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The Quantum Field Theory Associated with the Infinite Lattice Two-Dimensional Ising Model

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Abstract We show that the Schwinger functions associated with the infinite lattice correlation functions of the periodic two-dimensional Ising model can be represented by a Feynmann-Kac (F-K) formula in a Fermion Fock space H. The energy-momentum and field operators are expressed in terms of two sets of canonical Fermion free field operator-valued distributions acting in H. These two sets are related by a proper linear canonical transformation (pclt), i.e. there is a unitary operator U which implements the transformation. Series representation for the Schwinger functions are obtained by substituting the spectral representations of the energy-momentum operators in the F-K formula. Below the critical temperature $P_{\pm} = (I \pm U)/2$ are commuting orthogonal projections which reduce the algebra of observables and give an explicit decomposition of the periodic states into two, translationally invariant, pure states.

1. INTRODUCTION

Much attention has been devoted to the theoretical solution of the finite and infinite lattice two dimensional Ising model of nearest neighbor interacting spins (see [1-7]). Also the time- continuum and scaling limits of this model have been studied (see [8-11]). In ref. 4 for finite volume and periodic boundary conditions an efficient method for handling the algebraic complexity is developed which employs two finite auxiliary sets of operators, $\{\xi_k, \xi_k^*\}$ and $\{\xi_\ell, \xi_\ell^*\}$ (the $\{k\}$ and $\{\ell\}$ are wavenumbers belonging to distinct sets) which satisfy anticommutation relations. All the eigenvalues and eigenfunctions of the transfer matrix are obtained explicitly. In ref, 7 a linear relation between the $\{\xi_k, \xi_k^*\}$ and $\{\xi_\ell, \xi_\ell^*\}$ operators is exploited to obtain a series representation for the n-point spin correlation functions, for finite and infinite volume.

Here we construct the infinite lattice quantum field theory of this model starting from the LMS[4] solution (which is reviewed in section 2) preserving as much of the algebraic structure as possible. In section 3 for the infinite volume theory we introduce two sets of auxiliary free Fermion operator-valued distributions $\{\hat{\xi}(k),\hat{\xi}^*(k)\}$ and $\{\tilde{\xi}(k),\tilde{\xi}^*(k)\}$, $k\varepsilon[-\pi,\pi]$ acting in the Fermionic Fock space H. As in the finite volume case the ^ and ~ operators are not independent but are shown to be related by a proper linear canonical transformation (plct), J, i.e. the transformation is implemented by a unitary operator U acting on H and have vacuum vectors $\hat{\psi}$ and $\tilde{\psi} = U\hat{\psi}$, respectively. Also energy-momentum and spin or field operators are defined. A Feynman-Kac formula is obtained for the infinite volume Schwinger functions in section 4 and an infinite series expansion results by inserting the spectral representations of the energy-momentum operators. Using the explicit form of $\tilde{\psi}$ as given by the theory of plct (see Berezin[12]) evaluation of the series is in principle reduced to an application of Wick's theorem. However, in ref. 18 a generalization of Wick's theorem is proved and used to evaluate the terms of the series; this same expansion has been obtained by Abraham[7] using infinite systems of integral equations. The inverse of the plct, J, is J which implies that U can be chosen to satisfy $U^2 = I$ so that $U^{-1} = U^* = U$. In section 5 we show that below the critical temperature the periodic state $(\hat{\psi},.\hat{\psi})$ admits a non-trivial decomposition $(\hat{\psi},.\hat{\psi}) = 1/2\ (\psi_+,.\psi_+) + 1/2\ (\psi_-,.\psi_-)$ into the two translationally invariant states $(\psi_\pm,.\psi_+)$ where $\psi_\pm = \sqrt{2}\ P_\mp\ \hat{\psi} = (\hat{\psi}\pm\tilde{\psi})/\sqrt{2}$ and $P_\mp = (I+U)/2$ are orthogonal projections satisfying $P_+P_- = 0$. $P_\pm$ reduce the algebra of observables. This decomposition has also been considered in[15-17]

A proof of the convergence of the correlation function for all temperatures is given in appendix A. The operator U and vacuum vector $\tilde{\psi}$ are constructed explicitly in appendix B. In appendix C its is shown that the decomposition of the periodic state is non-trivial by showing that the magnetization is non-zero,

2. REVIEW OF THE LIEB-MATTIS-SCHULTZ SOLUTION OF THE TWO-DIMENSIONAL ISING MODEL

The partition function, $Z_{N,M}$, for the periodic Ising model with nearest neighbor spin interactions wrapped on a $N\times M$ torus is taken to be

$$Z_{N,M} = \sum_{\{\sigma\}} \exp\left[\sum_{m=1}^{M} \left(\sum_{n=1}^{N-1} K\sigma(n,m)\sigma(n+1,m) + K\sigma(N,n)\sigma(1,n)\right) + \sum_{n=1}^{N} \left(\sum_{n=1}^{M-1} K\sigma(n,m)\sigma(n,m+1) + K\sigma(n,M)\sigma(n,1)\right)\right]$$

where $K = J T^{-1} > 0$ with $J > 0$, a constant, and T the tempeirature. $\sigma(n,m)$ is the spin variable at the lattice point n,m which takes values ± 1 and $\sum_{\{\sigma\}}$ is the sum over all 2^{NM} spin configurations.

Letting

$$\tau^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2\tau^+\tau^- - I, \quad \tau^\pm = \frac{1}{2}(\tau^x \pm i\tau^y)$$

we define

$$\tau_n^i = 1 \otimes \ldots \otimes \tau^i \otimes \ldots \otimes \quad , \qquad i = x,\ y,\ z,\ +,\ - \ ,$$

where τ^i occurs in the m^{th} factor from the left. The $\tau_m^\pm$ obey a mixed set of commutation, anti-commutation relations

$$[\tau_m^\pm, \tau_n^\pm] = 0,\ m \neq n \ ; \ \{\tau_m^+, \tau_m^-\} = 1; \qquad (\tau_m^\pm)^2 = 0 \ .$$

These operators act in the 2^M dimensional Hilbert space $H_M = \prod_{m=1}^{M} \otimes H_m$ where H is the 2-dim. space generated by $\binom{1}{0}$ and $\binom{0}{1}$. In terms of $\tau_m^\pm$ the partition function can be written as the trace of an operator which we state as

Theorem 2.1 (see Huang[13]): $Z_{N,M} = \mathrm{Tr}(V_1^{1/2}\ V_2\ V_1^{1/2})^N \equiv \mathrm{Tr}\ V^N$

where

$$V_1 = (2\sinh 2K)^{M/2} \exp\left[-2K^* \sum_{m=1}^{M} (\tau_m^+ \tau_m^- - 1/2)\right]$$

and

$$V_2 = \exp\left[K \sum_{m=1}^{M} (\tau_m^+ + \tau_m^-)(\tau_{m+1}^+ + \tau_{m+1}^-)\right] , \quad \tau_{M+1}^{\pm} = \tau_1^{\pm} \quad \text{and}$$

$$\tanh K^* = e^{-2K} .$$

The critical temperature, T_c, is defined by $K^* = K$ or equivalently $\sinh (2J/T_c) = 1$.

Let $B_M = V_2^{1/2} V_1 V_2^{1/2}$ which is self-adjoint. By suitable transformations B_M can be written as

$$B_M = (2 \sinh 2K)^{M/2} e^{-H_M} .$$

H_M is identified as the finite volume Hamiltonian,

The first transformation is a Jordan Wigner transformation. Let

$$c_m = \exp\left[\pi i \sum_{j=1}^{m-1} \tau_j^+ \tau_j^-\right]\tau_m^- , \quad 1 < m \leq M ; \; c_1 = \tau_1^- ,$$

the inverse transformation is

$$\tau_m^- = \left[\exp(\pi i \sum_{j=1}^{m-1} c_j^* c_j)\right] c_m , \quad \tau_m^+ = \left[\exp(\pi i \sum_{j=1}^{m-1} c_j^* c_j)\right] c_m^* ,$$

thus

$$\tau_m^x = \left[\exp(\pi i \sum_{j=1}^{m-1} c_j^* c_j)\right] c_m^x , \quad c_m^x = c_m + c_m^* .$$

The advantage of this transformation is that the c_m obey anti- commutation relations such as $\{c_m, c_{m'}^*\} = \delta_{mm'}$, $\{c_m, c_{m'}\} = \{c_m^*, c_{m'}^*\} = 0$. The vector

$$\Omega = \binom{0}{1} \otimes \ldots \otimes \binom{0}{1} \in H_M$$

with M factors has the property $c_m \Omega = 0$ for all m. A basis for H_M is

$$\Omega \; ; \; \{c_m^* \Omega\} \; ; \; \{c_{m_1}^* c_{m_2}^* \Omega\} \; , m_1 < m_2 \; ; \; \ldots \; \{c_1^* \ldots c_M^* \Omega\} .$$

$H_{e(o)}$ denotes the subspace generated by the vectors with an even (odd) number of c_m's, the subspace corresponding to the eigenvalue 1 (-1) of the operator

$$N = \exp(\pi i \sum_{m=1}^{M} c_m^* c_m) = \exp(\pi i \sum_{m=1}^{M} \tau_m^+ \tau_m^-) = \sum_{m=1}^{M} (-\tau_m^z) \ .$$

Secondly, a linear canonical transformation is defined by

$$c_m = M^{-1/2} \ \alpha \sum_q e^{iqm} \eta_q \ , \quad \alpha = e^{-i\pi/4} \ ,$$

q $\in S^+$ or S^-, but not both, where $S^\pm$ are the M element sets

$$S^+ = \{q \equiv \ell = \pm\pi/M, \ \pm 3\pi/M, \ \ldots \pm \ (M-1)\ \pi/M \ \}$$

$$S^- = \{q \equiv k = 0, \ \pm 2\pi/M, \ \ldots \pm \ (M-2)\pi/M, \pi \ \}$$

and M is assumed to be even. Throughout this section the letter $\ell(k)$ will refer to elements of S^+ (S^-).

Finally, we make the linear canonical Valatin-Bogoliubov transformation

$$E_q = \cos \phi_4 \ \eta_q + \sin \phi_q \ \eta^*_{-q}$$

where

$$\tan \phi_4 = C_q/(e^{E_q} - A_4) = (e^{E_q} - B_4)/C_4 , \ q \in S^+ \cup S^-/\{0,\pi\},$$

or $\tan 2\phi_4 = 2C_q/(B_4 - A_4)$ and we require $\phi_q = -\phi_{-q}$, $\phi_q \geq 0$ for $q \geq 0$. For $q = 0$ we set $\phi_0 = 0$ for $T > T_c$; for $T < T_c$, $\phi_0 = \pi/2$, $\xi_0 = \eta_0^*$, and for $q = \pi$, $\phi_\pi = 0$. The choice $\phi_0 = \pi/2$ for $T < T_c$ differs from that of LMS[4] and is made to simplify the expression for H_M in Thm. I.3, i.e. $\varepsilon_0 \geq 0$. A_q, B_q, C_q, and ε_4 are given by

$$A_4 = e^{-2K^*} (\cosh K + \sinh K \cos q)^2 + e^{2K^*} (\sinh K \sin q)^2,$$

$$B_q = {}^{-2K^*} (\sinh K \sin q)^2 + e^{2K^*} (\cosh K - \sinh K \cos q)^2,$$

$$C_q = (2 \sinh K \sin q)(\cosh 2K^* \cosh K - \sinh 2K^* \sinh K \cos q),$$

$$\cosh E_4 = \cosh 2K^* \cosh 2K - \sinh 2K^* \sinh 2K \cos q, \ E_q \geq 0,$$

It is to be noted that for $T < T_c$, ϕ_q is discontinuous at $q=0$; for $T > T_c$, ϕ_q is continuous. $\Theta(e^{iq}) = e^{-2i\phi_q}$ admits an analytic continuation in the variable $z = e^{iq}$ which is

$$\Theta(z) = -z \left[\frac{(1-x_1^{-1}z)(1-x_2 z^{-1})}{(1-x_1^{-1}z^{-1})(1-x_2 z)}\right]^{1/2} \quad \text{for} \quad T < T_c ,$$

$$= \left[\frac{(1-x_1^{-1}z)(1-x_2^{-1}z)}{(1-x_1^{-1}z^{-1})(1-x_2^{-1}z^{-1})}\right]^{1/2} \quad \text{for} \quad T > T_c ,$$

where x_1 = ctnh K* ctnh K > 1, x_2 = (ctnh K/ctnh K^*) $\gtrless 1$ for $T \gtrless T_c$ and the phase angle in each factor of $\Theta(z)$ is taken to be in $(-\frac{\pi}{2},\frac{\pi}{2})$. The winding number of $\Theta(z)$ is +1(0) for $T < T_c$ ($T > T_c$). Even though c_m does not have a simple expression in terms of ξ_q, $c_m + c_m^*$ does, and we find

$$c_m + c_m^* = (\alpha \sum_q e^{iqm}\, e^{i\phi_q}\, \xi_q + \bar{\alpha} \sum_q e^{-iqm}\, e^{-i\phi_q}\, \xi_q^*)\, M^{-1/2}$$

where $q \in S^+$ or S^- but not both.

