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Hidden Symmetries of Szekeres Quasi-spherical Solutions

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Abstract It is well known that the mass distribution of the Szekeres quasi-spherical solutions can be reduced to a monopole plus a dipole on one, freely chosen but just one, r=const., t=const. spherical shell. We give global evidences that this is true for the whole mass distrlbution, i.e. for the t=const. 3-surfaces.

1. INTRODUCTION

The Szekeres metric[1,2] is an exact solution to Einsteiri field equations for a cloud of colapsing, non-rotating, pressureless dust. It has no Killing vector[3] and yet it does not radiate as it was shown by Bonnor[4], by smoothly matching a ball of a quasi-spherical Szekeres spacetime onto the Schwarzschild exterior solution. Covarrubia[5] has shown that the Einstein linear approximation formula for gravitational radiation gives zero energy loss. Berger, Eardley and Olson[6] showed that the comoving space slices t=constant are conformally flat.

In section 2 we review some results about Szekeres quasispherical solutions, SQSS. We concentrate on those ones we are going to use in the following section. Section 3 contains the paper main contribution. We present global evidences that the mass distribution of the SQSS is of a monopole plus a dipole. Szekeres has shown that the mass distribution on a spherical shell r=constant, t=constant, bounded by spheres r and r=dr, has a monopole plus a dipole configuratiori. He notes however that "the axes of the dipoles on different spheres are oriented arbitrarily (up to smoothness) relative to each other, in the sense that no coordinate transformation exist in general to bring the surface densities on different spheres simultaneously to this canonical form at any instant t=constant."

The mass distribution depends on four, not independent, real, arbitrary r-functions: $a(r)$, $c(r)$, $f(r)$, and $g(r)$. So that, for a given

$r=r_1$=constant, $t=t_1$=constant, they can be made into constants $a = c = 0$, $f = g = 0$. It is in these $r=r_1$, $t=t_1$ 2-surfaces S_{rt} that the mass distribution assumes this simple monopole plus dipole form. But this is a local property in the sense that in any other S_{rt}, $r=r_1+dr$, $t=t_1$, for instance, the mass distribution will have contributions from poles of all orders. This is simply because the r-derivative of a, c, f, and g are not null. Otherwise we would have a spherical distribution. So we cannot say that the bulk mass distribution is a monopole plus a dipole. The most we can say is that it can be reduced to that in anyone chosen 2-sphere S_{rt}.

In section 3 we give equivalent but global proof for this. First we separate in the mass distribution the spherical distribution, ρ_S, from the non-spherical one, $\Delta\rho$. Then we show the existence of two time-independent, open and planewise infinite hypersurfaces of symmetry: 1) a hypersurface of mirror symmetry, S_M; 2) a hypersurface of "quasi-antisymmetry" of $\Delta\rho$, S_S. When crossing S_S $\Delta\rho$ changes of sign and on S_S $\Delta\rho$ is null. And there is no other hypersurface with similar properties. This globally characterizes the dipole nature of $\Delta\rho$. We remark that nowhere is made use of the freedom of making a, c, f, and g constants on a given S_{rt}.

S_S can be visualized as the loci of the dipole zeroes on every S_{rt}, and S_M as the hypersurface that contains the monopole and the extrema of the dipole distributions on every S_{rt}. The fact that S_M and S_S are time-independent is an evidence of the conformal time evolution of the t=constant comoving space-slices.

We shouid emphasize that the mass distributions being a superposition of a monopole and a dipole distributions does not per si explain the absence of gravitational radiation. That is so becausethetwo distributions are not concentrical (otherwise there would be a Killing vector related to the azimuthal symmetry). Therefore, the mass distribution has contribution from quadrupole and higher order terms.

Szekeres obtained his solutions postulating the existence of coordinates in the space-time within which metric and matter fields assume a simple pre-established form. However this simple form is not always the simplest and the most convenient for physical interpretations. We believe that coordinates defined in terms of solution symmetries can make the physics more transparent even if the mathematical forms may become more complex.

