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Resolution of Hydrodynamical Equations for Transverse Expansions

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Abstract The three-dimensional hydrodynamical expansion is treated with a method similar to that of Milekhin, but more explicit. Althouqh in the final staqe we have to appeal to numerical calculation, the partial differential equations governing the transverse expansions are treated without transforming them into ordinary equations with an introduction of averaged quantities. The present paper is concerned with the formalism and we will report the numerical results in another paper.

1. INTRODUCTION

The hydrodynamical model for high-energy multiparticle production has been proposed by Landau[1] a long time ago and revived by some modern researchers, under a more current point of view, which has been acquired through recent progress in particle physics[2]. Due to their large mathematical complexity, until quite recently, the complete hydrodynamical equations have not been solved except for a very important case of one-dimensional motion for which Khalatnikov found an elegant exact solution[3]. That this solution is approximately valid in actual high-energy multiparticle production (evidently here we are not questioning the validity of the hydrodynamical model itself) follows from the flatness of the initial fluid due to the Lorentz contraction of the incident particles, so that the expansion occurs mainly in the incident direction. However, since the transverse dirnensions of such a state of fluid are finite, although large, there certainly exists a transverse expansion and there are indeed some empirical evidences of this phenomenon. Thus, in a previous paper one of the authors[4] has shown that within the framework of hydrodynamical description, the observed flattening of the large-p, inclusive pion distribution,

$$E\,\frac{d\sigma}{d\vec{p}}\bigg|_{\pi/2}\,,$$

with energy increase might well be attributed to the transverse expancriticise the use of hydrodynamics in processes such as hadron -hadron collisions, but others advocate its use even without local thermal equilibrium. One of the defenders of the last position is P. Carruthers who says "hydrodynamic behavior may exist without thermodynamic equilibrium"[15]. He argues that the local thermal equilibrium is not a pre-requisit to the use of collective variables, so formal hydrodynamic structures may exist even in the absence of this equilibrium and could provide useful information. In a recent paper[16], B.Lukács and K.Martinás have shown how to extend the thermodynamic formalism to situations where the distribution function deviates from equilibrium in momentum space. They conclude that the results are compatible with continuum mechanics. We accept these opinions in the present paper for our hydrodynamical study of hadron-hadron collisions.

In what follows, we present the method of resolution by starting from the choice of the coordinate system given in the next section. In sec. 3, we write down the equations of motion fortransverse flow in terms of this coordinate system and discuss the boundary conditions. In sec. 4, these equations are solved both in the "trivial" as well as in the "non-trivial" regions by reducing the equations to the canonical form. Contrary to the case of longitudinal expansion, the trivial region in the transverse expansion is much larger due to the initial dimensions R >> A and so much more important in the latter case than in the former. In sec. 5, we explain how to compute the physically observable quantities such as the inclusive distribution $E\frac{d\sigma}{d\vec{p}}$ from the knowledge of the solution of the hydrodynamical equations obtained above. We give additional remarks in sec. 6 and some mathematical details are gathered in the Appendices.

2. COORDINATE SYSTEM

The object whose expansion we would like to study is a flat disc of thickness 2R, radius R >> R, initial temperature $T_0 >> T_d$, where T_d is the temperature at which the dissociation into the final particles takes place. The expansion is assumed to have axial as well as forward-backward symmetry, just for the sake of simplicity. This is a quite natural assumption considering the large-cluster model we have been studying[12,5,7,13]. So, in the center-of-mass system of the fireball, the four velocity may conveniently be parametrized in terms of the rapidity variables (α,ξ) as

$$\bar{u}^{\mu}(\bar{x}) = (\mathrm{ch}\alpha\,\mathrm{ch}\xi,\ \mathrm{sh}\alpha\,\mathrm{ch}\xi,\ \mathrm{sh}\xi\cos\phi,\ \mathrm{sh}\xi\,\mathrm{sen}\phi), \qquad (2.1)$$

where ϕ is the azimutal angle around the symmetry $(x\text{-})$ axis. The equations of relativistic hydrodynamics are[1]

$$\partial_{\mu} T^{\mu\nu} = 0 \;, \qquad (2.2)$$

where

$$T^{\mu\nu} = (\varepsilon+p)\,u^{\mu}u^{\nu} - pg^{\mu\nu} \quad \text{and}$$
$$p = c_0^2\,\varepsilon \;, \quad (c_0 = \text{sound velocity}) \qquad (2.3)$$

which have been exactly solved by Khalatnikov in the case of the. one--dimensional expansion. If