In terms of the ξ_q operators we have:

Thm. 2.3.

$$H_M = H_M^+ + H_M^- , \quad H_+(-) = H_M \restriction H_e(o)$$

$$H_M^+ = \sum_{\ell \in S^+} \varepsilon_\ell \, (\xi_\ell^* \xi_\ell - 1/2)$$

$$H_M^- = \sum_{k \in S^-} \varepsilon_k \, (\xi_k^* \xi_k - 1/2)$$

The sets $\{\xi_\ell\}_{\ell\in S^+}$ and $\{\xi_k\}_{k\in S^-}$ obey anti-commutations relations, However they are not independent; from

$$c_m = M^{-1/2}\, \alpha \sum_{\ell \in S^+} e^{i\ell m}\, \eta_\ell$$

$$= M^{-1/2}\, \alpha \sum_{k \in S^-} e^{ikm}\, \eta_k$$

upon multiplying by $\bar{\alpha}\, e^{-ik'm}$, summing over M and replacing k' with k we obtain

$$\eta_k = \bar{\alpha}\,\alpha \frac{1}{M} \sum_m e^{-ikm} \sum_R e^{i\ell m}\, \eta_\ell \quad ,$$

which upon substituting in

$$\xi_k = \cos\phi_k\, \eta_k + \sin\phi_k\, \eta^*_{-k}$$

gives

$$\xi_k = t_{1k\ell}\, \xi_\ell + t_{2k\ell}\, \xi_\ell = \frac{1}{M} \sum_{\ell \in S^+} \{ (2e^{-i(k-\ell)}/(1-e^{-i(k-\ell)}))$$

$$\cdot (\cos(\phi_k-\phi_\ell)\xi_\ell + \sin(\phi_k-\phi_\ell)\, \xi^*_{-\ell}) \} ,$$

a linear canonical transformation between $\{\xi_R\}_{\ell\in S^+}$ and $\{\xi_k\}_{k\in S^-}$. If $T > T_c$ the same relation holds with k and ℓ interchanged. But for $T<T_c$:

$$\xi_\ell = \frac{1}{M} \sum_{k\in S^-} \left(\frac{2e^{i(k-\ell)}}{1-e^{i(k-\ell)}} \right) \left[\cos(\phi_\ell-\phi_k)\xi_k + \sin(\phi_\ell-\phi_k)\xi^*_{-k}\right]$$

$$+ \frac{4}{M} \frac{e^{-i\ell}}{1-e^{-i\ell}} \cos\phi_R\, \xi^*_\delta$$

$$= s_{1\,\ell k}\, \xi_k + s_{2\,\ell k}\, \xi^*_k \,.$$

The vacuum vectors $\psi_{S^+(S^-)}$ which satisfy $5_q\, \psi_{S^+S^-} = 0$ for all $q \in S^+(S^-)$ are given by

$$\psi_{S^+} = \prod_{\ell\in S^+} (\cos\phi_\ell + \sin\phi_\ell\, \eta^*_{-\ell}\, \eta^*_\ell)\Omega$$

$$\psi_{S^-} = \prod_{0<k\in S^-<\pi} \eta^*_0(\cos\phi_k + \sin\phi_k\, \eta^*_{-k}\, \eta^*_k)\Omega \,, \quad T < T_c$$

$$= \prod_{k\in S^-} (\cos\phi_k + \sin\phi_k\, \eta^*_{-k}\, \eta^*_k)\Omega \quad , \quad T > T_c$$

so that $\psi_{S^+} \in H_e$ and $\psi_{S^-} \in H_{o(e)}$ for $T<T_c$ $(T > T_c)$.

The lowest eigenvalue of H_M^+ is $-\frac{1}{2}\sum_\ell \epsilon_\ell$ and of H_M^- is $-\frac{1}{2}\sum_k \epsilon_k$. By the unitary transformation,

$$U = \prod_{m=1}^{M} U_m, \quad U_m = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

we have

$$U\tau_m^z U^{-1} = -\tau_m^x, \quad U\tau_m^x U^{-1} = \tau_m^y, \quad U\tau_m^x U^{-1} = \tau_m^z$$

and $Ue^{-H_M}U^{-1}$ har positive matrix elements. By the Perron-Frobenius theorem the eigenspace associated with the largest eigenvalue of $Ue^{-H_M}U^{-1}$ is one-dimensional and the corresponding eigenvector, Φ', can be chosen to have strictly positive components which implies $\Phi' \in UH_e = H_e'$.

Thus, the eigenvector associated with the lowest eigenvalue of H_M is $\psi_{S^+} \in H_e$.

In terms of the ξ_q operators, the momentum operator, P_M, is given in

Thm. 2.4.[19]

$$P_M = P_M^+ \oplus P_M^-, \quad P_M^{+(-)} = P_M \quad H_{e(0)}$$

$$P_M^+ = \sum_\ell \ell\, \xi_\ell^* \xi_\ell$$

$$P_M^- = \sum_k k\, \xi_k^* \xi_k$$

and the equations

$$e^{iP_M} \tau_m^x e^{-iP_M} = \tau_{m-1}^x, \quad 2 \leq m \leq M$$

$$e^{iP_M} \tau_1^x e^{-iP_M} = \tau_M^x$$

are satisfied.

The finite volume correlation functions are given by

Thm. 2.5. If $1 \leq n_1 \leq n_2 \ldots \leq n_k \leq N$, then

$$<\sigma(n_1,m_1)\ldots\sigma(n_k,m_k)>_{N,M} =$$

$$= \frac{Tr(e^{-H_M n_1}\tau^x_{m_1} e^{-H_M(n_2-n_1)}\tau^x_{m_2}\ldots\tau^x_{m_k} e^{-H_M(N-n_k)}}{Tr(e^{-H_M N})}$$

Let $\hat{H}_M = H_M -$ inf spec H_M and $\hat{H}_M^{+(-)} = \hat{H}_M \upharpoonright H_{e(0)}$. Letting $N\to\infty$ we obtain the finite volume Feynman-Kac formula as

Corollary 2.5.1. If $1 = n_1 \leqslant \ldots \leqslant n_k$, then

$$S_{Mk} \equiv <\sigma(n_1,m_1)\ldots\sigma(n_k,m_k)>_M =$$

$$= (\psi_{S^+}, \tau^x_{m_1} e^{-\hat{H}_M(n_2-n_1)}\tau^x_{m_2}\ldots\tau^x_{m_k}\psi_{S^+})$$

$$= (\psi_{S^+}, \tau^x_1 e^{-\hat{H}^-_M(n_2-n_1)} e^{-iP^-_N(m_2-m_1)}\tau^x_1 \cdot$$

$$e^{-\hat{H}^+_M(n_3-n_2)} e^{-iP^+_M(m_3-m_2)}\ldots\tau^x_1 e^{-\hat{H}^-_M(n_k-n_{k-1})} e^{-iP^-_M(m_k-m_{k-1})}\tau^x_1\psi_{S^+})$$

Remark: By making a 1-1 correspondence between S^+ and S^- Berezin's theory of plct[12] gives a unitary U_M such that $\xi_k = U_M\xi_\ell U_M^{-1}$. Knowing the null space of t_1, U_M and U_M^{-1} are given explicitly in terms of $\{\xi_R\}$ so that the energy-momentum factors with $\hat{H}^-_M$, P^-_M can be expressed in terms of only the one set of fermions $\{\xi_\ell\}$.

A series representation for S_{Mk} is obtained by inserting the spectral representation of H and P. For $T > T_c$,

$$\xi^*_{k_1}\psi_{S^-},\ \xi^*_{k_1}\xi^*_{k_2}\xi^*_{k_3}\psi_{S^-},\ldots,\xi^*_{k_1}\ldots\xi^*_{k_{M-1}}\psi_{S^-} \quad \text{and}$$

$$\psi_{S^+},\ \xi^*_{\ell_1}\xi^*_{\ell_2}\psi_{S^+},\ \ldots,\ \xi^*_{\ell_1}\xi^*_{\ell_2}\ldots\xi^*_{\ell_M}\psi_{S^+}$$

generate H_o and H_e, respectively. Thus we have

Corollary 2.5.2.

$$S_{Mk} = \sum_{\alpha_1\beta_2\ldots\alpha_{k-1}} (\psi_{S^+}, \tau_1^x e^{-\gamma_{\alpha_1}} \chi_{\alpha_1})(\chi_{\alpha_1}, \tau_1^x e^{-\gamma_{\beta_1}} \chi_{\beta_1}) \ldots$$

$$(\chi_{\beta_{k-1}}, \tau_1^x e^{-\gamma_{\alpha_{k-1}}} \chi_{\alpha_{k-1}})(\chi_{\alpha_{k-1}}, \tau_1^x \psi_{S^+})$$

where $\gamma_{\alpha_j}(\gamma_{\beta_j})$ is the eigenvalue of $\hat{H}_M^- + iP_M^-$ $(\bar{H}_M^+ + iP_M^+)$ corresponding to the eigenvector $\chi_{\alpha_j}(\chi_{\beta_j})$ with

$$\chi_{\alpha_j} = (n!)^{-1/2} \; \xi^*_{k_1} \ldots \xi^*_{k_n} \psi_{S^-} \qquad \alpha_j = n \quad \text{odd}$$

$$\chi_{\beta_j} = (n!)^{-1/2} \; \xi^*_{\ell_1} \ldots \xi^*_{\ell_n} \psi_{S^+} \qquad \beta_j = n \quad \text{even} .$$

The $\alpha_j(\beta_j)$ are multi-indices taking odd (even) values. A similar expansion holds for $T < T_c$ where

$$\psi_{S^-}, \; \xi^*_{k_1} \xi^*_{k_2} \psi_{S^-}, \ldots, \xi^*_{k_1} \xi^*_{k_2} \ldots \xi^*_{k_M} \psi_{S^-} \quad \text{and}$$

$$\psi_{S^+}, \; C^*_{\ell_1} \xi^*_{\ell_2} \psi_{S^+}, \ldots, \xi^*_{\ell_1} \ldots C^*_{\ell_M} \psi_{S^+}$$

generate H_o and H_e, respectively.

Remarks: 1. As noted by Abraham[7] by using the expression for τ_1^x in terms of the ξ_q's and the linear relations $\xi_\ell = s_{1\ell k}\,\xi_k + s_{2\ell k}\,\xi^*_k$, $\xi_k = t_{1k\ell}\,\xi_\ell + t_{2k\ell}\,\xi^*_\ell$, Corollary 2.5.2 reduces the determination of the correlation functions to the determination of a generic matrix element

$$(\psi_{S^-}, \xi_{k_1} \ldots \xi_{k_n} \psi_{S^+})$$

or

$$(\psi_{S^-}, \xi^*_{\ell_1} \ldots \xi^*_{\ell_n} \psi_{S^+})$$

since $\xi_\ell \psi_{S^+} = 0$ and $\xi_k \psi_{S^-} = 0$.

2. By Berezin's[12] theory of plct if we know the null space of t_1 or s_1 then we have explicit formulas for ψ_{S^-} and terms of $\{\xi_\ell\}$ and ψ_{S^+} and the above matrix elements can be obtained by applying Wick's theorem.

This can also be done in the infinite lattice case but in[18] a generalization of Wick's theorem is proved and used to obtain the matrix elements.

3. CONSTRUCTION OF INFINITE VOLUME OPERATORS AND THEIR PROPERTIES

In this section we define infinite volume analogs of the operators ξ_k, ξ_ℓ, η_k, η_ℓ, $H_M^\pm$, $P_M^\pm$, c_m τ_m^x of section 2 andstudy their properties. The theory of proper (improper) linear canonical transformations plct (ilct) as developed in Berezin[12] enters in an essential way. For relevant definitions, results and proofs we refer the reader to [12]. In section 4 these operators will be used to define the infinite volume correlation functions as a Feynmann-Kac foi-mula.

Let H be the anti-symmetric Fock space over $L^2(-\pi,\pi)\equiv L$ and let $\hat{\xi}(k)$, $\hat{\xi}^*(k)$ be the usual Fock annihilation and creation operator--valued distributions satisfying the canonical anti- commutation relations

$$\{\hat{\xi}(k),\ \hat{\xi}(k')\} = \{\hat{\xi}^*(k),\ \hat{\xi}^*(k')\} = 0, \quad \{\hat{\xi}(k),\ \hat{\xi}^*(k')\} = \delta(k-k')$$

and let $\hat{\psi}$ denote the vacuum vector, i.e. $\hat{\xi}(f)\hat{\psi}$ = for all $f\in L$.