2. SZEKERES' QUASI-SPHERICAL SOLUTIONS

Requiring that

$$ds^2 = dt^2 - X^2\, dr^2 - Y^2\, d\xi d\bar{\xi} \tag{1}$$

where X and Y are quite general functions of t, r, ξ, $\bar{\xi}$, be a solution of Einstein's equations

$$G_{\mu\nu} = \rho U_\mu U_\nu \tag{2}$$

in comoving coordinates, $U_\mu = \delta_{\mu 0}$, for a pressureless, non-rotating fluid, Szekeres calculated the explicit integration of (1.2) , obtaining an entire class of complete and exact solutions. Of these, we are interested here on just one subclass, the Szekeres' Quasi-sphericel Solutions (SQSS) , for which (1) is restricted by

$$Y \equiv \frac{\partial Y}{\partial r} \neq 0 \tag{3}$$

The solutions of (1-2) are

$$Y = \frac{\Phi(r,t)}{P(r,\xi,\bar{\xi})} \quad ; \quad X = \frac{PY_1}{W(r)} \tag{4}$$

with $W(r) > 1$, an arbitrary function, and

$$P = a(r)\xi\bar{\xi} + b(r)\xi + \bar{b}(r)\bar{\xi} + c(r) = a\{(\xi + \frac{\bar{b}}{a})(\bar{\xi} + \frac{b}{a}) + \frac{1}{4a^2}\} > 0 \tag{5}$$

where $b = f + ig$, and a, c, f, and g are real arbitrary functions of r, restricted by $ac - b\bar{b} = 1/4$. The mass density ρ, obtained from the zero-zero equation of (2) is given by

$$\rho = \frac{PS_1 - 3SP_1}{\Phi^2(P\Phi_1 - \Phi P_1)} \tag{6}$$

where S is also an arbitrary function of r.

We observe that it is the presence of P and P_1 that makes ρ asymmetrical. P_1 cannot be zero (that is a, b, and c cannot be constants) because this would reduce the Szekeres metric to a spherical solution. Also, for the same reason, we cannot have P_1 proportional to P, that is: $a_1/a \neq c_1/c \neq f_1/f \neq g_1/g$. However, for a fixed r = const., using the transformation of coordinates

$$\xi \to \xi' = 2a\xi + 2\bar{b} \quad , \tag{7}$$

we can absorb a, c, f, and g in the definition of the new coordinates in a such way that it is equivalent to make a = c = 1/2, and b = f = 0.

In this hypersurface r = const., the 2-surfaces S_{rt}, defined by =const., are spheres of radius $\Phi(r,t)$, as it can be shown with a new transformation of coordinates

$$\xi = e^{i\psi'} \cot(\theta'/2) \tag{8}$$

which puts the metric induced on S_{rt} in the standard spherical form

$$ds^2_{rt} = \Phi(r,t)^2 \quad (d\theta'^2 + \sin^2\theta' d\psi'^2) \tag{9}$$

However, the mass distribution on these 2-surfaces S_{rt} is not constant or spherically symmetric, being rather a dipole distribution. For different values of r, the spheres of S_{rt} (which are smooth functions of r) have different centers and different dipole orientations, and there is no one single transformation able to make concentric these 2-spheres and to align all these dipoles!

3. HIDDEN SYMMETRIES

In this section, we want to prove the main results of this paper, which are the following two:

i) the mass density $\rho(t,r,\xi,\bar{\xi})$ is constituted by the superposition of two mass distributions: a spherically symmetrical one, $\rho_S(r,t)$, and one with thesymmetries of a dipole distribution, $\Delta\rho(t,r,\xi,\bar{\xi})$;

ii) there are two time-independent, open and planewise infinite hypersurfaces of symmetry: a) hypersurface of mirror symmetry, S_M, and b) hypersurface of spherical symmetry, S_S, (where ρ reduces to ρ_S). S_S is a *$\Delta\rho$-hypersurface-of-"quasi-antisymmetry"*.

Let us start rewriting the mass density (6) as

$$\rho = \frac{S_1P-3SP_1}{\Phi^2(\Phi,P)} + \frac{H(r,t)}{\Phi^2} - \frac{H(r,t)}{\Phi^2} = \frac{H(r,t)}{\Phi^2} + \frac{P(S_1-H\Phi_1) - P_1(3S-\Phi H)}{\Phi^2(\Phi,P)} \tag{10}$$

where $H(r,t)$ is an arbitrary function to be determined later, and $(\Phi,P) = \Phi_1P-\Phi P_1$, the notation being that for any functions of r, A and B, $(A,B) = A_1B - AB_1$, the subindex 1 meaning r-derivative.

