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The Grassmann Mean Spherical Model

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Abstract We discuss the phase transition of a version of the spherical model with anticommuting variables. As with the usual spherical model it exhibits a phase transition with long range order and spontaneous magnetization for dimension $\nu \geqq 3$.

1. INTRODUCTION

There has been great interest in the study of the statistical mechanics of Grassmann variables. In fact these systems appear naturally in several situations in theoretical physics, for instance inthe study of disordered systems[1,2,3], in the euclidean formulation of field theories involving fermions[4,5], etc.

In this note we introduce and discuss the properties of a model similar to the usual (mean)-spherical model[6,7]. The main difference is that at each lattice site x we have a pair of anticommutating variables $\bar{\psi}(x)$, $\psi(x)$ instead of the usual spin variables. The spherical constraint is introduced in its average version , which in the usual model is equivalent (via a change of ensemble) to the original Kac-Berlin model. Due to the Gaussian nature of the Grassmann algebra integrations, the model cam be explicitly solved and its most striking feature is that as in the usual model the system exhibits a phase transition with long range order and spontaneous magnetization for $\nu \geqq 3$, with no phase transition for $\nu \leqq 2$. Apart from its intrinsic interest, this model, as discussed in[9] (whose notation and language we follow closely), should be the prototype of phase transition for a class models

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with Grassmann variables satisfying Infrared Bounds and Sum Rules (that might be dynamically generated). In fact, the nature of the phase transition is that of a Bose-Einstein condensation of "spin"-waves.

2. THE MODEL

Let Λ be a finite subset of $\mathbb{Z}^{\nu}$. At each lattice site x we have the anticommuting variables $\psi_\alpha(x)$, $\bar{\psi}_\beta(x)$, i.e. we consider the Grassmann algebra $\mathcal{a}_\Lambda$, generated by $\{\bar{\psi}_\alpha(x),\ \psi_\beta(y);\ x,y \in \Lambda,\ \alpha\beta = 1,\ldots, n\}$, with

$$\begin{aligned}
\bar{\psi}_\alpha(x)\bar{\psi}_\beta(y) + \bar{\psi}_\beta(y)\bar{\psi}_\alpha(x) &= 0\\
\psi_\alpha(x)\psi_\beta(y) + \psi_\beta(y)\psi_\alpha(x) &= 0 \qquad (1)\\
\bar{\psi}_\alpha(x)\psi_\beta(y) + \psi_\beta(y)\bar{\psi}_\alpha(x) &= 0
\end{aligned}$$

We define the integration on Grassmann algebra as usual[8]

$$\begin{aligned}
&\int \bar{\psi}\, d\bar{\psi} = \int \psi\, d\psi = 1\\
&\int d\bar{\psi} = \int d\psi = 0 \qquad (2)\\
&\int \bar{\psi}\psi\, d\bar{\psi}\, d\psi = -1\ , \quad \text{ets.}
\end{aligned}$$

The interaction is given by the Hamiltonian $H_\Lambda \in \mathcal{a}_\Lambda$

$$H_\Lambda = \sum_\mu \left(\bar{\psi},\ \left\{\frac{\partial_\mu - \partial^*_\mu}{2} + m_\Lambda\right\} \psi\right) \qquad (3)$$

with the notation

$$(f,g) = \sum_{\substack{x\in\Lambda\\ \alpha=1,\ldots,n}} f_\alpha(x) g_\alpha(x) \qquad (4)$$

$$\begin{aligned}
(\partial_\mu f)(x) &= f(x+e_\mu) - f(x)\\
&\qquad\qquad\qquad (5)\\
(\partial^*_\mu f)(x) &= f(x-e_\mu) - f(x)
\end{aligned}$$

where e_μ is the unit vector in the μ-th direction. in eq. (3) we use periodic boundary condition on $\Lambda = \{-2N,+2N\}^{\nu}$, $N \in Z$.