We define the Hilbert transform, H, as the $L^2(-\pi,\pi)$ closure of H_c where

$$f)(k) = \pi^{-1}\ P\int_{-\pi}^{\pi} (1-e^{i(k-q)})^{-1} f(q)\,dq\ ,$$

$f\in C\ [-\pi,\pi]$. H satisfies $|H| = 1$ and $H^2 = I$. For any bounded operator on $(-\pi,\pi)$ define $\bar{A}$ by $(\bar{A}f)(k) = (A\bar{f})(k)$ and $A' \equiv \bar{A}^* = \overline{A^*}$. Thus $H^* = H$. Also define the operator V by $(Vf)(k) = f(-k)$. Thus $\bar{V} = V = V^* = V'$.

$\tilde{\xi}(q)$ and $\tilde{\xi}^*(q)$, $q \in [-\pi,\pi]$, the analogs of ξ_k and ξ_k^*, are defined by

$$\tilde{\xi}(f) = \hat{\xi}(T_1 f) + \hat{\xi}^*(T_2 f)\ , \tag{2.1a}$$

$$\tilde{\xi}^*(f) = \hat{\xi}(T_2 f) + \hat{\xi}^*(\bar{T}_1 f)\ , \tag{2.1b}$$

$f \in L^2$; T_1, T_2 are the L closures of T_{1c}, T_{2c} where

$$(T_{1c}f)(k) = \pi^{-1} P \int_{-\pi}^{\pi} (1-e^{i(k-q)})^{-1} \cos(\phi_q-\phi_k) f(q)\,dq \qquad (2.2)$$

$$(T_{2c}f)(k) = \pi^{-1} P \int_{-\pi}^{\pi} (1-e^{i(k+q)})^{-1} \sin(\phi_q+\phi_k) f(q)\,dq \ , \qquad (2.3)$$

$f \in C\,[-\pi,\pi]$, and P denotes the principal value. Some of the properties of T_1 and T_2 are given by

Thm. 3.1,

i) $T_1 = \cos\phi H \cos\phi + \sin\phi H \sin\phi = 2^{-1}(e^{i\phi}He^{-i\phi} + e^{-i\phi}He^{i\phi})$

ii) $T_2 = \cos\phi VH \sin\phi + \sin\phi VH \cos\phi = (2i)^{-1}\ (e^{i\phi}VHe^{i\phi} - e^{-i\phi}VHe^{-i\phi})$

iii) $T_1' = \bar{T}_1$, $T_1 = T_1^*$, $T_1 \neq \bar{T}_1$

iv) $T_2' = T_2 = \bar{T}_2^*$, $T_2 \neq \bar{T}_2$

v) $T_1'T_2 + T_2'T_1 = 0$

vi) $T_1'\bar{T}_1 + T_2'\bar{T}_2 = I$

vii) dim ker T_1 = 1(0) for $T < T_c$ $(T > T_c)$
= index $O(z)$.

Prf, of Thm 3.1:

i) - iv) follow from the definitions of T_1 and T_2.

v) is proved by direct calculation. We have

$$\begin{aligned} T_1'T_2 + T_2'T_1 &= (\cos\phi VHV\cos\phi + \sin\phi VHV\sin\phi)\ . \\ &\quad (\cos\phi VH\sin\phi + \sin\phi VH\cos\phi) + \\ &\quad (\cos\phi VH\sin\phi + \sin\phi VH\cos\phi)\ . \\ &\quad (\cos\phi H\cos\phi + \sin\phi H\sin\phi) = \\ &= \cos\phi VH(V\cos^2\phi\, V + \sin^2\phi)H\sin\phi\ + \\ &\quad \sin\phi VH(V\sin^2\phi V + \cos^2\phi)H\cos\phi\ + \\ &\quad \cos\phi VH(V\cos^2\sin\phi V + \sin\phi\cos\phi)H\cos\phi\ + \\ &\quad \sin\phi VH(V\sin^2\cos\phi V + \cos\phi\sin\phi)H\sin\phi \end{aligned}$$

The last two terms vanish since $\sin\phi\cos\phi$ is odd. Since $\sin^2\phi$ and $\cos^2\phi$ are even and $H^2 = I$ the right side becomes

$$\cos\phi V \sin\phi + \sin\phi V \cos\phi = V(\cos\phi\sin\phi - \sin\phi\cos\phi) = 0 .$$

vi) is proved in a similar way. vii) is proved by observing $T_1 f = 0$ iff $H\Theta g + \Theta H g = 0$ where $g = e^{i\phi}f$ and $\Theta = e^{-2i\phi}$. Abraham has shown that the kernel of $H\Theta + \Theta H$ obeys the statement of vii).

Concerning the operators $\tilde{\xi}(f)$, $\tilde{\xi}^*(f)$ defined by eq. (2.1a,b) we have

Thm. 3.2,

i) $\tilde{\xi}(f)$, $\tilde{\xi}^*(f)$ satisfy CACR.

ii) eqs. (2.1a,b) hold with ~ and ^ interchanged.

iii) the transformation (2.1a,b) is a plct, i.e. there exists a unique (up to a constant λ, $|\lambda| = 1$) unitary U such that for all $f \in L$

$$\tilde{\xi}(f) = U\,\hat{\xi}(f)U^{-1} \quad , \quad \tilde{\xi}^*(f) = U\,\hat{\xi}^*(f)U^{-1}$$

and $\tilde{\psi} = U\hat{\psi}$ is the vacuum vector for $\tilde{\xi}(f)$, i.e. $\tilde{\xi}(f)\ \tilde{\psi} = 0$ for all $f \in L$.

iv) the U of iii) satisfies $U^2 = \alpha I$, $|\alpha| = 1$. U and $\tilde{\psi}$ can be redefined so that $U = U^*$ and $\tilde{\psi} = U\hat{\psi}$.

Remarks:

1. In the sequel we take $U = U^*$; this choice plays an important role in the 'one-Fermion' Schwinger functions (see Theorem 4.2) and in the decomposition of the periodic state in section 5.

2. Formulas for U and $\tilde{\psi}$ are given in Appendix B in terms of $\hat{\xi}(k)$, $\hat{\xi}^*(k)$ and $\hat{\psi}$.

Prf. of Thm. 3.2: i) Follows from v) and vi) of Thm. 3.1. ii). We see that

$$\tilde{\xi}(T_1 f) = \hat{\xi}(T_1{}^2 f) + \hat{\xi}^*(T_2 T_1 f) ,$$

$$\tilde{\xi}^*(T_2 f) = \hat{\xi}(\bar{T}_2 T_2 f) + \hat{\xi}^*(\bar{T}_1 T_2 f) .$$

From Thm, 3.1 vi), $T_1'\bar{T}_1 + T_2'\bar{T}_2 = I$ or $\bar{T}_1^{\,2} + T_2\bar{T}_2 = I$ (since $T_1' = \bar{T}_1$ and $T_2' = T_2$). Thus $T_1^{\,2} + \bar{T}_2 T_2 = I$. From Thm. 3.1 v), $T_2'T_1 + T_1'T_2 = 0$ or $T_2T_1 + \bar{T}_1T_2 = 0$. Thus

$$\tilde{\xi}(T_1f) + \tilde{\xi}^*(T_2f) = \hat{\xi}((T_1^2 + \bar{T}_2T_2)f) + \hat{\xi}^*((T_2T_1 + \bar{T}_1T_2)f) = \hat{\xi}(f).$$

Similarly one shows $\hat{\xi}^*(f) = \tilde{\xi}(\bar{T}_2f) + \tilde{\xi}^*(\bar{T}_1f)$.

iii) From[12], i) and ii) it is sufficient to show that T_1 has property A and that T_2 is Hilbert-Schmidt. As T_1 is self-adjoint it has property A. $T_2(k,q)$, the kernel of T_2, is given by

$$T_2(k,q) = \pi^{-1}\sin(\phi_q+\phi_k)(1-e^{-i(k+q)})^{-1}$$

Since $T_2(k,q)$ is bounded and piecewise continuous on $[-\pi,\pi] \times [-\pi,\pi]$ it is in $L^2((-\pi,\pi) \times (-\pi,\pi))$, thus it is Hilbert-Schmidt.

iv) By ii) and iii):

$$U\tilde{\xi}(f)U^{-1} = U(\hat{\xi}(T_1f) + \hat{\xi}^*(T_2f))U^{-1} =$$

$$= \tilde{\xi}(T_1f) + \tilde{\xi}^*(T_2f) = \hat{\xi}(f) = U^{-1}\tilde{\xi}(f)U$$

or $U^2\tilde{\xi}(f) = \tilde{\xi}(f)^2U$ for all $f\in L$. Similarly $U^2\tilde{\xi}^*(f) = \tilde{\xi}^*(f)U^2$. Thus U^2 comutes with all $\tilde{\xi}(f)$, $\tilde{\xi}^*(f)$ which (see ref.12) implies $U^2 = \lambda I$, $|\lambda|=1$. This result also follows abstractly from the following group property; associate the matrix

$$J = \begin{pmatrix} T_1' & T_2' \\ T_2^* & T_1^* \end{pmatrix}$$

with the transformation of eqs. (2.1a,b). The lct form a group (successive transformations corresponding to matrix multiplication of the associated matrices) and the inverse transformation has the matrix

$$J^{-1} = \begin{pmatrix} (T_1')^* & (T_2')' \\ (T_2^*)' & (T_1^*)^* \end{pmatrix}$$

By Thm. 3.1 iii) and iv) $T_1^* = T_1$ and $T_2' = T_2$ so that $J^{-1} = J$ or $JJ = I$. As the unitaries which implement the transformations form a ray representation of the lct group we have $U^2 = AI$, $|\lambda| = 1$.

It is always possible to redefine U and $\tilde{\psi} = U\hat{\psi}$ such that $U^2 = I$. Indeed, for the new U take the operetor $4 = \mu U$ with $\mu^2 = \lambda^{-1}$ ($|\mu|=1$) and for the new $\tilde{\psi}$ the vector $\tilde{\psi}_n = \mu\tilde{\psi}$. Thus $U_n^2 = I$, $V_n = U_n^*$ and $\tilde{\psi}_n = V_n\hat{\psi}$.

Let $\hat{H}_{e(o)}$ be the subspace of H with an even (odd) number of $\hat{\xi}$-particles. More precisely $\hat{H}_e(\hat{H}_o)$ is the eigenspace of $\exp(\pi i N)$ associated with the eigenvalue +I(-1) where

$$\hat{N} = \int_{-\pi}^{\pi} \hat{\xi}^*(k)\hat{\xi}(k)dk \quad .$$

Similarly define $\tilde{H}_{e(o)}$. We have

Thm. 3.3.

$$\tilde{H}_e = \hat{H}_o \quad , \quad \tilde{H}_o = \hat{H}_e \quad \text{for} \quad T < T_c \quad \text{and}$$

$$H_e = H_e \quad , \quad \tilde{H}_o = \hat{H}_o \quad \text{for} \quad T > T_c \quad .$$

Prf. of Thm. 3.3. Follows from the form of $\tilde{\psi}$ given in Appendix B.

In analogy with the finite lattice case we define infinite volume energy and momentum operators H and P, respectively, by

$$H \upharpoonright \hat{H}_e = \int \varepsilon(k)\hat{\xi}^*(k)\hat{\xi}(k)dk \upharpoonright \hat{H}_e$$

$$H \upharpoonright \tilde{H}_e = \int \varepsilon(k)\tilde{\xi}^*(q)\tilde{\xi}(q)dq \upharpoonright \tilde{H}_e$$

$$P \upharpoonright \hat{H}_e = \int k \; \hat{\xi}^*(k)\hat{\xi}(k)dk \upharpoonright \hat{H}_e$$

$$P \upharpoonright \tilde{H}_e = \int q \; \tilde{\xi}^*(q)\tilde{\xi}(q)dq \upharpoonright \tilde{H}_e$$

for $T < T_c$ and for $T > T_c$

$$H \upharpoonright \hat{H}_e = \int \varepsilon(k)\hat{\xi}^*(k)\hat{\xi}(k)dk \upharpoonright \hat{H}_e$$

$$H \upharpoonright \tilde{H}_o = \int \varepsilon(q)\tilde{\xi}^*(q)\tilde{\xi}(q)dq \upharpoonright \tilde{H}_o$$

$$P \upharpoonright \hat{H}_e = \int k \; \hat{\xi}^*(k)\hat{\xi}(k)dk \upharpoonright \hat{H}_e$$

$$P \upharpoonright \tilde{H}_o = \int q \; \tilde{\xi}^*(q)\tilde{\xi}(q)dq \upharpoonright \tilde{H}_o$$

It is interesting to note

Thm. 3.4. Let

$$\hat{\eta}(q) = \cos\phi_q \; \hat{\xi}(q) - \sin\phi_q \; \hat{\xi}^*(-q) \; ,$$

$$\tilde{\eta}(q) = \cos\phi_q \; \xi(q) - \sin\phi_q \; \tilde{\xi}^*(-q) \; .$$

These transformations are improper linear canonical transformations, i. e. there does not exist a unitary U which implements the transformation. Furthermore $\tilde{\eta}(f) = \hat{\eta}(Hf)$.

