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On a Generalized Elliptic-Type Integral

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Abstract In this paper we study a family of integrals of the form

$$K_\mu(k,m) = \int_0^\pi \frac{\cos^{2m}\theta \, d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}$$

where $0 \leq k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$ and m is a nonnegative integer. Such integrals occur in radiation field problems. We obtain a series expansion and establish its relationship with Gauss' hypergeometric function. Asymptotic expansions valid in the neighbourhood of $k^2=1$ are given. One of these formulas has been obtained by the use of an Abelian theorem. Some recurrence relations are established. Results obtained earlier by Epstein and Hubbell, and Weiss follow as particular cases of our formulae given here. Some numerical values of $K_\mu(k,m)$ for selected values of the parameter are tabulated, using different formulae.

1. INTRODUCTION

Epstein and Hubbell [1,p.1] have treated a family of integrals

$$\Omega_j(k) = \int_0^\pi (1-k^2\cos\theta)^{-j-\frac{1}{2}} \, d\theta \tag{1}$$

where $0 \leq k < 1$. Such integrals are found in the application of the Legendre polynomial expansion method [2, p.109] to certain problems involving computation of the radiation field off-axis from a uniform circular disc radiating according to an arbitrary distribution law [3 p.249]. In continuation of the Epstein-Hubbell work[1], Weiss [4,p.1] has obtained an expansion of (1) in the neighborhood of $k^2=1$ and established its relationship with Legendre and hypergeometric functions.

Recently, Kalla[5] and Kalla, Conde and Hubbell[6] have definedand studied certain generalized elliptic-type integrals.

In the present work, we study the family of integrals

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$$K_\mu(k,m) = \int_0^\pi \frac{\cos^{2m}\theta \; d\theta}{(1 - k^2 \cos\theta)^{\mu+\frac{1}{2}}} \tag{2}$$

where $0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$ and m is a non-negative integer. We observe that for $m=0$ and $\mu=j$, a positive integer

$$K_j(k,0) = \Omega_j(k) \tag{3}$$

and further

$$K_0(k,0) = \Omega_0(k) = (\sqrt{2}\lambda/k)K(\lambda) \tag{4}$$

$$K_1(k,0) = \Omega_1(k) = (\sqrt{2}\lambda/\, k(1-k^2))\; E(\lambda) \tag{5}$$

where

$$\lambda^2 = 2k^2/(1 + k^2) \tag{6}$$

and

$$K(\lambda) = \int_0^{(\pi/2)} (1 - \lambda^2 \sin^2\theta)^{-\frac{1}{2}} d\theta \tag{7}$$

$$E(\lambda) = \int_0^{(\pi/2)} (1 - \lambda^2 \sin^2\theta)^{\frac{1}{2}} d\theta \tag{8}$$

are the complete elliptic integrals of the first and second kind respectively [7,p.295; 8,p.587].

First, we obtain a series expansion of $K_\mu(k,m)$ for small values of k. We establish a relation between our generalized elliptic-type integral and Gauss' hypergeometric function. Asymptotic expansions of $K_\mu(k,m)$, valid in the neighborhood of $k^2=1$ are obtained. One of the results has been obtained by the use of the transformation formulae for the hypergeometric function, while the other, by an appeal to an Abelian theorem. Some recurrence formulae are established.

2. SERIES EXPANSION

We have

$$K_\mu(k,m) = \int_0^\pi \frac{\cos^{2m}\theta \, d\theta}{(1 - k^2 \cos\theta)^{\mu+\frac{1}{2}}}$$

where $0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$ and m is a non-negative integer. We use the binomial expansion for small values of k to obtain

$$K_\mu(k,m) = \sum_{r=0}^{\infty} \frac{k^{2r} \, (\mu + \frac{1}{2})_r}{r!} \int_0^\pi \cos^{2m+r}\theta \, d\theta \tag{9}$$