The expectation value of an observable $f \in \mathcal{a}_{\Lambda}$, at finite temperature β, is defined by

$$<f>_{\Lambda,\beta} = \frac{1}{Z} \int \prod_{x\in\Lambda} \prod_{\alpha=1}^{n} d\bar{\psi}_{\alpha}(x)\ d\psi_{\alpha}(x)\ f\ e^{-\beta H_{\Lambda}} \tag{6}$$

with

$$Z = \int \prod_{x\in\Lambda} \prod_{\alpha=1}^{n} d\bar{\psi}_{\alpha}(x) d\psi_{\alpha}(x)\ e^{-\beta H_{\Lambda}} \tag{7}$$

The model is defined choosing the parameter m_{Λ}, which plays the role of a chemical potential, in such way that

$$\sum_{\alpha} <\psi_{\alpha}(x)\psi_{\alpha}(x)>_{\beta,\Lambda} = 1 \tag{8}$$

which by translation invariance amounts to

$$\frac{1}{\Lambda} <(\bar{\psi},\psi)>_{\beta,\Lambda} = 1 \tag{9}$$

The "sum rule" (9) is the analogue of the spherical constraint in the mean spherical model[7,9]. It amounts to a change of ensemble in the original Kac-Berlin model.

We first remark that the "spherical condition" can always be met be an unique choice of m_{Λ}. In fact, introducing the Fourier transformed variables

$$\hat{\bar{\psi}}_{\alpha}(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_{x\in\Lambda} e^{ip.x}\ \bar{\psi}_{\alpha}(x)$$

$$\hat{\psi}_{\alpha}(p) = \frac{1}{\sqrt{|\Lambda|}} \sum_{x\in} e^{-ip.x}\ \psi_{\alpha}(x) \tag{10}$$

$$p \in \Lambda^* = \{-\pi, -(2N-1)\frac{\pi}{2N}, \dots, -\frac{\pi}{2N}, 0, \frac{\pi}{2N}, \dots, (2N-1)\frac{\pi}{2N}, \pi\}^{\nu}$$

we have

$$H_\Lambda = \sum_{p\in\Lambda^*} \sum_{\alpha=1}^{n} \hat{\bar{\psi}}_\alpha(p)\hat{\psi}_\alpha(p)\,(m_\Lambda + \sum_{\mu=1}^{\nu} \text{sen}\ p_\mu) \tag{11}$$

Therefore

$$<\bar{\psi}_\alpha(p)\psi_\beta(p)>_{\beta,\Lambda} = \frac{\delta_{\alpha\beta}}{\beta\,(m_\Lambda + D(p))} \tag{12}$$

where

$$D(p) = \sum_{\mu=1}^{\nu} \text{sen}\ p_\mu$$

and

$$\frac{1}{\Lambda} <(\bar{\psi},\psi)>_{\beta,\Lambda} = \frac{1}{\beta\Lambda} \sum_{p\in\Lambda^*} \frac{1}{m_\Lambda + D(p)} \equiv \frac{1}{\beta}\ f_\Lambda(m_\Lambda) \tag{13}$$

The function $f_\Lambda(.)$ has the following properties

a) it is monotonic in $x \in (+\nu,+\infty)$

b) $\lim\limits_{x\to\nu} f_\Lambda(x) = \infty$

c) $\lim f_\Lambda(x) = 0$

from a), b) and c) we conclude that there exists one and only $m_\Lambda(\beta) \in (\nu,\infty)$ s.t. $\frac{1}{\beta} f_\Lambda(m_\Lambda) = 1$.

3. PHASE TRANSITION - LONG RANGE ORDER

In the following we consider $n = 1$ but all results can be easily extended to any value of n.

For $\nu = 1,2$ there exists a@) > ν such that $m_\Lambda(\beta) \geqq a(\beta)$. This follows from the fact that

$$\lim_{x\downarrow\nu} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{p\in\Lambda^*} \frac{1}{(x + D(p))} = \lim_{x\downarrow\beta} \frac{1}{(2\pi)^\nu} \int_{[-\pi\ \pi]^\nu} \frac{d^\nu p}{(x + D(p))} = \infty \tag{14}$$

(the first equality follows from the dominated convergence theorem).

This implies $m(\beta) = \lim_{\Lambda\to\infty} m_\Lambda(\beta)$ (where the limit does exist along a convenient subsequence).