Prf. of Thm 3.4: Write $\hat{\eta}(f) = \hat{\xi}(f_1) + \hat{\xi}^*(f_2)$ for $f \varepsilon L^2$ and

$$f_1(k) \equiv (S_1 f)(k) = \cos\phi_k \; f(k)$$

$$f_2(k) \equiv (S_2 f)(k) = \sin\phi_k \; f(-k) \; .$$

By direct calculation we find

$$S_1' S_2 + S_2' S_1 = 0$$

$$S_1' \bar{S}_1 + S_2' \bar{S}_2 = I$$

and as in the proof of Thm. 3.3 the lct is invertible,

$$\hat{\xi}(f) = \hat{\eta}(S_1 f) - \hat{\eta}^*(S_2 f) \; ,$$

and $(\hat{\eta}(k), \hat{\eta}^*(k))$ satisfy canonical anti-commutation relations. The transformation is improper, i.e. not proper, since S_2 is not a Hilbert -Schmidt operator.

$\tilde{\eta}(f) = \hat{\eta}(Hf)$ follows from the explicit evaluation of

$$\begin{aligned} \tilde{\eta}(f) &= \tilde{\xi}(S_1 f) + \tilde{\xi}^*(S_2 f) \\ &= \hat{\xi}((T_1 S_1 + \bar{T}_2 S_2) f) + \hat{\xi}^*((T_2 S_1 + \bar{T}_1 S_2) f) \\ &= \hat{\eta}((S_1 T_1 S_1 + S_1 \bar{T}_2 S_2 - \bar{S}_2 T_2 S_1 - \bar{S}_2 \bar{T}_1 S_2) f) + \\ &\quad \hat{\eta}^*((\bar{S}_1 T_2 S_1 + \bar{S}_1 \bar{T}_1 S_2 - S_2 T_1 S_1 - S_2 \bar{T}_2 S_2) f) \; . \end{aligned}$$

We now define

$$\hat{c}_m = \frac{\alpha}{2\pi} \int e^{iq(m-1)} \; \hat{\eta}(q) \, dq \; , \qquad m \geq 1 \qquad \text{and}$$

$$\tilde{c}_m = \frac{\alpha}{2\pi}\int e^{iq(m-1)}\,\tilde{\eta}(q)dq, \quad m \geq 1, \quad \alpha = e^{-i\pi/4}.$$

We have

Thm. 3.5. $\tilde{c}_m = \hat{c}_m$, $m \geq 1$.

Prf. of Thm. 3.5: Follows from $H\,e^{iqm} = (\mathrm{sgn}\ m)e^{iqm}$. Since there is no distinction between $\tilde{c}_m$ and $\hat{c}_m$ we denote these operators by c_m. We define the field operators, σ^x_m, by

$$\sigma^x_1 = c^x_1 = \frac{1}{2\pi}\int\left[\alpha e^{i\phi_q}\,\hat{\xi}(q) + \bar{\alpha}\,e^{-i\phi_q}\,\hat{\xi}^*(q)\right]dq$$

and

$$\sigma^x_m = e^{-iPm}\,\sigma^x_m\,e^{iPm}$$

where $c_m = c_m + c^*_m$.

We believe, but have been unable to prove that linear combinations of vectors of the forms

$$(\prod_{i=1}^{N} e^{-H|n_i|}\,e^{iPm_i}\,\sigma^x_1)\hat{\psi}$$

are dense in H. Assuming this result, we have

Thm. 3.6: $[\sigma^x_m, \sigma^x_n] = 0$.

Proof: It is sufficient to show

$$((\prod_{i=1}^{N} e^{-H|n_i|}\,e^{iPm_i}\,\sigma^x_1)\hat{\psi},\ \sigma^x_m\,\sigma^x_n\,(\prod_{j=1}^{N'} e^{-H|n'_j|}\,e^{iPm'_j}\,\sigma^x_1)\hat{\psi}))$$

is symmetric in m, n. But this follows from the Feynmann-Kac formula (whose proof will be given in sec. 4) which relates the above inner-product to an $N+N'+2$ - point correlation function of the periodic Ising model.

4. INFINITE VOLUME SCHWINGER FUNCTIONS AND THEIR REPRESENTATION

In terms of the vectors and operators introduced in section 3, define the k-point Schwinger function, S_k, as

$$S_k = (\hat{\psi}, \sigma_1^x e^{-H(n_2-n_1)} e^{-iP(m_2-m_1)} \sigma_1^x \ldots$$
$$e^{-H(n_k-n_{k-1})} e^{-iP(m_k-m_{k-1})} \sigma_1^x \hat{\psi}) \quad (4.1)$$

where $n_1 \leq n_2 \ldots \leq n_k$. We have

Thm. 4.1, S_k of eq. (4.1) is indeed the k-point correlation function of the periodic Ising model, i.e.

$$S_k = \lim_{M\to\infty} (S_{Mk} = \langle\sigma(n_1,m_1)\ldots\sigma(n_k,m_k)\rangle_M) \equiv \langle\sigma(n_1,m_1)\ldots\sigma(n_k,m_k)\rangle.$$

We defer the proof of Thm. 4.1 to the end of this section.

From the definition of H and P in section 3 we see that the $\hat{}$ or $\tilde{}$ operators occur in H or P depending on the parity of the vector on which it acts. However, S_k of eq. (4.1) can be expressed in terms of the $\hat{}$ operators only as seen in

Thm. 4.2. Let

$$\hat{H}(\hat{P}) = \int_{-\pi}^{\pi} \varepsilon(k) \quad (k) \quad \hat{\xi}^*(k)\hat{\xi}(k)dk \text{ and } \sigma_0 = \sigma_1^x U, \quad \sigma_1^x$$

and U being expressed in terms of the $\hat{}$ operators. Then $\sigma_0 = U^{-1}\sigma_1^x$ and

$$S_k = (\hat{\psi}, \sigma_0 e^{-\hat{H}(n_2-n_1)} e^{-i\hat{P}(m_2-m_1)} \sigma_0 \ldots$$
$$e^{-\hat{H}(n_k-n_{k-1})} e^{-i\hat{P}(m_k-m_{k-1})} \sigma_0 \hat{\psi}) .$$

Proof of Thm. 4.2. Let $\tilde{H}(\tilde{P})$

$$\tilde{H}(\tilde{P}) = \int_{-\pi}^{\pi} \varepsilon(k) \quad (k)\tilde{\xi}^*(k)\tilde{\xi}(k)dk \quad .$$

Then eq. (4.1) can be written as

$$S_k = (\hat{\psi},\sigma_1^x e^{-\tilde{H}(n_2-n_1)} e^{-i\tilde{P}(m_2-m_1)} \sigma_1^x e^{-\hat{H}(n_3-n_2)} e^{-i\hat{P}(m_3-m_2)}$$
$$\ldots e^{-\tilde{H}(n_k-n_{k-1})} e^{-i\tilde{P}(m_k-m_{k-1})} \sigma_1^x \hat{\psi}) .$$

Substituting $\tilde{}$ operators by $\hat{}$ operators according to Thm. 3.2 iii gives

$$S_k = (\hat{\psi},\sigma_1^x U e^{-\hat{H}(n_2-n_1)} e^{-i\hat{P}(m_2-m_1)} U^{-1} \sigma_1^x e^{-\hat{H}(n_3-n_2)} e^{-i\hat{P}(m_3-m_2)}$$

$$\ldots U e^{-i\hat{H}(n_k-n_{k-1})} e^{-i\hat{P}(m_k-m_{k-1})} U^{-1} \sigma_1^x \hat{\psi}) \ .$$

The theorem follows since $U^{-1} \sigma_1^x = \sigma_1^x$ using Thm. 3.5 and $U = U^* = U^{-1}$.

Remark: If we are considering vacuum expectation values of Heisenberg operators (replace n_j by $-it_j$) then the energy-momentum operators can be incorporated in $\sigma_1^x U$ by using the relation

$$e^{-iF} \hat{\xi}(k) e^{iF} = e^{if(k)}\hat{\xi}(k)$$

and its adjoint where

$$F = \int_{-\pi}^{\pi} f(q) \ \hat{\xi}^*(q)\hat{\xi}(q)dq,$$

f real.

An infinite series representation for S_k is obtained by inserting the spectral representation of H and P in eq. (4.1). We have

Thm. 4.3, For $T > T_c$

$$\tilde{\xi}^*(f_i)\tilde{\psi},\ldots, \ \tilde{\xi}^*(f_{i_1})\ldots \tilde{\xi}^*(f_{i_n})\tilde{\psi},\ldots \ n \text{ odd}$$

and

$$\hat{\psi},\ldots, \ \hat{\xi}^*(f_{i_1})\ldots \hat{\xi}^*(f_{i_n})\hat{\psi},\ldots \ n \ \text{ even}$$

generate $\hat{H}_o$ and $\hat{H}_e$, respectively, where $\{f_i\}$ is a complete orthonormal set in L. Thus

$$S_k = \sum_{\alpha_1\beta_2\ldots\beta_{k-2}\alpha_{k-1}} \int dq^{\alpha_1} \int dq^{\beta_1}\ldots \int dq^{\alpha_{k-1}} (\hat{\psi}, \sigma_1^x e^{-\gamma_{\alpha_1}} \chi_{\alpha_1})$$

$$(\chi_{\alpha_1},\sigma_1^x e^{-\gamma_{\beta_1}} \chi_{\beta_1}) \ \ldots \ (\chi_{\beta_{k-2}}, \sigma_1^x e^{-\gamma_{\alpha_{k-1}}} \chi_{\alpha_{k-1}})(\chi_{\alpha_{k-1}},\sigma_1^x \hat{\psi})$$

where

$$\gamma_{\alpha_j}(\gamma_{\beta_j}) = (\sum_{i=1}^{n} \varepsilon(q_i))(n_{j+1}-n_j) + i(\sum_{i=1}^{n} q_i)(m_{j+1}-m_j) \quad \text{for } \alpha_j = n \ (\beta_j=n)$$

and

$$\chi_{\alpha_j} = (\alpha_j!)^{-1/2}\ \tilde{\xi}^*(q_1)\ldots\tilde{\xi}^*(q_{\alpha_j})\tilde{\psi}\ ,\ \alpha_j \ \text{odd}\ ,$$

$$\chi_{\beta_j} = (\beta_j!)^{-1/2}\ \hat{\xi}^*(q_1)\ldots\hat{\xi}^*(q_{\beta_j})\hat{\psi}\ ,\ \beta_j \ \text{even}\ ,$$

and the integral $\int dq^{\alpha}$ $(\int dq^{\beta})$ extends over the α (β) q variables occuring in χ_{α} and χ_{β}.

Similar results hold for $T < T_c$ where

$$\tilde{\psi},\ \ldots\ ,\ \tilde{\xi}^*(f_{i_1})\ \ldots\ \tilde{\xi}^*(f_{i_n})\ \tilde{\psi},\ \ldots n \ \text{even}$$

and

$$\hat{\psi},\ \ldots\ ,\ \hat{\xi}^*(f_{i_2})\ \ldots\ \hat{\xi}^*(f_{i_1})\ldots\hat{\xi}^*(f_{i_n})\hat{\psi},\ldots n \ \text{even}$$

generate $\hat{H}_o$ and $\hat{H}_e$, respectively.

Remark: Using the explicit form of $\tilde{\psi}$ in terms of $\hat{\xi}$, $\hat{\xi}^*$ and $\hat{\psi}$ the above matrix can in principle be evaluated by applying Wick's theorem. However in ref.18 a generalization of Wick's theorem is proved and used to evaluate $(\chi_{\alpha}, \sigma_1^x \chi_{\beta})$.