$$\zeta^2 - c_0^2\alpha^2 \gg 1 \quad \text{and}$$
$$\zeta^2 \gg \alpha^2, \qquad (2.4)$$

where

$$\zeta = \log \frac{T}{T_0} \;, \qquad (2.5)$$

his solution may be approximately written as

$$a = \frac{1}{2} \log \frac{t+x}{t-x} \qquad (2.6)$$

$$\zeta \cong -\frac{1+c_0^2}{4} \log \frac{t^2-x^2}{\Delta^2} + \frac{1-c_0^2}{4} \left[\log^2 \frac{t^2-x^2}{\Delta^2} - \log^2 \frac{t+x}{t-x}\right]^{1/2} \qquad (2.7)$$

where

$$\Delta = \sqrt{\frac{1-c_0^2}{\pi}}\; \ell \;. \qquad (2.8)$$

If $a \ll \log$, A we may rewrite (2.7) as

$$\zeta \cong - c_0^2 \log \sqrt{t^2-x^2} \;. \qquad (2.9)$$

Here, eq. (2.6) and (2.9) appear as the solution of (2.2), showing an approximate scale invariance. In ref. 11, the scale invariance is imposed as an externa1 condition instead.

Now, in accordance with the Milekhin's method, we introduce the following system of coordinates which is suggested by the solution above for the one-dimensional motion and will prove to be useful in solving three-dimensional problems:

$$\begin{cases} \tau = \sqrt{t^2-x^2} \quad , \; \alpha_0 = \text{th}^{-1}\dfrac{x}{t} \quad , \\ r = \sqrt{y^2+z^2} \quad , \; \phi = \tan^{-1}\dfrac{z}{y} \; . \end{cases} \tag{2.10}$$

in terms of which we have

$$\begin{cases} t = \tau \text{ ch } \alpha_0 \; , \quad x = \tau \text{ sh } \alpha_0 \; , \\ y = r \cos\phi \; , \quad z = r \sin\phi \; . \end{cases} \tag{2.11}$$

In (2.10), a, represents the rapidity of a fluid element in the absence of the transve'rse expansion and τ is the corresponding proper time. The introduction of these variables reflects our expectation that theradial motion is much smaller than the longitudinal one and so does not modify the latter by any considerable amount. Actually, we will assume, **a** = **a,** in the derivation of the radial equations below (3.2). The metric tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ in the new coordinate system are written as

$$\begin{cases} g^{00} = 1 \; , \quad g^{11} = -\dfrac{1}{\tau^2} \quad , \quad g^{22} = -1 \quad , \quad g^{33} = -\dfrac{1}{r^2} \\ \text{and} \qquad g^{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \end{cases} \tag{2.12}$$

and

$$\begin{cases} g_{00} = 1 \quad , \quad g_{11} = -\tau^2 \quad , \quad g_{22} = -1 \quad , \; g_{33} = -r^2 \\ \text{and} \qquad g_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \quad . \end{cases} \tag{2.13}$$

So,

$$g = \det|g_{\mu\nu}| = -\tau^2 r^2 . \tag{2.14}$$

Let us now rewrite the four velocity given by (2.1) in the new coordinate system. We have

$$
\begin{cases}
u^{\mu}(x) = \left(\mathrm{ch}(\alpha-\alpha_0)\,\mathrm{ch}\,\xi,\ \dfrac{\mathrm{sh}(\alpha-\alpha_0)}{\tau}\,\mathrm{ch}\,\xi,\ \mathrm{sh}\,\xi,\ 0\right) \\
\text{and} \\
u_{\mu}(x) = g_{\mu\nu}u^{\nu} = \left(\mathrm{ch}(\alpha-\alpha_0)\,\mathrm{ch}\,\xi,\ -\tau\,\mathrm{sh}(\alpha-\alpha_0)\,\mathrm{ch}\,\xi,\ -\mathrm{sh}\,\xi,\ 0\right).
\end{cases}
\qquad (2.15)
$$

3. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Equations of Motion

In curvilinear coordinates, the equations of hydrodynamics (2.2) must be rewritten by replacing the derivatives which appear there by the covariant derivatives. More explicitly, the generalization of (2.2) is

$$
\frac{1}{\sqrt{-g}}\,\frac{\partial(\sqrt{-g}\,T^{\mu}_{\nu})}{\partial x^{\mu}} - \frac{1}{2}\,\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\,T^{\mu\lambda} = 0\,. \qquad (3.1)
$$