The last term on the right hand side of (10) contains all the assymetrical parts of ρ,

$$\Delta\rho \equiv \frac{P(S_1-H\Phi_1) - P_1(3S-H\Phi)}{\Phi^2(\Phi,P)} \tag{11}$$

while the other term represents a spherical mass distribution,

$$\rho_S(r,t) \equiv \frac{H(r,t)}{\Phi^2} \tag{12}$$

We want now to find the loci of null Δρ. They are given by

$$P(S_1-H\Phi_1) = P_1(3S-\Phi H) \tag{13}$$

On the other hand, using (5) and (8) we can write for P

$$2rP\sin^2\theta'/2 = r\Sigma + Dz' + 2fx' - 2gy' \tag{14}$$

where $\Sigma = a + c \neq 0$, $D = a - c$, and (x', y', z') are quasi- Cartesian coordinates defined by

$$\begin{aligned} x' &= r\sin\theta'\cos\psi' \\ y' &= r\sin\theta'\sin\psi' \\ z' &= r\cos\theta' \end{aligned} \tag{15}$$

So, with (14-15) in (13), we have

$$(S_1-H\Phi_1)\{Dz'+2fx'-2gy'\} + (H\Phi-3S)\{D_1z'+2f_1x'-2g_1y'\}$$

$$+ r\{(S_1-H\Phi_1)\Sigma + (H\Phi-3S)\Sigma_1\} = 0 \tag{16}$$

which we can identify as the equation of a 3-surface in spacetime. It is now clear the role of $H(r,t)$. It gives the interception of this 3-

surface with the coordinate lines. Since we do not lose generality, we can choose a coordinace system with origin on this 3-surface, demanding that

$$(S_1 - H\Phi_1)\Sigma = \Sigma_1 (3S - H\Phi) \tag{17}$$

There is a theorern demonstrated by Szekeres[2] which says that $\Phi(r,t) = \Phi(r)\Phi(t)$ would reduce Szekere' metric to a Friedmann model. Besides that, we are interested only on open and infinite planewiçe surfaces. Therefore, we can say that

$$(\Phi,\Sigma) \neq 0, \quad \text{and} \quad 3S\Phi_1 \neq S_1\Phi \tag{18}$$

With (18) in (17) and (16), we have

$$\rho_S = \frac{H(r,t)}{\Phi^2} = \frac{\Sigma S_1 - 3SC_1}{\Phi^2 (\Phi,\Sigma)} \tag{19}$$

$$\Delta\rho = \frac{(3S\Phi_1 - S_1\Phi)(\Sigma,P)}{\Phi^2 (\Phi,\Sigma)(\Phi,P)} \tag{20}$$

and the Ap = 0 3-surface is defined by

$$(\Sigma,P) = 0, \quad \text{or} \quad 2(\Sigma,f)x' - 2(\Sigma,g)y' + (\Sigma,D)z' = 0 \tag{21}$$

The coeficients of x', y' and z' in (21) are regular functions of r. If they were constant we would have just an Euclidean hyperplane; being r-dependent they just create circular ripples on it.

We want now to rotate our coordinates lines:

$$\begin{aligned} x' &= \cos\mu\{\cos\nu x - \sin\nu(\sin\eta y + \cos\eta z)\} - \sin\mu(\cos q y - \sin\eta z) \\ y' &= \sin\mu\{\cos\nu x - \sin\nu(\sin\eta y + \cos\eta z)\} + \cos\mu(\cos\eta y - \sin\eta z) \\ z' &= \sin\nu x + \cos\nu(\sin\eta y + \cos\eta z) \end{aligned} \tag{22}$$

where

$$\tan\mu = \frac{(D,\Sigma)}{(f,\Sigma)} \quad ; \quad \tan\nu = \frac{(g,\Sigma)}{\{(a,c)^2 + (f,\Sigma)^2\}^{1/2}} \quad ; \quad \tan\eta = \frac{4gL\Sigma - g_1\Sigma_1}{2m(f,D)} \tag{23}$$

and

$$a^2L = (f,a)^2 + (g,a)^2 + (a_1/2)^2 \; ; \; m^2 = (f,\Sigma)^2 + (g,\Sigma)^2 + (a,c)^2 \neq 0 \tag{24}$$