Consequently, there is no long-range order since the two point function

$$<\bar{\psi}(x)\psi(y)>_\beta = \frac{1}{(2\pi)^\nu}\int \frac{e^{ip(x-y)}}{m + D(p)}\, d^\nu p$$

($\nu = 1,2$) decays exponentially with $|x-y|$. Again the existence of the thermodynamic limit for the correlation functions

$$<\bar{\psi}(x)\psi(y)>_\beta = \lim_{\Lambda\to\infty} <\bar{\psi}(x)\psi(y)>_{\beta,\Lambda} = \frac{1}{(2\pi)^\nu}\int_{[-\pi,\pi]^\nu} \frac{e^{ip(x-y)}d_p}{m_\Lambda + D(p)}\,, \qquad (15)$$

follows from the dominated convergence theorem.

For $\nu \geqq 3$ the system will exhibit long-range order provided

$$\beta > \beta_c = \frac{1}{(2\pi)^\nu}\int_{[-\pi,\pi]^\nu} \frac{d^\nu p}{\nu + D(p)} \qquad (16)$$

To prove this we first show that, if $\beta > \beta_c$ then

$$\rho_{-\pi/2} = \lim_{\Lambda\to\infty} \frac{1}{\Lambda} <\hat{\bar{\psi}}(-\tfrac{\pi}{2})\ \hat{\psi}(-\tfrac{\pi}{2})>_{\beta,\Lambda} > 0 \qquad (17)$$

This follows from the sum rule

$$\frac{1}{\Lambda}\sum_{p\in\Lambda^*} <\hat{\bar{\psi}}(p)\hat{\psi}(p)>_{\beta,\Lambda} = \frac{1}{\Lambda}\sum_x <\bar{\psi}(x)\psi(x)>_{\beta,\Lambda} = 1 \qquad (18)$$

In fact, since $m_\Lambda(\beta) > \nu$, and $\lim_{\Lambda\to\infty} m_\Lambda(\beta) = m(\beta) \geqq \nu$, (the limit taken along a convenient subsequence), the dominated convergence theorem gives for $\nu \geqq 3$

$$\lim_{\Lambda\to\infty}\frac{1}{\Lambda}\sum_{\substack{p\in\Lambda^* \\ p\neq -\frac{\pi}{2}}} \frac{1}{m_\Lambda(\beta) + D(p)} = \frac{1}{(2\pi)^\nu}\int_{[-\pi,\pi]^\nu} \frac{d^\nu p}{m(\beta) + D(p)} \qquad (19)$$

Now

$$\frac{1}{(2\pi)^{\nu}} \int_{[-\pi,\pi]^{\nu}} \frac{d^{\nu}p}{m(\beta) + D(p)} < \beta_c$$

therefore

$$\rho_{-\pi/2} = \frac{1}{\beta} \lim_{\Lambda\to\infty} \frac{1}{\Lambda} \sum_{\substack{p\in\Lambda^* \\ p\neq -\frac{\pi}{2}}} \frac{1}{m_{\Lambda}(\beta) + D(p)} = \tag{20}$$

$$= 1 - \frac{1}{\beta} \frac{1}{(2\pi)^{\nu}} \int_{[-\pi,\pi]^{\nu}} \frac{d^{\nu}p}{m(\beta) + D(p)} > 0 \quad \text{if} \quad \beta > \beta_c$$

From the Riemann-Lebesgue lemma we have

$$\lim_{|x-y|\to\infty} \langle\bar{\psi}(x)\psi(y)\rangle_{\beta} = \rho_{-\pi/2} \tag{21}$$

concluding the proof.