Proof of Thm. 4.1: The proof is given for the two-point function. A similar argument holds for the $k \geq 2$ point functions, We begin by showing that both systems of point functions can be expressed in terms of an infinite set

$$\{G(k_1,\ldots,k_n)\}_{n=0}^{\infty}\ ,\ \{F(k_1,\ldots,k_n)\}_{n=0}^{\infty}\ ;$$

the first set for the Schwinger functions, the second for the correlation functions. The functions within each set will be shown to satisfy a set of coupled integral equations identical in form to those described in[7]. This will imply that the two expectations, call them $\langle\cdot\rangle_1$ and $\langle\cdot\rangle_2$ are related by $\langle\cdot\rangle_1 = c\langle\cdot\rangle_2$; c = 1 follows from $\langle\sigma_i\sigma_i\rangle_1 = \langle\sigma_i\sigma_i\rangle_2 = 1$.

We establish a useful representation of the two-point function of the periodic state as in[7]. From Cor. 2.5.1, assuming $n_2 \geq n_1$, and inserting a complete set of energy-momentum eigenvectors in S_{M2} we have

(a) $T < T_c$

$$S_{M2} = \sum_{n \text{ even}} \sum_{k_i \in S^-} \frac{1}{n!} \left| (\tau_1^x \psi_{S^+}, \xi^*_{k_1} \ldots \xi^*_{k_n} \psi_{S^-}) \right|^2 .$$

$$\exp\{-(\sum_{j=1}^{n} \varepsilon(k_j) + \delta_M)(n_2-n_1) - i(\sum_{j=1}^{n} k_j)(m_2-m_1)$$

(b) $T > T_c$

S_{M2} = (same as above with n odd).

In these formulas,

$$\delta_M = \frac{1}{2}\left[\sum_{\ell \in S^+} \varepsilon_\ell - \sum_{k \in S^-} \varepsilon_k\right] \to 0 \quad \text{as } M \to \infty.$$

The last assertion follows from

$$\lim_{M \to \infty} \left(\sum_{\ell \in S^+ (k \in S^-)} \varepsilon_{\ell(k)} - \frac{M}{2\pi} \int_{-\pi}^{\pi} \varepsilon(q)\, dq \right) = 0 .$$

Let, as in[7],

$$F_M^x((k)_{1,n}) = M^{n/2} e^{-in\pi/4} \exp(i \sum_{j=1}^{n} (k_j - \phi_{k_j}))(\psi_{S^-}, \xi_{k_n} \ldots \xi_{k_1} \tau_1^x \psi_{S^+})$$

and

$$F_M((k)_{1,n}) = M^{n/2} e^{-in\pi/4} \exp(i \sum_{j=1}^{n} (k_j - \phi_{k_j}))(\psi_{S^-}, \xi_{k_n} \ldots \xi_{k_1} \psi_{S^+}).$$

Following Abraham[7] we use the notation $\Delta_{i_1 \ldots i_\ell}(k)_{1,n}$ to mean the set

$$\{k_1, \ldots, k_n\} / \{k_{i_1}, \ldots, k_{i_\ell}\} .$$

Theorem 4.4. Ref.7 If $\theta(e^{ik}) = e^{2i\phi_k}$ then

$$F_M^x((k)_{1,n}) = \frac{1}{M} \sum_{k_0 \in S^-} \theta(e^{ik_0}) \, F_M((k)_{0,n}) \; + \sum_{j=1}^{n} (-1)^{j-1} \, \theta(e^{ik_j})^{-1} \, F_M(\Delta_j(k)_{1,n})$$

Prf: Follows immediately from the representation

$$T_1^x = M^{-1/2} \sum_{k \in S^-} \left[e^{-i\pi/4} \, e^{i(k+\phi_k)} \xi_k \; + e^{i\pi/4} \, e^{-i(k+\phi_k)} \; \xi_k^* \right].$$

Theorem 4.5. Let

$$h(k,1) = 2(1-e^{-i(k+\ell)})^{-1} \; (\theta(e^{i\ell})^{-1}(\theta(e^{ik})^{-1} - 1) \; .$$

Then

$$2M^{-1} \sum_{k_0 \in S^-} F_M((k)_{0,n})(e^{i(k_0-\ell)} -1)^{-1} \; (1 + \theta(e^{ik_0})\theta(e^{i\ell})^{-1}) =$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \; h(k_j, \ell) \; F_M(\Delta_j(k)_{1,n}) \; .$$

Prf: From

$$(\psi_{S^-}, \xi_{k_n}, \ldots, \xi_{k_1} \xi_\ell \psi_{S^+}) = 0$$

together with the relation ($T < T_c$)

$$\xi_\ell = S_{1\ell k}\xi_k + S_{2\ell k}\xi_k^*$$

we have

$$0 = \sum_{k_0 \in S^-} 2M^{-1} \, e^{i(k_0-\ell)} \, (1-e^{i(k_0-\ell)})^{-1} \; \cos(\phi_\ell - \phi_{k_0})(\psi_{S^-}, \xi_{k_n} \ldots \xi_{k_0}\psi_{S^+}) \; +$$

$$+ \sum_{k_0 \in S^-} 2M^{-1} \, e^{-i(k_0+\ell)} \, (1-e^{-i(k_0+\ell)}) \; \sin(\phi_\ell + \phi_{k_0})(\psi_{S^-}, \xi_{k_n} \ldots \xi_{k_1} \xi_{k_0}^* \psi_{S^+}) \, .$$

After eliminating $\xi_{k_0}^*$ in the second term using anti-commutation the result follows by direct substitution. For $T > T_c$ the relation between ξ_ℓ

and ξ_k does not contain the last term involving ξ_0^* but still the above formula is obtained.

It has been argued in[7] that the functions F_M and F_M^x have limiting values F and F^x, respectively, as $M\to\infty$, satisfying

$$F^x((k)_{1,n}) = (2\pi)^{-1}\int dk_0\ \theta(e^{ik_0})F((k)_{0,n}) +$$

$$+\sum_{j=1}^{n}(-1)^{j-1}\ \theta(e^{ik_j})^{-1}\ F(\Delta_j(k)_{1,n}),\quad n\geq 1 \qquad (4.2)$$

and

$$P\frac{1}{\pi}\int dk_0\ F((k)_{0,n})(e^{i(k_0-\ell)}-1)^{-1}\ (1+\theta(e^{ik_0})\theta(e^{i\ell})^{-1}) =$$

$$=\sum_{j=1}^{n}(-1)^{j-1}\ h(k_j,\ell)F(\Delta_j(k)_{1,n}),\quad n\geq 1\ . \qquad (4.3)$$

In addition, the two-point function can be written as

(a) $T < T_c$

$$\langle\sigma(n_1,m_1)\sigma(n_2,m_2)\rangle \equiv \lim_{M\to\infty} S_{M2} =$$

$$=\sum_{n\ \text{even}}(2\pi n!)^{-n}\int\dots\int|F^x((k)_{1,n})|^2\exp\{-(n_2-n_1).$$

$$(\sum_{j=1}^{n}\varepsilon(k_j)) - i(m_2-m_1)(\sum_{j=1}^{n}k_j)\}\ dk_1\dots dk_n\ . \qquad (4.4)$$

(b) T > T_c, the sum on the RHS above is now over odd n's.

Since we have not been able to confirm the argument in ref.7 a proof of the above facts is presented in Appendix A.

Remarks: 1. Assume $T<T_c$. The n=O term in (4.4) is $|F^x(\phi)|^2$, where $F^x(\phi) = \lim_{M\uparrow\infty}(\psi_S-,\ \tau_1^x\ \psi_S+) \equiv m^*$. Eq, (4.2) is replaced by

$$(2\pi)^{-1}\int dk_0\ \theta(e^{ik_0})F(k_0) = m^* \qquad (4.5)$$

The functions $F^x((k)_{1,2n})$ for $m \geq 1$ are determined through (4.2) once we know the set $\{F((k)_{1,n})\}$ for odd n. These obey the systern

$$YF(\cdot,(k)_{1,n}) = \sum_{j=1}^{n} (-1)^{j-1} h(k_j,\cdot)F(\Delta_j(k)_{1,n})$$

where Y: $L^2(-\pi,\pi) \to L^2(-\pi,\pi)$ is the bounded operator given by

$$(Yf)(q) = P\frac{1}{\pi}\int dk\; f(k)(e^{i(k-q)}-1)^{-1}\;(1+\theta(e^{ik})\theta(e^{iq})^{-1}). \qquad (4.6)$$

It has heen shown in[7] that dimker $Y = 1$ if $T < T_c$ and $= 0$ if $T > T_C$. In addition, it has been proved that the system (4.3) has a unique anti-symetric solution once $F(k)$ is given. This function satisfies $YF=0$ and therefore is therefore is uniquely specified except for a constant, which is then fixed by (4.5).

2. $T > T_c$. In this case, $<\sigma(n_1,m_1)\sigma(n_2,m_2)>$ is determined from $\{F^x((k)_{1,n})\}$, n odd, or $\{F((k)_{1,n})\}$ with even n. Since now dim ker $Y=0$, these functions are uniquely specified onte $F(\phi)$ is given. It is easy to show that the appropriate value is $F(\phi) = \lim_{M\uparrow\infty}(\psi_S-,\psi_S+)$.

To show that

$$S_2 = <\sigma(n_1,m_1)\sigma(n_2,m_2)>$$

we proceed as follows:

(a) $T < T_c$. Inserting a complete set of energy-momentum eigenvectors

$$S_2 = |(\tilde{\psi},\sigma_1^x\hat{\psi})|^2 + \frac{1}{2!}\int\int dk_1dk_2\; e^{-(\varepsilon(k_1)+\varepsilon(k_2))(n_2-n_1)}\,e^{-i(k_1+k_2)(m_2-m_1)}$$

$$\cdot\;|(\tilde{\psi},\tilde{\xi}(k_1)\tilde{\xi}(k_2)\,\sigma_1^x\,\hat{\psi})|^2 + \ldots$$

(b) $T > T_c$

$$S_2 = \int dk\; e^{-\varepsilon(k)(n_2-n_1)}\,e^{-ik(m_2-m_1)}\,|(\tilde{\psi},\tilde{\xi}(k)\sigma_1^x\,\hat{\psi})|^2 + \ldots$$

Let

$$G^x((k)_{1,n}) = (2\pi)^{n/2}\, e^{-in\pi/4}\, e^{-i(\sum_{j=1}^{n} \phi_{k_j})}\, (\tilde\psi, \xi(k_n)\ldots\tilde\xi(k_1)\sigma_1^x\,\hat\psi)$$

and

$$G((k)_{1,n}) = (2\pi)^{n/2}\, e^{-in\pi/4}\, e^{-i(\sum_{j=1}^{n} \phi_{k_j})}\, (\tilde\psi, \tilde\xi(k_n)\ldots\tilde\xi(k_1)\sigma_1^x\,\hat\psi).$$

Theorem 4.6.

$$G^x((k)_{1,n}) = (2\pi)^{-1}\int \theta(e^{ik_0})\,G((k)_{0,n})\,dk_0 + \sum_{j=1}^{n} (-1)^{j-1}\,\theta(e^{ik_j})^{-1}\, G(\Delta_j(k)_{1,n}) .$$

Prf: Follows from direct substitution of $\sigma_1^x = \tilde\xi^*(g) + \tilde\xi(\bar g)$, where $g(k) = e^{i\pi/4}\, e^{-i\phi_k}/\sqrt{2\pi}$ in the expression for $G^x((k)_{1,n})$.

Theorem 4.7.

$$\pi^{-1}\, P\int dk_0\, [e^{i(k_0-\ell)}-1)^{-1}\, G((k)_{0,n})(1+ (e^{ik_0})\theta(e^{i\ell})^{-1}) = $$

$$= \sum_{j=1}^{n} (-1)^{j-1}\, h(k_j,\ell)\,G(\Delta_j(k)_{1,n})$$

Prf: Let $g_1,\ldots, g_n$ be arbitrary L functions and $f \in C$ such that $e^{i\phi} f$ is smooth. Then

$$0 = (\tilde\psi,\tilde\xi(g_n)\ldots\tilde\xi(g_1)\hat\xi(f)\hat\psi) =$$

$$= \int g_n(k_n)\ldots g_1(k_1)\,(T_1 f)(k_0)\,(\tilde\psi,\tilde\xi(k_n)\ldots\tilde\xi(k_0)\hat\psi)\,dk_n\ldots\, dk_0$$

$$+ \sum_{j=1}^{n} (-1)^{j-1}\, \left(\int g_j(k_j)\,(T_2 f)(k_j)\,dk_j\right)$$

$$\left(\prod_{\substack{\ell=1\\ \ell\neq j}}^{n} g_\ell(k_\ell)\right)(\tilde\psi, \prod_{k\neq j} \tilde\xi(k_\ell)\hat\psi)\left(\prod_{\ell\neq j} dq_\ell\right) \,..$$

Using the definition of G, together with

$$e^{i\phi_k}(T_1 f)(k) = (2\pi)^{-1}\ P\int (1-e^{i(u-k)})^{-1}(1+\theta(e^{iu})\theta(e^{ik})^{-1})(e^{i\phi_u} f(u))du$$

and

$$e^{-i\phi_k}(T_2 f)(k) =$$

$$= (2\pi i)^{-1}\ P\int (1-\theta(e^{iu})^{-1}\ \theta(e^{ik})^{-1})(1-e^{-i(k+u)})^{-1}\ (e^{i\phi_u} f(u))\, du$$

one gets

$$\int du(e^{i\phi_u} f(u))\ P\int \frac{1}{2\pi} dk_0\ G((k)_{0,n})(1-e^{i(k_0-u)})^{-1}(1+\theta(e^{ik_0})\theta(e^{iu})^{-1}) =$$

$$\sum_{j=1}^{n} (-1)^{j-1}\int du\ (e^{i\phi_u} f(u))(1-\theta(e^{iu})^{-1}\ \theta(e^{ik_j})^{-1})(1-e^{-i(u+k_j)})^{-1}$$

$$\times\ G(\Delta_j(k)_{1,n})\ .$$

The theorem follows.