By putting eqs. (2.3) into this equation and using thermodynamical relations, we may rewrite it as

$$
\frac{\partial\zeta}{\partial x^{\nu}} = -\frac{c_0^2 u_{\nu}}{\sqrt{-g}}\,\frac{\partial}{\partial x^{\mu}}\left[\sqrt{-g}\,u^{\mu}\right] + u^{\mu}\,\frac{\partial u_{\nu}}{\partial x^{\mu}} - \frac{u^{\mu}u^{\lambda}}{2}\,\frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\,. \qquad (3.2)
$$

In the specific case of our interest, if we assume

$$
\alpha = \alpha_0 \qquad (3.3)
$$

as discussed before, the introduction of (2.10), (2.13-15) into eq.(3.2) leads to

$$
\begin{cases}
\dfrac{\partial\zeta}{\partial\tau} = -\dfrac{c_0^2\,\mathrm{ch}^2\xi}{\tau} - \dfrac{c_0^2\,\mathrm{sh}\xi\,\mathrm{ch}\xi}{r} + (1-c_0^2)\,\mathrm{sh}\xi\,\mathrm{ch}\xi\,\dfrac{\partial\xi}{\partial\tau} + (\mathrm{sh}^2\xi - c_0^2\,\mathrm{ch}^2\xi)\,\dfrac{\partial\xi}{\partial r}\,, \\
\dfrac{\partial\zeta}{\partial\alpha_0} = 0\,, \\
\dfrac{\partial\zeta}{\partial r} = \dfrac{c_0^2\,\mathrm{sh}\xi\,\mathrm{ch}\xi}{r} + \dfrac{c_0^2\,\mathrm{sh}^2\xi}{r} - (\mathrm{ch}^2\xi - c_0^2\,\mathrm{sh}^2\xi)\,\dfrac{\partial\xi}{\partial\tau} - (1-c_0^2)\,\mathrm{sh}\xi\,\mathrm{ch}\xi\,\dfrac{\partial\xi}{\partial r}\,, \\
\dfrac{\partial\zeta}{\partial\phi} = 0\,.
\end{cases}
\qquad (3.4)
$$

The second (of. eqs.) (3.4) is actually not entirely compatible with (2.6,7) in the one-dimensional flow limit and the origin of this inconsistency is traced back to our assumption (3.3). However, the main part of the entropy is concentrated in the region $\alpha_0 \sim 1$ and one may show that there

$$\alpha - \alpha_0 \sim \frac{1}{\log \frac{\tau}{\Delta}} \qquad (3.5)$$

when $\log \frac{\tau}{\Delta} >> \alpha_0$. Since we will mainly be concerned with this region, we may neglect the small difference given above.

The system of equations (3.4) represents an enormous simplification compared with (3.1). We have now a system of two equations in two independent variables (τ, r) and the unknown functions are ζ and ξ. Thus, the transverse motions have been separated from the longitudinal ones. In order to solve this system, we must now specify the boundary conditions.

Boundary Conditions

All our approximation scheme is based on a fundamental assumption, namely, as far as the central region of the disc is concerned and for $t \lesssim R$ the one-dimensional solution is a good description of the phenomena and the deviation from this behaviour appears first at the boundary of the disc and it propagates from outside towards the center of the disc. This is Milekhin's picture and, in our opinion, it is both an intuitive and correct image of the phenomena. Accordingly, the fluid in three-dimensional flow would be bounded by the surface

$$r = R + \tau \qquad (3.6)$$

on the vacuum side and would contact the one-dimensional flow region on

$$r = R - c_0 \tau \qquad (3.7)$$

(see appendix A for the derivation of this equation, even though it is more or less self-evident). We illustrate this picture by figs. 1 and 2.

However, Yotsuyanagi in his paper[9] gives another version for the boundaries. His argument goes as follows. At the moment when the fireball is formed, a weak discontinuity would leave the initial surface and go to the inside of the fluid. At a very early stage, $t \sim \frac{\Delta}{c_0}$, this discontinuity would reach the symmetry planes = 0 and vanish so that it could not be the surface of separation between the region of one-dimen-

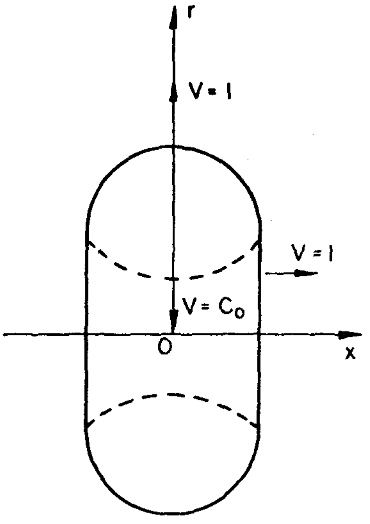

Fig.1 - Contour of the fluid as seen in the center-of-mass frame of M, at an instant $t < R/c_0$. The brokeii lines indicate the boundary between the three-dimensional-flow region (outside) and the one-dimensional-flow region (inside).