With these new coordinates (x,y,z), (21) becomes

$$S_S : x = 0 \tag{25}$$

So, S_S becomes the $x = 0$ coordinates hypersurface. It is time-independent and there is only one such a $\Delta\rho = 0$ hypersurface. For a fixed time t, it reduces to a spacelike 2-surface (quasi-plane). From (20) and (22) we have for $\Delta\rho$, written in terms of these new coordinates

$$\Delta\rho(t,r,\xi,\bar{\xi}) = -x \frac{4nm^2(3S_1\Phi - S_1\Phi)}{(\Phi,\Sigma)^2\Phi^2\{x(\Phi_1\Sigma_1 - 4L\Sigma\Phi)n + (\Phi,\Sigma)(pz+2rmn)\}} \tag{26}$$

where

$$n^2 = (f,\Sigma)^2 + (a,c)^2 \; ; \; p^2 = (g_1\Sigma_1 - 4gL\Sigma)^2 + 4m^2(f,D)^2 \tag{27}$$

It becomes clear that

$$\Delta\rho(x=0,y,z,t) = 0 \tag{28}$$

and

$$\mathrm{Sign}\{\Delta\rho(x,y,z,t)\} = \mathrm{Sign}(x) \tag{29}$$

Therefore, we can say that S_S is a sypersurface of "quasi-antisymmetry" of $\Delta\rho$. The word "quasi" is necessary because the denominator in the right hand side of (26) is not an even function of X. Again $\Phi_1\Sigma_1 - 4L\Sigma\Phi \neq 0$. Equations (28) and (29) reveal the dipole-like character of $\Delta\rho$.

On the other hand, from (26) we conclude that $\Delta\rho$ is an even function of y. Therefore, we can confirm our next statement:

"There is a time-independent hypersurface of mirror symmetry:

$$S_M : y = 0 \tag{30}$$

$$\rho(x,y,z,t) = \rho(x,-y,z,t) \tag{31}$$

For a fixed t, S_M reduces to a spacelike 2-surface."

The existence of S_M could have been inferred from the invariance of (1) under the following transformation, for a fixed r,

$$\xi \to \xi' = \frac{\bar{b}}{b}\ \bar{\xi} = e^{-2i\alpha}\ \bar{\xi} \tag{32}$$

where α is the argument of the complex function $b = f + ig$, and $\bar{b}$ its complex conjugate. With (8) we can see that (32) corresponds to a reflection

$$\psi + \alpha(r) \to -\{\psi + \alpha(r)\} \tag{33}$$

4. CONCLUDING REMARKS

We have found some new symmetries for the Szekeres quasi-spherícal solutions. The fact that they do not have any Killing vector does not preclude the presence of a still very much symmetric configuration. The mass density ρ is a superposition of two mass distributions, ρ_S and $\Delta\rho$, a spherical and a dipole-like distribution, respectively. A mass distribution so symmetrical and so simple that it should not surprise us anymore that it does not radiate. A more detailed analysis of the mass configuration described by ρ can substantially help in this understanding[7].

The existence of the two time-independent hypersurfaces S_M and S_S, sends light on the spacetime configuration of ρ. S_M being a "quasi-antisymmetry" hypersurface of $\Delta\rho$ reveals its dipole-like character. The arbitrary functions a, c, f, and g of SQSS are indeed director cosines of S_M and S_S. With the knowledge of these symmetries it is possible to elaborate a mechanical model of mass distribution having the same dynamics and symmetries of the SQSS. Then the physical picture of the SQSS will be very clear. A Newtonian model of this sort is being completed and will be presented and discussed in a subsequent paper[7].

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Resumo

Sabe-se que as distribuições de massa das soluções quase-esféricas de Szekeres podem ser reduzidas a um monopólo mais um dipólo, em qualquer uma, mas apenas uma, camada esférica r=const. e t=const.. Mostra-se que isso é globalmente verdadeiro para toda hypersuperfície t = const..