Assume $T < T_c$. The function $G(k)$ satisfies YG= 0 and $(2\pi)^{-1}$.

$$\int\theta(e^{ik})G(k)dk = (\tilde{\psi},\sigma_1^x\ \hat{\psi})\ .$$

This implies

$$G^x((k), \quad = (\tilde{\psi},\sigma_1^x\ k \qquad _{1,n})/m^*, \quad (n\ \text{even})$$

and therefore

$$S_2 = |(\tilde{\psi},\sigma_1^x\ \hat{\psi})/m^*|^2\ \langle\sigma(n_1,m_1)\sigma(n_2,m_2)\rangle\ .$$

By evaluating the correlation functions at coincident points we conclude

$$|(\tilde{\psi},\sigma_1^x\hat{\psi})/m^*|^2 = 1\ ,$$

hence

$$S_2 = \langle\sigma(n_1,m_1)\sigma(n_2,m_2)\rangle\ .$$

A similar argument holds for $T > T_c$.

5. DECWPOSITION OF THE PERIODIC STATE

In addition to properties iii) and iv) of Thm. 3.2 enjoyed by U we have

Thm. 5.1. $[\sigma_1^x, U] = 0$ and for $T < T_c$

$$[e^{imP}, U] = [e^{imH}, U] = 0.$$

Prf. of Thm. 5.1.: As $\sigma_1^x = c_1 + c_1^*$ it follows from Thm. 3.5 that

$$U\sigma_1^x U^{-1} = \sigma_1^x \quad \text{or} \quad [\sigma_1^x, U] = 0 .$$

If (for $T < T_c$)

$$\hat{\chi} = \sum_{i=1}^{2N} \hat{\xi}^*(f_i)\, \hat{\chi} \in \hat{H}_e$$

then

$$e^{imP}\hat{\chi} = \exp(im \int q\ \xi^*(q)\hat{\xi}(q)dq)\hat{\chi} .$$

Furthermore, if

$$\tilde{\chi} = \sum_{i=1}^{2N} \tilde{\xi}^*(f_i)\tilde{\psi} \in \tilde{H}_e$$

then

$$U^{-1} e^{imP} U \hat{\chi} = U^{-1} e^{imP} \tilde{\chi} = U^{-1} \exp(im \int q\ \tilde{\xi}^*(q)\tilde{\xi}(q)dq)\tilde{\chi}$$

$$= \exp(im \int q\ \hat{\xi}^*(q)\hat{\xi}(q)dq)\hat{\chi} = e^{imP}\ \hat{\chi} .$$

Thus we have shwn that

$$U^{-1}\ e^{imP}\ U \restriction \hat{H}_e = e^{imP} \restriction \hat{H}_e$$

and similary one shows the equality for $\hat{H}_0$ so that $[e^{imP}, U] = 0$. Substituting H for P in the above arguments shows that $[e^{imH}, U] = 0$.

Since $U^2 = I$ and thus $U^* = U, P_\pm = \frac{L \pm U}{2}$ are orthogonal projections and $P_+P_- = 0$. Thus for $T < T_c$ the subspace $H_\mp = P_\mp H$ reduce σ_1^x, e^{imP} and e^{imH}. We have

Thm. 5.2. Let $T < T_c$ and $\psi_\mp = \sqrt{2}\, P_\pm\, \hat{\psi} = \frac{1}{\sqrt{2}} (\hat{\psi} \pm \tilde{\psi})$ so that $|\psi_\mp| = 1$.

$(\hat{\psi}, 0\,\hat{\psi})$ admits the decomposition

$$(\hat{\psi}, O\,\hat{\psi}) = \frac{1}{2}\,(\psi_+, O\,\psi_+) + \frac{1}{2}\,(\psi_-, O\,\psi_-)$$

where o is a generic product

$$O = \prod_j \sigma_1^x\, e^{-H|n_j|}\, e^{-iPm_j}$$

and $(\psi_+, \cdot\, \psi_\pm)$ are translationally invariant states. Furthermore, the decomposition is non-trivial since $(\psi_+, \sigma_1\, \psi_+) = -\,(\psi_-, \sigma_1\, \psi_-) \neq 0$. If O has an even number of factors

$$(\hat{\psi}, O\,\hat{\psi}) = (\psi_+, O\,\psi_+) = (\psi_-, O\,\psi_-) = (\tilde{\psi}, O\,\tilde{\psi})$$

and for an odd number

$$(\hat{\psi}, O\,\hat{\psi}) = 0, \quad (\psi_+, O\,\psi_+) = (\hat{\psi}, O\,\tilde{\psi}) = (\tilde{\psi}, O\,\hat{\psi})\ .$$

$(\psi_+, \cdot\,\psi_+)$ and $(\psi_-, \cdot\,\psi_-)$ have the clustering property; thus they are indecomposable.

Remarks:

1. By redefining $\tilde{\psi}$, if necessary, $(\psi_+, a, \psi_+) = (\hat{\psi}, \sigma_1 \tilde{\psi})$ can be taken to be positive and is usually called the spontaneous magnetization.

2. Let H' denote the closure of the subspace of H generated by applying polynomials in $\prod_{i \in Z} D_i$ (finite product), where Z is the set of integers and

$$D_i = e^{-H|n_i|}\, e^{im_i P}\, \sigma_1\, e^{-im_i P}\ ,$$

to $\hat{\psi}$. Let H'' be the physical Hilbert space obtained from the GNS construction applied to the state $(\hat{\psi}, \cdot\,\hat{\psi})$. We expect, but have been unable to show that $H' = H$ or $H'' = H$.

Prf.: We have, using $P_+P_- = [P_\pm, O] = 0$,

$$\begin{aligned}(\hat{\psi}, O\hat{\psi}) &= ((P_+ + P_-)\hat{\psi}, O(P_+ + P_-)\hat{\psi}) \\ &= (P_+\hat{\psi}, O\,P_+\hat{\psi}) + (P_-\hat{\psi}, O\,P_-\hat{\psi}) \\ &= \frac{1}{2}\,(\psi_+, O\,\psi_+) + \frac{1}{2}\,(\psi_-, O\,\psi_-)\end{aligned}$$

which shows the decomposition. If O has an even number of factors then $O.\ \hat{H}_e \to \hat{H}_e$ and $\hat{H}_o \to \hat{H}_o$ so that

$$(\psi_\pm, O\,\psi_\pm) = \frac{1}{2}\,((\hat{\psi} \pm \tilde{\psi}),\ \ (\hat{\psi} \pm \tilde{\psi})) = \frac{1}{2}\,[(\hat{\psi}, O\,\hat{\psi})+(\tilde{\psi}, O\,\tilde{\psi})]$$

$$= (\hat{\psi}, O\ \hat{\psi}) = (\tilde{\psi},\ O\ \tilde{\psi}).$$

If O has an odd number of factors then

$$(\psi_\pm, O\,\psi_\pm) = \pm\,\frac{1}{2}\,[(\hat{\psi}, O\ \tilde{\psi}) + (\tilde{\psi}, O\ \hat{\psi})] = \pm(\hat{\psi}, O\,\tilde{\psi})\ ,$$

since

$$(\tilde{\psi},\ O\ \hat{\psi}) = (U\hat{\psi}, O\,U^{-1}\ \tilde{\psi}) = (\hat{\psi},\ U^{-1}\,O\ \ U^{-1}\ \bar{\psi}) = (\hat{\psi}, U\ O\,U^{-1}\ \tilde{\psi}) = (\hat{\psi},\ O\ \tilde{\psi})\cdot$$

If $O = \sigma_1^x$ then $(\psi_\pm,\ \sigma_1^x\ \psi_\pm) = \pm(\hat{\psi},\sigma_1^x\ \tilde{\psi})$ is shown to be non-zero in Appendix C which implies that the ■ states are distinct. The clustering of these states is a consequence of the uniqueness of the vacuum in $H_\pm = P_\pm H$ (which follows from the known double degeneracy of the ground state of (H,P)).

APPENDIX A: THERMODYNAMIC LIMIT

We remark that it can be shown that, for high temperatures, the sequence $\{S_{Mk}\}$ converges to S_k, given by (4.1). For all temperatures compactness can be used to extract a convergent subsequence. In this appendix we show that any subsequence of $\{S_{M2}\}$ where M is of the for 2^n contains a convergent subsequence with limit given by (4.4) if $T < T_c$ or the appropriately modified (4.4) for $T > T_c$ where F^x is given by (4.2) and F satisfies (4.3). The idea of the proof is to show that the sequence $\{F_M\}$ is a normal family of analytic functions in the variable $z = e^{iq}$ in a neighborhood (independent of M) of $|z| = 1$. Then, we show that the limit F of any convergent subsequence satisfies (4.3) and in addition the same subsequence of correlation functions converges to (4.4). We will assume $T < T_c$ but similar arguments apply for $T > T_c$ and k > 2.

Recall the definition

$$F_M(k) = M^{1/2} e^{-i\pi/4} \exp(i(k-\phi_k))(\psi_{S^-},\xi_k\psi_{S^+}) .$$

Thm.A.1. F_M as a function of $z = e^{ik}$ extends as an analytic function in an annulus, independent of M but dependent on T, around $|z| = 1$ and is uniformly (in M) bounded on compact subsets.

Proof: Using the relationship between ξ_k and ξ_R we write

$$F_M(z) = e^{-iz\pi/4} M^{-1/2} z \sum_{\ell\in S^+} e^{i(\ell+\phi_\ell)}(z-e^{i\ell})^{-1}(\theta(e^{i\ell})^{-1} - \theta(z)^{-1}) .$$

$$(\psi_{S^-}, \xi^*_{-\ell}\psi_{S^+}) .$$

Let the annulus of analyticity be described by $r < |z| < R$, with $r < 1$ and $R > 1$. The function $(\theta^{-1}(w) - \theta^{-1}(z))/(z-w)$ is analytic in both variables in $r < |z|, |w| < R$ and therefore uniformly bounded in $r < r' \leq |z|, |w| \leq R' < R$, with $r' < 1$ and $R' > 1$ independent of M. Let C be the bound. Then for $r' \leq |z| \leq R'$,

$$|F_M(z)| \leq R'M^{-1/2} C \sum_{\ell\in S^+} |(\psi_{S^-},\xi^*_{-\ell}\psi_{S^+})| \leq CR' \left(\sum_{\ell\in S^+}|(\psi_{S^-},\xi^*_{-\ell}\psi_{S^+})|^2\right)^{1/2}.$$

Since the set

$$\{\xi^*_\ell \psi_{S^+}\}_{\ell\in S^+} , \{\xi^*_{\ell_1}\xi^*_{\ell_2}\xi^*_{\ell_3}\psi_{S^+}\}_{\ell_i\in S^+} , \dots ,$$

$\ell_1 < \ell_2 < \ell_3 < \dots$ form an orthogonal basis in H_e, we have

$$1 = (\psi_{S^-},\psi_{S^-}) =$$

$$= \sum_{\ell\in S^+} |(\psi_{S^-},\xi^*_\ell\psi_{S^+})|^2 + \frac{1}{3!}\sum_{\ell_i\in S^+} |(\psi_{S^-},\xi^*_{\ell_1}\xi^*_{\ell_2}\xi^*_{\ell_3}\psi_{S^+}) + \dots$$

It follows that $|F_M(z)| \leq CR'$.

