{d^D \vec{k}}{(2\pi)^D} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + V''_{\text{eff}}(\phi_{\text{VAC}}) + \vec{k}^2\right]^j} \tag{B.1}$$

where D is the number of spatial dimensions,

Lets obtain the dependence with β of

$$I_j = \frac{1}{\beta} \sum_n \int \frac{d^D \vec{k}}{(2\pi)} \frac{1}{\left[\left(\frac{2\pi n}{\beta}\right)^2 + k^2 + m^2\right]^j} , \tag{B.2}$$

where $m^2 = V''_{\text{eff}}(\phi_{\text{VAC}})$, when $\beta \to 0$.

Performing the scaling $\vec{t} = \beta\vec{k}$ we can write

$$I_j = \beta^{2j-(D+1)} \sum_n \int \frac{d^D \vec{t}}{(2\pi)^D} \frac{1}{\left[(2\pi n)^2 + t^2 + m^2\beta^2\right]^j} . \tag{B.3}$$

Now it is easy to see that

$$I_j = \frac{(-1)^{j-1}}{(j-1)!} \beta^{2j-(D+1)} \frac{d^{j-1}}{dx^{j-1}} f(x)\Big|_{x=m^2\beta^2} \tag{B.4}$$

where

$$f(x) = \sum_n \int \frac{d^D t}{(2\pi)^D} \frac{1}{[(2\pi n)^2 + t^2 + x]} . \tag{B.5}$$

For $D \geqq 3$ we have that

$$\lim_{\beta\to 0} f(m^2\beta^2) = \text{constant} .$$

Then, for $D \geqq 3$, the term $j=1$ is the most important term of (B.1) in the limit $\beta\to 0$.

If we have $D<3$, $f(x)$ diverges as x goes to zero due to the infrared of the theory. For example for $D=1$

$$\lim_{\beta\to 0} f(m^2\beta^2) \cong \frac{C}{m} .$$

Using (B.4) we get that is proportional to β^{-1}. Since all terms in the series (B.1) have the same temperature dependence with temperature, we have to sum the whole series then our formal expansion (4.2) (B.1) does not lead to a simple result to the determinant ratio.

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Resumo

Investigamos o comportamento de alta temperatura da taxa de decaimento (Γ) do vácuo metaestável em teoria de campos utilizando a aproximação semiclássica. Exibimos exemplos solúveis (na aproximação semiclássica) em (1+1) e (3+1) dimensões e desenvolvemos uma expressão formal para Γ no limite de altas temperaturas.