Evaluating the θ-integral, by using

$$\int_0^\pi \cos^n\theta \, d\theta = \left[1 + (-1)^n\right] \, (\sqrt{\pi}/2) \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})} \tag{10}$$

we get the required expansion

$$K_\mu(k,m) = \sum_{r=0}^{\infty} W_r(\mu,m) k^{4r} \tag{11}$$

where

$$W_r(\mu,m) = \frac{(\mu + \frac{1}{2})_{2r} \, \sqrt{\pi} \, \Gamma(m + r + \frac{1}{2})}{(2r)! \, \Gamma(m + r + 1)} \tag{12}$$

From (11), we can derive

$$K_j(k,\ 0) = \sum_{r=0}^{\infty} \frac{k^{4r} \, \pi \, \Gamma(j+1)\Gamma(2j+4r+1)}{(64)^r \, (r!)^2 \, \Gamma(2j+1)\Gamma(j+2r+1)} \tag{13}$$

a result given earlier by Epstein and Hubbell [1,p.3].

3. RELATION WITH GAUSS HYPERGEOMETRIC FUNCTION

We use the well known result

$$\frac{1}{z^{\mu+\frac{1}{2}}} = \frac{1}{\Gamma(\mu + \frac{1}{2})} \int_0^\infty t^{\mu-\frac{1}{2}} e^{-zt} \, dt, \mathrm{Re}(z) > 0,\ \mathrm{Re}(\mu) > -\frac{1}{2} \tag{14}$$

to obtain

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$$K_\mu(k,m) = \int_0^\pi \cos^{2m}\theta \left[\frac{1}{\Gamma(\mu+\frac{1}{2})} \int_0^\infty t^{\mu-\frac{1}{2}} e^{-(1-k^2\cos\theta)t} dt\right] d\theta$$

Interchanging the order of integration, which is justified due to the absolute convergence of the integrals involved, we get

$$K_\mu(k,m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \int_0^\infty e^{-t} t^{\mu-\frac{1}{2}} \left[\int_0^\pi \cos^{2m}\theta \; e^{k^2 t \cos\theta} d\theta\right] dt \qquad (15)$$

Now, expressing $\cos^{2m}\theta$ as $(1 - 2\sin^2\frac{\theta}{2})^{2m}$ and using the binomial expansion, the θ-integral reduces to:

$$\sum_{r=0}^{2m} (-1)^r \, 2^r \binom{2m}{r} e^{-k^2 t} \int_0^1 u^{-\frac{1}{2}} (1-u)^{r-\frac{1}{2}} e^{2k^2 tu} du \qquad (16)$$

This can be rewritten as,

$$\sum_{r=0}^{2m} \frac{(-1)^r \, 2^r \binom{2m}{r} e^{-k^2 t} \, \Gamma(r+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(r+1)} \; \Phi(\tfrac{1}{2}, \; r+1; \; 2k^2 t)$$

by virtue of the integral representation for the confluent hypergeometric function [9,p.255, 10,p.266]

$$\Phi(\alpha,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zt} \, t^{\alpha-1} (1-t)^{\gamma-\alpha-1} dt \qquad (17)$$

$$\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$$

Hence, we have

$$K_\mu(k,m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} \frac{(-1)^r \, 2^r \binom{2m}{r} \Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{\Gamma(r+1)}$$

$$\int_0^\infty e^{-(1+k^2)t} \, t^{\mu-\frac{1}{2}} \, \Phi(\tfrac{1}{2}, \; r+1; \; 2k^2 t) dt$$

$$= \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r} \Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{(2k^2)^{\mu+\frac{1}{2}}\,\Gamma(r+1)}$$

$$\int_0^\infty e^{-(\frac{1+k^2}{2k^2})x}\, x^{\mu-\frac{1}{2}}\, \Phi(\tfrac{1}{2},r+1;\, x)\, dx \qquad (18)$$

Evaluating the x-integral, we obtain

$$K_\mu(k,m) = \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r}\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{(1+k^2)^{\mu+\frac{1}{2}}\;\Gamma(r+1)}$$

$${}_2F_1\,(\tfrac{1}{2},\mu+\tfrac{1}{2},r+1;\; 2k^2/1+k^2) \qquad (19)$$

From (19), we deduce that

$$K_j(k,0) = \Omega_j(k) = \frac{\pi}{(1+k^2)^{j+\frac{1}{2}}}\; {}_2F_1\,(\tfrac{1}{2},\; j+\tfrac{1}{2};\; 1;\; 2k^2/(1+k^2)) \;, \qquad (20)$$

a result given by Weiss [4,p.2 (9)].