A similar result holds for $\{F_M(z, \dots, z_{2n+1}\}$. Thus, any, subsequence of M contains a subsequence for which $F_M(z_1,\dots,z_{2n+1})$ converges uniformly oncornpact subsets of $r < |z_i| < R$, For notational convenience, we still call the converging subsequence $\{F_M\}$ and call the limit $F(z_1,\dots,z_{2n+1})$. We next show that F satisfies (4.3). Since M has the form 2^n this implies $S_M^- \subset S_{M'}^-$, if $M' > M$. Let $R^- = \bigcup_M S_M^-$, and let $h(z)$ be an arbitrary analytic function on $r < |z| < R$. If

$$g_M(k) = 2M^{-1} \sum_{\ell \in S^+} h(e^{i\ell})(e^{i(k-\ell)}-1)^{-1}(1+\theta(e^{ik})/\theta(e^{i\ell})),\quad k \in R^-,$$

then

Lemma A.2.

$$\lim_{M\to\infty} g_M(k) = g(k) = \pi^{-1}\, P \int_{-\pi}^{\pi} h(e^{i\ell})(e^{i(k-\ell)}-1)^{-1}(1+\theta(e^{ik})/\theta(e^{i\ell}))\,d\ell ,$$

uniformly in $k \in R^-$.

Proof: Write

$$g_M(k) = 2M^{-1} \sum_{\ell \in S^+} h_1(e^{i\ell})(e^{i(k-\ell)}-1)^{-1}$$

$$+\ \theta(e^{ik})\, 2M^{-1} \sum_{\ell \in S^+} h_2(e^{i\ell})(e^{i(k-\ell)}-1)^{-1}, \qquad \text{(A.1)}$$

where

$$h_1(e^{i\ell}) = h(e^{i\ell}) \quad \text{and} \quad h_2(e^{i\ell}) = \theta(e^{i\ell})^{-1} h(e^{i\ell}) .$$

Let

$$B_k^+ = \{\ell \in S^+ : \operatorname{Im} e^{i(\ell-k)} > 0\} .$$

Then,

$$2M^{-1} \sum_{k \in S^+} h_1(e^{i\ell})/(e^{i(k-\ell)}-1) =$$

$$= 2M^{-1} \sum_{\ell \in B_k^+} \left[h_1(e^{i\ell})(e^{i(k-\ell)}-1)^{-1} + h_1(e^{i(2k-\ell)})(e^{i(\ell-k)}-1)^{-1} \right] =$$

$$= 2M^{-1} \sum_{\ell \in B_0^+} \left[h_1(e^{i(\ell+k)}) - e^{-i\ell} h_1(e^{i(k-\ell)}) \right] / (e^{-i\ell}-1) .$$

Consider the function of two variables

$$(zh_1(wz)-h_1(w/z))/(1-z), \qquad z\neq 1 ;$$

$$-(h_1(w)+2wh_1'(w)), \qquad z=1 .$$

This is analytic in $r < |w|$, $|z| < R$, and in particular uniformly continuous for $|w| = |z| = 1$. From this, it follows that

$$\lim_{M\to\infty} 2M^{-1} \sum_{\ell\in S^+} h_1(e^{i\ell})(e^{i(k-\ell)} - 1)^{-1} =$$

$$= \pi^{-1} \int_{-\pi}^{\pi} \frac{[e^{i\ell} h_1(e^{i(\ell+k)}) - h_1(e^{i(k-\ell)}) d\ell]}{1 - e^{i\ell}}$$

$$= \pi^{-1} \int_{-\pi}^{\pi} h(e^{i\ell})(e^{i(k-\ell)} - 1)^{-1} d\ell,$$

uniformly in $k\in R-$. The second term in (A.1) is treated similarly and the proof is complete,

Thm.A.3:

$$(YF)(\ell) = \pi^{-1} P \int_{-\pi}^{\pi} dk\, F(k)(e^{i(k-\ell)}-1)^{-1}(1+\theta(e^{ik})/\theta(e^{i\ell})) = 0.$$

Proof: Since

$$2M^{-1} \sum_{k\in S^-} F_M(k)(e^{i(k-\ell)}-1)^{-1} \left(1 + \frac{\theta(e^{ik})}{\theta(e^{i\ell})}\right) = 0 \quad (\ell \in S^+),$$

we have, upon multiplying by $2M^{-1}$ $h(\ell)$ and summing over $R \in S^+$,

$$2M^{-1} \sum_{k\in S^-} F_M(k) g_M(k) = 0 .$$

Using the uniformity of convergence of F_M and g_M, it is easy to show that

$$0 = \lim_{M\to\infty} 2M^{-1} \sum_{k\in S^-} F_M(k) g_M(k) = \lim_{M\to\infty} 2M^{-1} \sum_{k\in S^-} F(k) g(k) =$$

$$= \pi^{-1} \int F(k) g(k) dk .$$

We see from Lemma A.2 that $(\bar{F},\bar{Y}h) = 0$, where the operator $\bar{Y}$ is defined by $(\bar{Y}f) = \overline{(Y\bar{f})}$. Using self-adjointness of $\bar{Y}$ and the fact that analytic functions on a neighborhood of $|z| = 1$ are dense in L^2 (torus) we conclude that YF = 0.

Using the same argument, it is easy to show that

$$\pi^{-1}\, P\int_{-\pi}^{\pi} F((k)_{0,n})(e^{i(k_0-\ell)}-1)^{-1}\ (1+\theta(e^{ik_0})/\theta(e^{i\ell}))dk_0 =$$

$$= \sum_{j=1}^{n} (-1)^{j-1}\ h(k_j,\ell)F(\Delta_j(k)_{1,n})$$

first for all $k_1,\dots,k_n \in R^-$ and then for all k_i by continuity. Thus, we see that the set of limiting functions F satisfies (4.3).

Recall from section 4 that ($T < T_c$, $n_2 \geq n_1$)

$$S_{M2} = \sum_{n\ \text{even}} \dots = \sum_n \alpha^M_{n,n_2-n_1,m_2-m_1}$$

where

$$F^x_M((k)_{1,n}) = M^{-1}\sum_{k_0\in S^-} \theta(e^{ik_0})F_M((k)_{0,n}) +$$

$$+ \sum_{j=1}^{n} (-1)^{j-1}\ \theta(e^{ik_j})^{-1}\ F_M(\Delta_j(k)_{1,n})$$

(see Thm. 4.4). From the results above we see that F^x_M converges uniformly in $k_i \in R^-$ to

$$F^x((k)_{1,n}) = (2\pi)^{-1}\int_{-\pi}^{\pi} \theta(e^{ik_0})F((k)_{0,n})dk_0 +$$

$$+ \sum_{j=1}^{n} (-1)^{j-1}\ \theta(e^{ik_j})^{-1}\ F(\Delta_j(k)_{1,n}) \quad .$$

This last formula allows a continuous extension of F^x to all $k_i \in [-\pi,\pi]$. It follows that

$$\lim_{M\to\infty} M^{-n} \sum_{k_i\in S^-} |F^x_M((k)_{1,n})|^2$$

$$\times \exp[-(n_2-n_1)(\sum_{j=1}^{n} \varepsilon(k_j)) - i(m_2-m_1)(\sum_{j=1}^{n} k_j)]\ e^{-\delta_M(n_2-n_1)}$$

$$= (2\pi)^{-n}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi} |F^x((k)_{1,n})|^2 \exp[-(n_2-n_1)(\sum_{j=1}^{n}\varepsilon(k_j))$$

$$- i(m_2-m_1)\ (\sum_{j=1}^{n} k_i)]\ dk_1\dots dk_n \equiv n!\ \alpha_{n,n_2-n_1,m_2-m_1} \quad .$$

Thm.A.4. There exists a constant c (independent of M) such that

$$\alpha^M_{n,0,0} \leq c^{n+1}(n+1)^{(n+5)/2}/n! \ .$$

Corollary A.5. $\lim_{M\to\infty} S_{M2} = \sum_{n \text{ even}} a_{n,n_2-n_1,m_2-m_1}$

Proof: Follows from the dominated convergence theorem.

Proof of Thm.A.4: Starting from the definition of $F_M(k)_{1,n})$ we get

$$|F_M|^2_M \equiv M^{-n} \sum_{k_i\in S^-} |F_M(k)_{1,n})|^2 =$$

$$= \sum_{k_i\in S^-} |(\psi_{S^-}, \xi_{k_n} \ldots \xi_{k_1} \psi_{S^+})|^2 \leq \sum_{k_i\in S^-} (\psi_{S^+}, \xi^*_{k_1} \ldots \xi^*_{k_n} \xi_{k_n} \ldots \xi_{k_1} \psi_{S^+}) .$$

The right hand side can be calculated by Wick's theorem. Terms involving contractions like $(\psi_{S^+}, \xi^*_{k_i} \xi^*_{k_j} \psi_{S^+})$ or $(\psi_{S^+}, \xi_{k_i} \xi_{k_j} \psi_{S^+})$ do not contribute because of the summation over the k's. Thus we see that

$$|F_M|^2 \leq \sum_{k_i\in S^-} \sum_{\text{perm.}} (-1)^P (\psi_{S^+}, \xi^*_{k_1} \xi_{k_{i_1}} \psi_{S^+}) \ldots (\psi_{S^+}, \xi^*_{k_n} \xi_{k_{i_n}} \psi_{S^+}) = \sum_{k_i\in S^-} \det A$$

where the self-adjoint matrix A has elements

$$A_{ij} = (\psi_{S^+}, \xi^*_{k_i} \xi_{k_j} \psi_{S^+}) \ .$$

We calculate, using $\xi_k = t_{1k\ell}\xi_\ell + t_{2k\ell}\xi^*_\ell$,

$$A_{ij} = 4M^{-2} \sum_{\ell\in S^+} e^{-i(\ell-k_i)} (1-e^{-i(\ell-k_i)})^{-1} e^{i(\ell-k_j)} (1-e^{i(\ell-k_j)})^{-1}$$

$$\times \sin(\phi_{k_i} - \phi_\ell) \sin(\phi_{k_j} - \phi_\ell) \ .$$

Let

$$c_1 = \sup_{\ell,k\in[-\pi,\pi]} |\sin(\phi_k - \phi_\ell)/(1-e^{i(k-\ell)})^{-1}| < \infty$$

(c_1 independent of M). Then $|A_{ij}| \leq 4c_1^2/M$ and by Hadamard's inequality

$|\det A| \leqslant (4c^2/M)^n . n^{n/2}$ and therefore $|F_M|_M^2 \leqslant (4c_1^2)^n n^{n/2} \equiv c_2^n n^{n/2}$, where $c_2 = 4c_1^2$. Now, majorizing F^x, we have

$$|F_M^x|_M \leqslant (c_2^{(n+1)}(n+1)^{(n+1)/2})^{1/2} + n(c_2^{(n-1)}(n-1)^{(n-1)/2})^{1/2}.$$

This implies

$$|F_M^x|_M^2 \leqslant 2(c_3^{(n+1)}(n+1)^{(n+1)/2} + n^2 c_3^{(n+1)}(n+1)^{(n+1)/2}), \quad c_3 = \max\{1,c_2\},$$

$$|F_M^x|_M^2 \leqslant 4(n+1)^2 c_3^{(n+1)}(n+1)^{(n+1)/2} \leqslant c^{n+1}(n+1)^{(n+5)/2}, \quad c = 4 c_3,$$

and completes the proof of the theorem.

APPENDIX B

The structure of U and $\tilde{\psi}$ depends on the null space of T_1 which in turn depends on the temperature by Thm. 3.1 vii, Let $N_0(\ ,\ )$ denote the normal ordering operation with the $\hat{\psi}$ vacuum projection occuring after the last creation operation. For $T > T_c$ we have

Thm.B.I.

a) $$\tilde{\psi} = (\det T_1'^* T_1')^{1/4} \exp(-\frac{1}{2}\int \hat{\xi}^*(k)(T'^{-1}T')(k,k')\hat{\xi}^*(k') \, dk\, dk')\hat{\psi} \qquad (B.1)$$

and $(\hat{\psi},\tilde{\psi}) = (\det T_1'^* T_1')^{1/4}$

b) $$U = (\det T_1'^* T_1')^{1/4} N_0(\exp\{-\frac{1}{2}\int (\hat{\xi}(k),\ \hat{\xi}^*(k)) . \begin{pmatrix} \bar{T}_2' T_1'^{-1} & T_1^{-1} \\ -T_1'^{-1} & T_1'^{-1} T_2' \end{pmatrix} (k,k') \begin{pmatrix} \hat{\xi}(k') \\ \xi^*(k') \end{pmatrix} dk\, dk'\}) \qquad (B.2)$$

c) $U = U^*$, $\tilde{\psi} \in H_e$, $U\hat{H}_e = \hat{H}_e$ and $U\hat{H}_o = \hat{H}_o$.

