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The Vlasov-Maxwell System for the Darwin Model

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Abstract A simple derivation of the Vlasov-Maxwell system of equations is presented using Darwin's approximation for the electromagnetic interaction. The one-particle averages are calculated directly from the expressions for the microscopic electric and magnetic field as opposed to working with the scalar and vector potentials. This formulation brings out an interesting property of the Vlasov equation according to which time derivatives of average fields can be evaluated using the Heisenberg or the Schrödinger picture.

1. INTRODUCTION

The Darwin model for the electromagnetic interaction consists in neglecting the transverse part of the displacement current in in Ampère's law[1,2,3]. This is a self-consistent model that is correct to order $(v/c)^2$, where v is a characteristic velocity of particles or waves and c is the speed of light. Although electromagnetic radiation is not included in this model, it has been found to be quite useful to describe lowest-order relativistic effects in atomic physics calculations[4] and in particule simulation of low-frequency wave phenomena in plasmas[5,6]. In particular, in the latter application Darwin's model allows the inclusion of the self-consistent magnetic field generated by plasma currents without the requirement of using very small time steps to follow high-frequency electromagnetic phenomena.

To compare the results of numerical particle simulations with approximate analytical results, one usually uses the Vlasov-Maxwell system of equations dropping the transverse part of the displacement current from Ampère's law, However, there is a subtle point in this procedure that is not fully recognized. From a kinetic point of view, the electric and magnetic fields in Darwin's model are "microscopic" fields[3]. On the other hand, the fields that appear in Vlasov's equation are "macroscopic" fields obtained from averaging over all two-particle interaction processes. Thus, to fully justify the procedure of dropping the transverse part of the displacement current from the Vlasov-Maxwell system, one has to demonstrate that in Darwin's model the average fields also satisfy Faraday's law and the approximate form of Ampère's law.

It is interesting to notice that although there are many books in Plasma Physics, we could find only the one of Gartenhaus that explicitly deals with this point . In this reference the author derives the average equations using the scalar and vector potentials. Although this procedure is straighforward, it obscures the link between the microscopic and macroscopic fields and the role of time derivatives in the averaging process. In this paper we carry out an equivalent calculation working directly with the electric and magnetic fields. In our derivation, the problem of commuting time derivatives with the averaging operator appears explicitly and brings out an interesting property of the Vlasov equation. Namely, for calculating the time derivative of average fields, the Heisenberg picture of classical nonequilibrium statitiscal mechanics is equivalent to the Schrödinger picture[8]. This property is not only of academic relevance, but it also allows more flexibility when applying quantum-mechanical methods to solve the Vlasov equation[9].

In the next section we briefly recall the basic procedure for deriving the Vlasov equation from the Liouville equation[10,11]. In section 3 we derive the Vlasov-Maxwell system for the average field using the Darwin model. Finally, in section 4 we conclude by pointing out the analogy between Quantum Mechanics and Plasma Kinetic Theory with respect to the pictures of Heisenberg and Schrödinger to calculate time derivatives of average quantities.

2. THE VLASOV EQUATION

Classical statistical mechanics of many-body systems is rigorously based upon the Liouville equation for the probability density $D(\xi_1,\xi_2,\ldots,\xi_N;t)$ in the $6N$-dimensional space, where N is the number of particles in the system and $\xi_i = (\vec{x}_i,\vec{v}_i)$ are the coordinates of a point in the phase space of the i^{th} particle[12]. The propability density is defined such that $D(\xi_1,\xi_2,\ldots,\xi_N;t)d\xi_1\ldots d\xi_N$ is the probability of finding a given system in the volume $d\xi_1 d\xi_2\ldots d\xi_N$. The Liouville equation can be written in the form[11]

$$\frac{\partial D}{\partial t} + \sum_{i=1}^{N} \vec{v}_i \cdot \frac{\partial D}{\partial \vec{x}_i} \quad \sum_{i=1}^{N} \left[\frac{\vec{F}_i^{(0)}}{m} + \sum_{j\neq i} \frac{\vec{F}_{ij}}{m} \right] \cdot \frac{\partial D}{\partial \vec{v}_i} = 0 \tag{1}$$