4. ASYMPTOTIC EXPANSIONS

By recast (19) by using [9, p.105 (1)]

$$K_\mu(k,m) = (1+k^2)^{-\mu-\frac{1}{2}} \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r}\Gamma(\frac{1}{2})\;\Gamma(r+\frac{1}{2})}{\Gamma(r+1)}$$

$$\left(\frac{1-k^2}{1+k^2}\right)^{r-\mu} {}_2F_1(r+\tfrac{1}{2},\; r-\mu+\tfrac{1}{2};\; r+1;\; 2k^2/(1+k^2)) \qquad (21)$$

and then [9,p.105 (3)] leads to

$$K_\mu(k,m) = (1-k^2)^{-\mu-\frac{1}{2}} \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r}\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{\Gamma(r+1)}$$

$${}_2F_1(r+\tfrac{1}{2},\; \mu+\tfrac{1}{2};\; r+1;\; 2k^2/(k^2-1)) \qquad (22)$$

Now the formula [9,p.107 (34)] can be used to obtain

$$K_\mu(k,m) = \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r} \Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{\Gamma(r+1)}$$

$$\left[\frac{\Gamma(\mu-r)\Gamma(r+1)}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})} \frac{(1-k^2)^{r-\mu}}{(2k^2)^{r+\frac{1}{2}}}\ {}_2F_1\left(r+\tfrac{1}{2},\ \tfrac{1}{2};\ r-\mu+1;\ (k^2-1)/2k^2\right)\right. \tag{23}$$

$$\left. + \frac{\Gamma(r-\mu)\Gamma(r+1)}{\Gamma(r-\mu+\frac{1}{2})\Gamma(r+\frac{1}{2})} \frac{1}{(2k^2)^{\mu+\frac{1}{2}}}\ {}_2F_1\left(\mu-r+\tfrac{1}{2},\ \mu+\tfrac{1}{2};\ \mu-r+1; (k^2-1)/2k^2\right)\right]$$

Hence an expansion formula valid in the neighbourhood of $k^2=1$, can be expresses as

$$K_\mu(k,m) = \sum_{r=0}^{2m} (-1)^r\, 2^r \binom{2m}{r} \Gamma(\tfrac{1}{2})\Gamma(r+\tfrac{1}{2})$$

$$\left[\frac{\Gamma(\mu-r)\Gamma(1-k^2)^{r-\mu}}{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})(2k^2)^{r+\frac{1}{2}}} + \frac{\Gamma(r-\mu)\ (2k^2)^{-\mu-\frac{1}{2}}}{\Gamma(r-\mu+\frac{1}{2})\Gamma(r+\frac{1}{2})}\right] \tag{24}$$

A particular case of (24)

$$K_\mu(k,0) = \sqrt{\pi}\left[\frac{\Gamma(\mu)(1-k^2)^{-\mu}}{\Gamma(\mu+\frac{1}{2})(2k^2)^{\frac{1}{2}}} + \frac{\Gamma(-\mu)(2k^2)^{-\mu-\frac{1}{2}}}{\Gamma(\frac{1}{2}-\mu)}\right] \tag{25}$$

(provided μ is not an integer) is in agreement with a special case of a result given by Kalla, Conde and Hubbell[6].

Now we shall obtain an another asymptotic expansion valid in the neighbourhood of $k^2=1$. To do so, we rewrite (18) in the following form

$$K_\mu(k,m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} \frac{(-1)^r\, 2^r \binom{2m}{r}\Gamma(\frac{1}{2})\Gamma(r+\frac{1}{2})}{(2k^2)^{\mu+\frac{1}{2}}\ \Gamma(r+1)}$$

$$\int_0^\infty e^{-(\frac{1-k^2}{2k^2})x}\ x^{\mu-\frac{1}{2}}\ e^{-x}\ \Phi(\tfrac{1}{2},\ r+1;x)\ dx \tag{26}$$