4. SPONTANEOUS BREAKDOWN OF SYMMETRY

The model has an antiferromagnetic nature as suggested by the condensate at $p_i = -\pi/2$, $i = 1,\dots,\nu$. Therefore we introduce an external staggered anticommuting source

$$H_{\Lambda} = (\bar{\psi},\{m_{\Lambda}(\beta,\lambda) + D\}\psi) + \lambda\bar{h} \sum_{x\in\Lambda} e^{-i\frac{\pi}{2}p(x)} \psi(x) + \lambda \sum_{x\in\Lambda} e^{i\frac{\pi}{2}p(x)} \bar{\psi}(x)h \tag{22}$$

where

$$D = \sum_{\mu=1} \frac{(\partial_{\mu} - \partial^{*}_{\mu})}{2i} \quad \text{and} \quad p(x) = \sum_{i=1}^{\nu} x_i$$

Here we are consider'ing an enlarged Grassmann algebra generated by $\{\bar{\psi}(x),\psi(y),h,\bar{h},x,\ y\ \in \Lambda\}$, with $m_{\Lambda}(\beta,\lambda)$ being again uniquely defined by the spherical constraint

$$\frac{1}{\Lambda} \langle(\bar{\psi},\psi)\rangle_{\beta,\Lambda,\lambda} = 1 \tag{23}$$

An explicit computation shows that

$$<\hat{\bar\psi}(-\tfrac{\pi}{2})\,\hat\psi(-\tfrac{\pi}{2})>_{\beta,\Lambda,\lambda} = \frac{\lambda^2\,\bar h\, h}{\beta(m_\Lambda(\beta,\lambda)-\nu)} + \frac{1}{\beta\, m_\Lambda(\beta,\lambda)} \tag{24}$$

$$p \neq -\frac{\pi}{2}\,,\ <\hat{\bar\psi}(p)\hat\psi(p)>_{\beta,\Lambda,\lambda} = \frac{1}{\beta(m_\Lambda(\beta,\lambda)+D(p))} \tag{25}$$

and

$$\begin{aligned} <\hat\psi(-\tfrac{\pi}{2})>_{\beta,\Lambda,\lambda} &= -\frac{\sqrt{\Lambda}\,\lambda\, h}{m_\Lambda(\beta,\lambda)-\nu} \\ <\hat{\bar\psi}(-\tfrac{\pi}{2})>_{\beta,\Lambda,\lambda} &= -\frac{\sqrt{\Lambda}\,\lambda\,\bar h}{m_\Lambda(\beta,\lambda)-\nu} \end{aligned} \tag{26}$$

The sum rule (23) reads then

$$\frac{1}{\beta\Lambda\, m_\Lambda(\beta,\lambda)} + \frac{\lambda^2\,\bar h\, h}{\beta(m_\Lambda(\beta,\lambda)-\nu)} + \frac{1}{\beta\Lambda}\sum_{\substack{p\in\Lambda^* \\ p\neq -\frac{\pi}{2}}} \frac{1}{m_\Lambda(\beta,\lambda)+D(p)} = 1 \tag{27}$$

Therefore for $\lambda \neq 0$ there exists $a(\beta,\lambda) > \nu$ such that $m_\Lambda(\beta,\Lambda) > a(\beta,\lambda)$, $\forall \Lambda$, and so

$$m(\beta,\lambda) = \lim_{\Lambda\to\infty} m_\Lambda(\beta,\lambda) > a(\beta,\lambda) \tag{28}$$

From this and the dominates convergence theorem it follows that

$$<\bar\psi(0)>_{\beta,\lambda}\,<\psi(0)>_{\beta,\lambda} = 1 - \frac{1}{\beta}\cdot\frac{1}{(2\pi)^\nu}\int \frac{d^\nu p}{m(\beta,\lambda)+D(p)} \tag{29}$$

for $\beta > \beta_c$, $\forall \lambda \neq 0$.

Therefore

$$<\bar\psi(0)>_{\beta,\lambda}\,.\,<\psi(0)>_{\beta,\lambda} > 1 - \frac{\beta_c}{\beta} \tag{30}$$

Since the r.h.s. of (31) is independent of λ, in particular we have

$$\lim_{\lambda \downarrow 0} \langle\bar{\psi}(0)\rangle_{\beta,\lambda} \cdot \langle\psi(0)\rangle_{\beta,\lambda} \geqq 1 - \frac{\beta_c}{\beta} > 0$$

if $\beta > \beta_c$, that is, the symmetry is spontaneously broken.

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Resumo

Discute-se uma versão do modelo esférico com variáveis grassmannianas. Assim como no modelo esférico usual esse sistema apresenta uma transição de fase com ordem de longo alcance e quebra expontânea de simetria para dimensão $\nu \geqq 3$