where $\vec{F}_i^{(0)}$ is the external force on the particle (i), $\vec{F}_{ij}$ is the force on particle (i) due to particle (j), and m is the mass of the particle (here we are consideringq a plasma made of electrons with a uniform backqround of neutralizing ions). The Vlasov equation is derived via the BBGKY hierarchy of equations as an approximation to the Liouville equation[11,13,14]. It can be written in the form

$$\frac{\partial f(\xi_1)}{\partial t} + \vec{v} \cdot \frac{\partial f(\xi_1)}{\partial \vec{x}_1} + \left[\frac{\vec{F}_1^{(0)}}{m} + \frac{n}{m} \int \vec{F}_{12}\, f(\xi_2;t)d\xi_2 \right] \cdot \frac{\partial f(\xi_1)}{\partial \vec{v}_1} = 0 \;, \tag{2}$$

where $f(\xi_1)$ denotes the one-particle distribution function, normalized such that $n\int f(\xi;t)d\xi = N$, where n is the average particle density.

The quantity

$$\langle\vec{F}\rangle = n \int \vec{F}_{12} f(\xi_2)d\xi_2 \tag{3}$$

gives the average force on particle 1 due to the remaining particles in the plasma. In the sequel, we shall employ the following convenient notation for averages:

$$\langle h/f\rangle = n \int h(\xi,\xi';t)f(\xi';t)d\xi' \quad , \tag{4}$$

where $\xi = \xi_1$ and $\xi' = \xi_2$. Using eq. (3), the Vlasov equation can be simply written as

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{g} \cdot \frac{\partial f}{\partial \vec{v}} = 0 \;, \tag{5}$$

where $\vec{g} = (\vec{F}^{(0)} + \langle\vec{F}\rangle)/m$ denotes the average acceleration of each particle. Thus, the Vlasov equation describes the interaction between psrticles through only an average field, neglecting contributions from close encounters. This point is already well known and it is clearly discussed in standard textbooks[10].

To calculate the average force, eq. (3), it is necessary to specify $\vec{F}_{12}$, i.e., the expression for the force between two particles in the plasma. In this paper we assume that this force is derived from the Darwin approximate electromagnetic potentials, as described in the next section.

3. DARWIN'S APPROXIMATION

Let us consider a particle of charge q moving with velocity $\vec{v}'$ through the plasma. We assume that its acceleration is so mild that radiative effects are negligible. In this case, the electromagnetic field produced by the particle can be obtained from the scalar potential

$$\phi(\vec{x},t) = \frac{q}{4\pi\varepsilon_0} \frac{1}{|\vec{x}-\vec{x}'(t)|} \tag{6}$$

and the approximate vector potential

$$\vec{a}(\vec{x},t) = \frac{\mu_0 q}{4\pi} \frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|} + \frac{\mu_0 q}{8\pi} \nabla \frac{\partial}{\partial t} |\vec{x}-\vec{x}'(t)| \tag{7}$$

introduced by Darwin[3]. These potentials belong to the Coulomb gauge so that

$$\nabla . \vec{a}(\vec{x},t) = 0 .$$

This means that the expansion for $\phi(\vec{x},t)$ would be the correct one even if fully relativistic effects were included. However, eq. (7) for $\vec{a}$ is an approximate one and eqs. (6) and (7) cannot be rewritten in the form of the proper Liénard-Wiechert potentials. The "microscopic" fields are given by the usual expressions

$$\vec{e}(\vec{x},t) = - \nabla\phi - \frac{\partial\vec{a}}{\partial t} \tag{8}$$

and

$$\vec{b}(\vec{x},t) = \nabla\times\vec{a} \tag{9}$$

and the force on another particle of charge q (unprimed) is simply given by

$$\vec{F}_{12}(\xi,\xi';t) = q\vec{e}(\vec{x},t) + q\vec{v}\times\vec{b}(\vec{x},t) \tag{10}$$