We have therefore expressed $K_\mu(k,m)$ as a Laplace transforms in which the coefficient in the first exponential, $(1-k^2)/k^2$, approaches zero as k^2 tends to 1. Hence we can now apply an Abelian theorem [11, p. 281] for Laplace transform to determine the behaviour of $K_\mu(k,m)$ in the neighbourhood of $k^2=1$. We know that the asymptotic expansion of the confluent hypergeometric function is [10, p.271 or 12, p.256]

$$\Phi(\alpha,\gamma;x) \sim \frac{\Gamma(\gamma)\ e^x\ x^{\alpha-\gamma}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\gamma-\alpha)_n\ (1-\alpha)_n}{n!}\ x^{-n} \tag{27}$$

as x tends to infinity. Consequently an expansion formula for $K_\mu(k,m)$ valid in the neighbourhood of $k^2=1$ is

$$K_\mu(k,m) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} (-1)^r\ 2^r\ \binom{2m}{r} \sum_{n=0}^{\infty} \frac{\Gamma(r+n+\frac{1}{2})\Gamma(\frac{1}{2}+n)\Gamma(\mu-n-r)}{\Gamma(\frac{1}{2})n!(1-k^2)^{\mu-n-r}(2k^2)^{n+r+\frac{1}{2}}} \tag{28}$$

provided that (v-r) is neither zero nor a positive integer. We observe that

$$K_\mu(k,0) = \frac{1}{\Gamma(\mu+\frac{1}{2})} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(\mu-n)}{\Gamma(\frac{1}{2})n!(1-k^2)^{\mu-n}\ (2k^2)^{n+\frac{1}{2}}} \tag{29}$$

in the neighbourhood of $k^2=1$.

Let $p-r=\nu_r$, a positive integer. In this case, we decomposethe infinite sum of (28) into two parts, $\sum_{n=0}^{\nu_r-1}$ and $\sum_{n=\nu_r}^{m}$. As the second sum presents an indeterminate form, we take the limit and use the formula [13, p.13],

$$\psi(m-n)/\Gamma(m-n) \simeq (-1)^{n-m+1}\ (n-m)! \qquad n \geq m \tag{30}$$

to obtain an asymptotic expansion valid in the neighbourhood of $k^2=1$,

$$K_\mu(k,m) = \frac{k^{-2\mu-1}}{\Gamma(\mu+\frac{1}{2})} \sum_{r=0}^{2m} (-1)^r \binom{2m}{r}$$

$$\left[\sum_{n=0}^{\nu_r-1} \frac{\Gamma(r+n+\frac{1}{2})\Gamma(\frac{1}{2}+n)\Gamma(\nu_r-n)}{\Gamma(\frac{1}{2})\ n!\ 2^{n+\frac{1}{2}}} \left(\frac{1-k^2}{k^2}\right)^{n-\nu_r} \right. \tag{31}$$

$$\left. + \sum_{n=\nu_r}^{\infty} \frac{(-1)^{n-\nu_r+1}\Gamma(r+n+\frac{1}{2})\Gamma(\frac{1}{2}+n)}{\Gamma(\frac{1}{2})\ n!\ 2^{n+\frac{1}{2}}\ (n-\nu_r)!} ((1-k^2)/k^2)^{n-\nu_r} \log\left[\frac{1-k^2}{k^2}\right] \right]$$

If we set $m=0$, $\mu=j$, in (31) we get

$$K_j(k,0) = \Omega_j(K) = \frac{k^{-2j-1}}{\Gamma(j+\frac{1}{2})} \left[\sum_{n=0}^{j-1} \frac{\Gamma(n+\frac{1}{2})\Gamma(j-n)(-1)^n\pi}{\Gamma(\frac{1}{2}-n)\Gamma(\frac{1}{2})\ n!\ 2^{n+\frac{1}{2}}} \left(\frac{k^2}{1-k^2}\right)^{j-n} \right.$$

$$\left. + \sum_{n=j}^{\infty} \frac{\pi(-1)^{j+1}\ \Gamma(n+\frac{1}{2}) \left(\frac{1-k}{k^2}\right)^{n-j} \log\left(\frac{1-k^2}{k^2}\right)}{\Gamma(\frac{1}{2})\ (\frac{1}{2}-n)\ n!\ 2^{n+\frac{1}{2}}\ (n-j)!} \right] \tag{32}$$

a result given earlier by Weiss [4,p.1].