Remark: 1. By expanding the exponential in (B.2) we see that

$$(\hat{\psi},\hat{\xi}(k_1)\hat{\xi}(k_2)U\hat{\psi})/(\hat{\psi},\tilde{\psi}) = (T_1'^{-1}T_2')(k_1,k_2) = (\bar{T}_2 T_1'^{-1})^* \ (k_1,k_2)$$

and

$$(\hat{\xi}^*(q)\hat{\psi}, U\hat{\xi}^*(k)\hat{\psi}/(\hat{\psi},\tilde{\psi}) = T_1'^{-1}(q,k) \ .$$

These functions are obtained explicitly in ref. 18 by Wiener-Hopf methods.

2. With our convention for $\tilde{\xi}(f)$ we associate the matrix

$$J = \begin{pmatrix} T_1' & T_2' \\ \bar{T}_2' & \bar{T}_1' \end{pmatrix}$$

with the transformation eqs. (2.1a,b) which differs from that of ref. 12. Our operators are the transpose of those in ref. 12. If B , D are matrices associated with pict then BD corresponds to the transforrnation first C followed by B, and in terms of the corresponding unitaries $U_{BD} = U_B U_D$.

Proof: a),b) follow from[12] since dim ker T_1 = 0.

c) $(T_1'^{-1} T_2)^* = \bar{T}_2 T_1'^{-1}$ and $T_1^* = T_1$ irnply U = U*.

Take T < T_c in what follows.

Lemma 8.2. The matríx

$$\mathbf{J} = \begin{pmatrix} T_1' & T_2' \\ \bar{T}_2' & \bar{T}_1' \end{pmatrix}$$

associated with the plct of eqs. (2.1a,b) is the product of the three plct v,, a and v„ i.e.

$$J = \nu_2 a \nu_1 = \begin{pmatrix} V_2'\Phi' V_1' & V_2'\psi'\bar{V}_2' \\ \bar{V}_2'\bar{\psi}' V_1' & \bar{V}_2'\bar{\Phi}' V_1' \end{pmatrix} \qquad (B.3)$$

where

$$\nu_1 = \begin{pmatrix} V_1' & 0 \\ 0 & \bar{V}_1' \end{pmatrix}, \quad a = \begin{pmatrix} \Phi' & \psi' \\ \bar{\psi}' & \bar{\Phi}' \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} V_2' & 0 \\ 0 & \bar{V}_2 \end{pmatrix}$$

and V_1, V_2 are the unitary operators defined by

$$(V_1 \phi)(k) = \sqrt{i}\, e^{-ik/2}\phi(k) \qquad (B.4)$$

$$(V_2 \phi)(k) = \sqrt{i}\, e^{-iq/2}\phi(-q) = \sqrt{i}\, e^{-iq/2}(V\phi)(q) \ ; \qquad (B.5)$$

Φ is the integral operator

$$(\Phi\ \phi)(k) = \pi^{-1}\ P\int (2-2\cos(k+q))^{-1/2}\ \varepsilon(k+q)\cos(\phi_q+\phi_k)\phi(q)dq, \tag{B.6}$$

$$\Phi = \Phi^* = \bar{\Phi}\ ,\quad \dim\ \ker\Phi = 1$$

and $\psi = \bar{V}_1^*\ T_2\ V_2^*$, $\varepsilon(x) = \pm 1$ if $x \gtrless 0$.

Proof:

Using the group property of plct and the fact that T_1 can be written as

$$(T_1 f)(k) =$$
$$= P\int \sqrt{i}\ e^{-ik/2}\ \frac{1}{\pi}\ (2-2\cos(k-q))^{-1/2}\ \varepsilon(k-q)\cos(\phi_q-\phi_k)(\sqrt{i}\ e^{iq/2})f(q)dq$$
$$\equiv \int T_1(k,q)\ f(q)\ dq = \int T_1(k,-q)f(-q)dq$$

the decompositions for T_1 and J follow.

As $\Phi = \Phi^* = \bar{\Phi}$ and $\dim\ker\Phi \neq 0$ the plct a of Lemma B.2. is a special case ($\dim\ker\Phi = 1$) of a decomposable plct in the sense of ref. 12:

1. $L = L_1 \oplus L_2$ where $L_i = \ker(\Phi)$. L_i are invariant under $\Phi, \psi, \bar{\psi}, \psi^*$ thus $\Phi = \Phi_1 \oplus \Phi_2$, $\psi = \psi_1 \oplus \psi_2$, ψ_1 is unitary on L_1. Φ_2^{-1} is bounded on L_2.

2. The transformation

$$b(\phi_i) = a(\Phi_i\phi_i) + a^*(\psi_i\phi_i) \tag{B.7a}$$

$$b^*(\phi_i) = a(\bar{\psi}_i\phi_i) + a^*(\bar{\Phi}_i\phi_i) \tag{B.7b}$$

$i = 1,2$, $\phi = \phi_1 + \phi_2$, is a plct in H_i, the subspace of H generated by $a^*(\phi)$, $\phi \in L_i$, with matrix $\begin{pmatrix}\Phi_i & \psi_i\\ \bar{\psi}_i & \bar{\Phi}_i\end{pmatrix}$.

Thm. B.3. Let $\phi_1 = \alpha_1 h$, $h = \bar{h} \in \ker\Phi$, $|h| = 1$. Let $F_0(F)$ be the vacuum vector for the a's (b's) of eq. (B.7a,b) satisfying $|F| = |F_0| = 1$, and let W be the unitary which implements a. Then

a) $\psi_1 \ \phi_1 = \gamma\phi_1$ with $|\gamma| = 1$,

b) $F = R_1 \ R_2 \ F_0$

where $R_, = a^*(h)$ and $R_a F_0$ is given by eq. (B.1) with Φ_2, ψ_2, $F_{,}$, $a^*(k)$ replacing

$$T_1', \ T_2', \ \hat{\psi}, \ \hat{\xi}^*(k) \ .$$

c) Let a, β satisfy $|\alpha| = |\beta| = 1$, $\beta = a\gamma$, then

$$W = W_1 W_2 ,$$

where $W_1 = a(\alpha h) + a^*(\beta h)$ and W_2 is given by eq. (B.2) with $-\Phi_2$, $-\psi_2$, $a(q)$, $a^*(q)$ replacing T_1', T_2', $\hat{\xi}(q)$, $\hat{\xi}^*(q)$ and N_0 referring to the a's.

Proof: b) First we show $b(\phi) \ F = 0$ for all ϕ,

$$b(\phi) = b(\phi_1)+b(\phi_2), \ b(\phi_1) = a^*(\psi\phi_1)=\gamma\alpha_1 \ a^*(h)$$

so that $b(\phi_1) \ F = 0$ since $b(\phi_1) \ R_1 = 0$. Now

$$b(\phi_2)F = -R_1 \quad b(\phi_2)R_2F_0 \quad \text{as} \quad \{b(\phi_2), R_1\} = 0$$

and $b(\phi_2)R_2F_0$ is zero by Thm. B.1. *c)*. W_1 satisfies $b(\phi_1)W_1 = W_1 \ a(\phi_1)$, $b^*(\phi_1)W_1 = W_1a^*(\phi_1)$ and W_2 satisfies $-b(\phi_2)W_2 = W_2a(\phi_2)$, $-b^*(\phi_2) \ W_2 = = W_2a^*(\phi_2)$ and as $b(\phi_2)W_1 = -W_1b \ (\phi_2)$ we have

$$\begin{aligned} b(\phi)W &= (b(\phi_1) + b(\phi_2))W_1W_2 \\ &= W_1(a(\phi_1) - b(\phi_2))W_2 \\ &= W_1W_2(a(\phi_1) + a(\phi_2)) = Wa(\phi) \ . \end{aligned}$$

A similar argument shows that $b^*(\phi)W = Wa^*(\phi)$.

Thm. B.4. Let U_1, U_2, $W_a = W_{a_1} \ W_{a_2}$, $U = U_2 \ W_a \ U_1$ *be* the unitaries corresponding to the plct v,, v„ a, J. Then

a) $\tilde{\psi} = U\hat{\psi} = U_2 \ R, \ R, \ \hat{\psi}$

and

$$U = U_2 \ W_{a_1} \ W_{a_2} \ U_1$$

where U_1, U_2 are the usual Fockified operators of V_1 and V_2; R_1, R_2, W_{a_1} W_{a_2} are as in Thm. **8.3.** with $U_1\ \hat{\xi}(k)U_1^{-1}$, $U_1\hat{\xi}^*(k)U_1^{-1}$ replacing $a(k)$, $a^*(k)$ and W_{a_i} replacing W_i,

b) $\tilde{\psi} \in \hat{H}_o$, $U\hat{H}_e = \hat{H}_o$ and $U\hat{H}_o = \hat{H}_e$.

Proof: a) Follows from Lemma 8.2 and Thm. 8.3 since the unitaries corresponding to the plct give a ray representation of the plct group. b) Holds since $U_i : \hat{H}_{e(o)} \to \hat{H}_{e(o)}$ and $W_a : \hat{H}_{e(o)} \to \hat{H}_{o(e)}$.

APPENDIX C: $(\tilde{\psi}, \sigma_1^x \hat{\psi}) \neq 0$ for $T < T_c$.

$$(\tilde{\psi}, \sigma_1^x \hat{\psi}) = (\tilde{\psi}, (\hat{\xi}^*(h) + \hat{\xi}(\bar{h}))\hat{\psi}) = (\tilde{\psi}, \hat{\xi}^*(h)\hat{\psi})$$

where $h(k) = e^{i\pi/4}\ e^{-i\phi_k}/2\pi$. $\tilde{\psi}$ is given by Thm. B.4 or $\tilde{\psi} = \tilde{\Omega}/|\tilde{\Omega}|$ where

$$\tilde{\Omega} = \sum_{n\ \text{odd}} \hat{\xi}^*_{(n)}(c_n)\ \hat{\psi}\ ,$$

$$\hat{\xi}^*_{(n)}(c_n) = \int c_n(k_1,\ldots k_n)\ \frac{1}{\sqrt{n!}}\ \hat{\xi}^*(k\ )\ldots\ \hat{\xi}^*(k_n)dk_1\ldots dk_n\ ,$$

and c_n is an anti-symmetric $L^2([-\pi,\pi]^n)$ function.

$$\tilde{\xi}(f)\tilde{\Omega} = 0 \quad \text{for all } f \in L \quad \text{implies} \quad (T_1 f, \bar{c}_1) = 0\ ,$$

or $\bar{c}_1 \in \ker T_1$, and we have

$$(\tilde{\psi}, \sigma_1^x \hat{\psi}) = |\tilde{\Omega}|^{-1}\int \bar{c}_1(\ell)h(\ell)d\ell = |\tilde{\Omega}|^{-1}\ (c_1,h)\ .$$

Letting $g = e^{i\phi}\ \bar{c}_1$ the requirement that $\bar{c}_1 \in \ker T_1$ becomes

$$T_1(e^{-i\phi}g) = \frac{1}{2}\ (H\Theta g + \Theta H g) = 0, \quad \Theta = e^{-2i\phi_\ell} \equiv \Theta(e^{i\ell})$$

or passing to Fourier coeficients

$$\sum_{n=0}^{\infty} \Theta_{n-m}\, g_m = 0 \qquad n \geq 0\ ,$$

$$\sum_{n=0}^{\infty} \Theta^{-1}_{n-m}\, g_{-(m+1)} = 0 \quad n \geq 0\ ,$$

where

$$g(e^{i\omega}) = \sum_{n=-\infty}^{\infty} g_n e^{in\omega}, \quad \Theta_{nm} = \Theta_{n-m} = \int e^{-i(n-m)\omega} \Theta(e^{i\omega}) d\omega .$$

By using Wiener-Hopf factorization (see Krein[14]) we find

$$g(z) = \frac{(z - x_2^{-1})^{1/2}}{(z - x_1)^{1/2}}$$

and it is confirmed that

$$(c_1, h) = \int g(\ell) e^{-2i\phi_\ell} d\ell \neq 0 .$$

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Resumo

Mostramos neste trabalho que as funções de Schwinger associadas as funções correlação do modelo de Ising bidimensional com condições de fronteira periódicas no limite termodinâmico podem ser representadas por uma fórmula de Feynmann - Kac (F-K) num espaço de Fock fermiônico H. Os operadores de campo e energia-momentum são expressos em termos de dois conjuntos de campos livres fermiônicos agindo em H. Estes dois conjuntos estão relacionados entre si por uma transformação linear canônica própria (pclt), i.e. existe um operador unitário U que implementa a transformação. Representações em séries para as funções de Schwinger são obtidas substituindo as representações espectrais dos operadores de energia-momentum na fórmula de F-K. Abaixo da temperatura critica $P_{\pm} = (I \pm U)/2$ são projeções ortogonais que comutam entre si e que reduzem a álgebra de observáveis, fornecendo a decomposição explicita do estado periódico em dois estados puros, invariantes por translação.