We have to keep in mind that in eqs. (7) and (8) the observation point $\vec{x}$ is assumed fixed. Thus, the time derivative in these equations and in the ones following from them apply only tothe variables $\vec{x}'(t)$ and $\vec{v}'(t)$ referring to the source. Calculating the average of eq. (10), we obtain

$$<\vec{F}> = q\vec{E}_p(\vec{x},t) + q\vec{v}\times\vec{B}_p(\vec{x},t) \quad , \tag{11}$$

where

$$\begin{aligned} \vec{E}_p(\vec{x},t) = &- \frac{nq}{4\pi\varepsilon_0}\nabla <\frac{1}{|\vec{x}-\vec{x}'(t)|}/f> \\ &- \frac{n\mu_0 q}{4\pi} <\frac{\partial}{\partial t}\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> \\ &- \frac{n\mu_0 q}{8\pi}\nabla <\frac{\partial^2}{\partial t^2}|\vec{x}-\vec{x}'(t)|/f> \end{aligned} \tag{12}$$

and

$$\vec{B}_p(\vec{x},t) = \frac{\mu_0 nq}{4\pi}\nabla\times <\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> \tag{13}$$

The field equations, relating theaverage fields $\vec{E}_p$ and $\vec{B}_p$ and the sources can be calculated directly from eqs. (12) and (13). In particular, it is straightforward to obtain Poisson's equation by calculating the divergence of eq. (12), i.e.,

$$\nabla.\vec{E}_p(\vec{x},t) = \frac{nq}{\varepsilon_0}\int f(\vec{x},\vec{v};t)d^3v \ . \tag{14}$$

The derivatipn of Faraday's law is, however, more involved. Taking the curl of eq. (12), we obtain

$$\nabla\times\vec{E}_p(\vec{x},t) = -\frac{n\mu_0 q}{4\pi}\nabla\times<\frac{\partial}{\partial t}\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> . \qquad (15)$$

From this equation and the expression for $\vec{B}_p$, eq. (13), we see that, to obtain Faraday's law, we have to show that the time derivative can be taken out of the averaging brackets. First we note that, if $h(\xi,\xi')$ does not depend explicitly on time, it follows that

$$\frac{\partial}{\partial t}<\vec{h}/f> = <\vec{h}/\frac{\partial f}{\partial t}> . \qquad (16)$$

Since this is the case for $\vec{h} = \vec{v}'/|\vec{x}-\vec{x}'(t)|$, we have to simply prove that

$$<\frac{\partial}{\partial t}\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> = <\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/\frac{\partial f}{\partial t}> . \qquad (17)$$

The left-hand side of this equation can be written as

$$<\frac{\partial}{\partial t}\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> = <\frac{\vec{g}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> + <\vec{v}'\vec{v}' . \frac{\partial}{\partial\vec{x}'}\frac{1}{|\vec{x}-\vec{x}'(t)|}/f>, \qquad (18)$$

where $\vec{g}'(t) = \partial\vec{v}'(t)/\partial t$ is the acceleration of the primed particle. Using the identity $\delta_{ij} = \partial v_i'/\partial v_j'$, we can wrice $\vec{g}'f$ as $g_i'(\partial\vec{v}'/\partial v')f$ and integrate by parts the velocity integral appearing in the first term in the right-hand side of eq, (18). Then, we obtain

$$<\frac{\vec{g}'(t)}{|\vec{x}-\vec{x}'(t)|}/f> = -\int d^3x'd^3v'\frac{\vec{v}'}{|\vec{x}-\vec{x}'|}\vec{g}' . \frac{\partial f}{\partial\vec{v}'} = -<\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}\vec{g}' . \frac{\partial f}{\partial\vec{v}'}> \qquad (19)$$