5. RECURRENCE FORMULAE

From the definitions of $K_\mu(k,m)$ we can obtain a number of recurence relations. For example, if we replace the numerator of (2) by $(1 - \sin^2\theta).\cos^{2m-2}\theta$ and then integrate by parts, we obtain

$$K_\mu(k,m) = K_\mu(k,m-1) + \frac{(2m-2)}{k^2(\mu-\frac{1}{2})} K_{\mu-1}(k,m-\tfrac{3}{2})$$

$$+ \frac{(1-2m)}{k^2(\mu-\frac{1}{2})} K_{\mu-1}(k,m-\tfrac{1}{2}) \tag{33}$$

If we multiply the numerator and denominator of (2) by $(1 - k^2\cos\theta)$ and then decompose into two integrals, we get the following formula

$$K_\mu(k,m) = K_{\mu+1}(k,m) - k^2 K_{\mu+1}(k,m+\tfrac{1}{2}) \tag{34}$$

$$K_\mu(k,m) = K_\mu(k,m+1) + \frac{2m}{k^2(\frac{1}{2}-\mu)} K_{\mu-1}(k,m-\tfrac{1}{2})$$

$$- \frac{(1+2m)}{k^2(\frac{1}{2}-\mu)} K_{\mu-1}(k,m+\tfrac{1}{2}) \tag{35}$$

From (33) and (35), we get

$$k^2(\tfrac{1}{2}-\mu) \left[K_\mu(k,m-1) - K_\mu(k,m+1)\right]$$

$$= K_{\mu-1}(k,m-\tfrac{1}{2}) - (1+2m) K_{\mu-1}(k,m+\tfrac{1}{2}) \tag{36}$$

$$+ 2(1-m) K_{\mu-1}(k,m-\tfrac{3}{2})$$

6. NUMERICAL COMPUTATION

In table 1, we compute the generalized elliptic-type integrals $K_\mu(k,m)$ on an VAX/VMS electronic computing machine, using equations (11) and (12). The results are given to eight significant digits.

In table 2, we compare some values of $K_\mu(k,m)$ from eqs.(11) and (12) ($m=0$, $\mu=j$, non negative integer) with the results given by (Epstein and Hubbell[1], Table 1).

In table 3, we compute the asymptotic approximations (24) and (28) of $K_\mu(k,m)$; K1 and K2 represent equations (24) and (28) respectively; the results are given in double precision.

In table 3, we observe a little difference in the entries of the two columns K1 and K2, and this is to be expected, due to the fact that the formula (24) is an approximate one, as only the first term of the hypergeometric series has been taken into consideration.

We propose to study some more properties and the numerical tabulation of $K_\mu(k,m)$ using the representation (19) and (31) in a subsequent paper.

Table 1

k	m	μ	$K_\mu(k,m)$
0.0	0	0.0	3.1415927
0.1	0	0.0	3.1416516
0.1	0	1.0	3.1418871
0.3	0	2.5	3.21910763
0.4	0	5.0	3.93922114
0.5	0	5.0	5.4212322
0.05	1	0.5	3.14160013
0.05	1	7.0	3.14182734
0.15	1	1.0	3.14271164
0.15	1	1.5	3.14338326
0.25	1	4.0	3.19930172
0.2	1	8.0	3.21872687
0.5	1	0.5	3.2192810
0.5	1	7.5	7.5090652
0.05	2	0.0	3.14159513
0.1	2	0.0	3.14162946
0.1	2	7.5	3.14512992
0.15	2	1.0	3.14252520
0.15	2	3.5	3.14657044
0.2	2	7.5	3.19887257
0.5	2	0.0	3.1655648
0.5	2	6	5.3373113
0.1	3	0.0	3.1416249
0.1	3	8.0	3.14506412
0.2	3	8.0	3.1978967
0.2	5	4.5	3.15872002
0.1	5	1.0	3.1417255
0.15	6	2.0	3.14305162
0.30	6	7.5	3.3473928
0.05	7	6.0	3.1416866
0.25	7	1.5	3.14886713

Table 2

k	j	$K_j(k,0)$	$\Omega_j(k)$
0.1	0	3.1416516	3.1416516
0.2	0	3.1425359	3.1425360
0.1	3	3.1428299	3.1428299
0.4	7	4.65059471	4.6505957
0.3	6	3.4647567	3.4647569
0.2	4	3.17284656	3.1728466

Table 3

k	m	μ	K1	K2
0.9	0	0.4	.51673832D 01	0.53422165D 01
0.93	0	0.4	0.58063225D 01	0.60271455D 01
0.99	0	0.4	0.12212198D 02	0.12537396D 02
0.90	0	1.9	0.26462283D 02	0.25033953D 02
0.99	0	1.9	0.16737238D 04	0.16771601D 04
0.90	0	8.6	0.76841550D 06	0.77144110D 06
0.99	0	8.6	0.18678756D 15	0.18684947D 15
0.99	1	7.6	0.39494292D 13	0.39509315D 13
0.99	1	1.1	0.89914795D 02	0.82548744D 02
0.96	2	1.6	0.45686106D 02	0.49430874D 02
0.99	2	1.6	0.53002991D 03	0.53419490D 03
0.99	2	8.1	0.27019611D 14	0.27028985D 14
0.93		8.6	0.13102725D 08	0.13131210D 08
0.99		8.6	0.18529050D 15	0.18535068D 15
0.90		8.6	0.69726588D 06	0.69930287D 06

REFERENCES

1. L.F. Epstein and J.H. Hubbell, J.Res.NBS B.V.*67*, 1 (1963).
2. M.J. Berger and J.C. Lamkin, J.Res.NBS, V.60, 109 (1958).
3. J.H. Hubbell, R.L. Bach and R.J. Herbold, J.Res.NBS c.*65*,249(1961).
4. G.H. Weiss, J.Res.NBS, B, V.68, 1 (1964).
5. S.L. Kalla, Results on generalized elliptic-type integrals.Bulgarian Acad. Sci. (special volume) 216-219 (1983).
6. S.L.Kalla, S. Conde and J.H. Hubbell, Some results on generalized elliptic-type integrals. Appl. Analysis.
7. A. Erdelyi, *Higher Transcendental Functions,* Vol.II McGraw-Hill,New York (1954).
8. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions,* Dover Publications Inc. New York (1972).
9. A. Erdelyi, *Higher Transcendental Functions*, Vol.I, McGraw-Hill, New York (1953).
10. N.N. Lebedev, *Special Functions and their Applications*, Prentice-Hall, Englewood Cliffs, N.J. (1965).

11. G. Doetsch, *Introduction to the Theory and Application of the Laplace Transformation,* Springer-Verlag, N.Y. (1974).

12. F.W.J. Olver, *Asymptotics and Special Functions,* Academic Press, New York (1974).

13. Y. L. Luke, *The Special Functions and Their Approximations.* Vol. I, Academic Press, New York (1969).

Resumo

Neste trabalho nós estudamos uma família de integrais de forma

$$K_{\mu}(k,m) = \int_0^{\pi} \frac{\cos^{2m}\theta \, d\theta}{(1-k^2\cos\theta)^{\mu+\frac{1}{2}}}$$

onde $0 \leqslant k < 1$, $\mathrm{Re}(\mu) > -\frac{1}{2}$ and m é um inteiro não negativo. Nós obtemos uma expansão em série e estabelecemos sua relação com a função hypergeométrica de Gauss. Expansões assintóticas válidas em uma vizinhança de $k=1$ são fornecidas. Uma dessas fórmulas é obtida usando-se um teorema abeliano. Algumas relações de recorrência são estabelecidas. Resultados obtidos anteriormente por Epstein e Hubbell e por Weiss se seguem como casos particulares de nossas fórmulas. Alguns valores numéricos de $K_{\mu}(k,m)$ para valores selecionados dos parametros são tabelados, usando diferentes fórmulas.