The second term in the right-hand side of eq. (18) can also be easily calculated by integration by parts with respect to the space coordinate. Combining these results, we have

$$<\frac{\partial}{\partial t}\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/f = -<\frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|}/(\vec{v}' . \frac{\partial f}{\partial\vec{x}'}+\vec{g}' . \frac{\partial f}{\partial\vec{v}'}) , \qquad (20)$$

If follows from Vlasov's equation, eq. (5), that the term in parenthesis in the right-hand side of eq. (20) is just $-\partial f/\partial t$. Thus

$$\frac{\partial}{\partial t} < \frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|} / f> = <\frac{\partial}{\partial t} \frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|} / f> = < \frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|} / \frac{\partial f}{\partial t}> . \quad (21)$$

Using eqs. (13), (15), and (21), we finally obtain Faraday's law

$$\nabla x \vec{E}_p = - \frac{\partial \vec{B}_p}{\partial t} . \quad (22)$$

To obtain Ampere's law, we take the curl of eq. (13). Since $\vec{v}'$ and $\vec{x}'$ are independent variables, it follows that

$$\nabla \times \nabla \times \frac{\vec{v}'(t)}{|\vec{x}-\vec{x}'(t)|} = \nabla \left[\vec{v}'.\nabla \frac{1}{|\vec{x}-\vec{x}'(t)|} \right] + 4\pi \vec{v}' \delta(\vec{x}-\vec{x}') . \quad (23)$$

Substituting this expression into the curl of eq. (13), it follows that Ampere's law can be written in the form

$$\nabla \times \vec{B}_p(\vec{x},t) = \mu_0 \vec{J}(\vec{x},t) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}_L(\vec{x},t)}{\partial t} , \quad (24)$$

where

$$\vec{J}(\vec{x},t) = nq \int \vec{v} f(\vec{x},\vec{v};t) d^3v \quad (25)$$

is the plasma current density and

$$\vec{E}_L(\vec{x},t) = - \nabla \left[\frac{nq}{4\pi\varepsilon_0} \int \frac{f(\vec{x}',\vec{v}';t)}{|\vec{x}-\vec{x}'(t)|} d^3x' d^3v' \right] \quad (26)$$

is the longitudinal component of the electric field.

Equations (5), (14), (22), (24), (25), and (26) form the basic set of equation for the Vlasov-Maxwell system in the Darwin model. Clearly, the only approximation in this model is to neglect the transverse part of the displacement current in Ampère's law.

4. THE HEISENBERG AND SCHRÖDINGER PICTURES

A basic point in the calculations just presented is the proof that the time derivative commutes with the average over the one-particle phase space, as indicated in eq, (21). In analogy with Quantum Mechanics, we can say that the first form of calculating the time derivative of an average quantity corresponds to the Heisenberg picture,

$$\frac{\partial}{\partial t} \langle h/f \rangle = \langle \frac{\partial h}{\partial t} / f \rangle ,$$

whereas the second form corresponds to the Schrödinger picture,

$$\frac{\partial}{\partial t} \langle h/f \rangle = \langle h / \frac{\partial f}{\partial t} \rangle .$$

Actually, it can be rigorously shown that the Heisenberg and Schrödinger pictures of nanequilibrium statistical mechanics are equivalent at the level of Liouville's equation[8]. However, this is not necessarily true at the kinetic level described by reduced distribution functions and approximate equations. Since the Vlasov equation can be viewed as anone -particle Liouville equation, one could suspect that the same property would hold for that equation. In this paper we have shown that this is indeed true and leads to a correct derivation of the equations relating the average field quantities in the Darwin model.

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Resumo

O sistema de equações de Vlasov e Maxwell é obtido de uma maneira simples utilizando a aproximação de Darwin para a interação eletromagnética. As médias de uma partícula são calculadas diretamente das expressões para os campos microscópicos elétrico e magnético ao invés de utilizar os potenciais escalar e vetorial. Esta formulação permite realçar uma propriedade interessante de equação de Vlasov segundo à qual as derivadas temporais de campos médios podem ser calculadas utilizando o procedimento de Heisenberg ou o de Schrödinger.