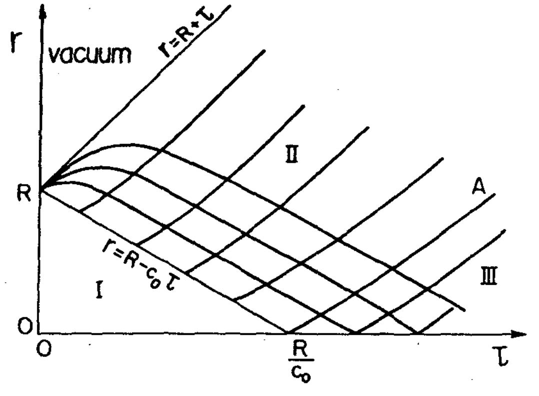

Fig.2 - (Proper-) time evolution of the boundaries among the vacuum, the "trivial" three-dimensional-flow region (II), non-trivial three-dimensional-flowregion (III) and the one-dimensional-flow region (I). The two families of characteristics given by eqs. (4.7) are also schematically shown.

sional and three-dimensional flow. Notice that if one assumes a very flat spheroidal fireball, as he did, and observe the motion of the above mentioned discontinuity at 90° in the center of mass frame, eq. (3.7) would describe its motion. His proposal is to take the surface of separation as starting from the origin, $r = 0$ at $t \sim \frac{\Delta}{c_0}$, and travelling outward in the transverse directions as t increases. We cannot, nevertheless, agree with this view because we think it is in contradiction with the very basic assumption which lies under this kind of approximation and which has been stated at the top of this subsection. For, according to his version, all the fluid would be in three-dimensional flow at the beginning (here we neglect a very small interval of time $t \sim \Delta$). As t increases, the purely one-dimensional-flow region would appear behind a surface of discontinuity and would increase indefinitely.

In our opinion, $t \lesssim \Delta << R$ is a rather smal interval of time so that we do not even know whether we can justify treating our fireball using eq. (3.4)*. We would restrict their use to $t \gtrsim \Delta$. Observe that the weak discontinuity mentioned above does not reach the center of the spheroid in the transverse direction but in the longitudinal direction (because $\Delta << R$) and along the axis, so when it reaches the symmetry plane, a new discontinuity begins to travel outward in the longitudinal direction. It is clear that in this treatment, a small transverse inhomogeneity is, as usual, completely neglected and within this approximation, the central fluid will continue to expand longitudinally until the surface given by eq. (3.7) reaches the fluid element in question. Note that what we are considering is not a discontinuity in a strict mathematical sense but has a certain width $\sim \Delta$, which is neglected so as to simplify the calculations in this treatment

The boundary conditions of our problem may now be written

$$\xi = \infty \quad \text{and} \quad \zeta = -\infty \quad , \tag{3.8}$$

when $r = R + \tau$, and

$$\xi = 0 \quad \text{and} \quad \zeta = - c^2 \log \frac{\tau}{\Delta} \tag{3.9}$$

---

*As mentioned in the *Introduction*, there are criticisms to applying hydrodynamics even at $t \sim R$.

when $r = R - c_0\tau$ $(\tau \leqq \frac{R}{c_0})$. Along the axis and for $\tau \geqq \frac{R}{c_0}$, we have

$$\xi = 0 \ , \qquad (3.10)$$

which will see below in sec. 4 implies

$$\frac{\partial\zeta}{\partial r} = 0 \ . \qquad (3.11)$$

As mentioned in the Introduction, the initial conditions are not clearly stated in ref. 10, but it seems that there the authors assume that the transverse expansion starts only after $\tau = \tau_0 \sim 1$ fm, when the materialization would occur and, at that instant, ξ and ζ would approximately be given by equations (3.8) and (3.9) at $\tau \sim \tau_0$ as they are in our case.

4. RESOLUTION OF HYDRODYNAMICAL EQUATIONS

Reduction to Canonical Forms

We are now ready to solve the system of equations (3.4), satisfying the boundary conditions specified in the last section. This will be done by the method of characteristics. First of all, we write

$$\zeta = \zeta_\perp - c^2 \log \frac{\tau}{\Delta} \ . \qquad (4.1)$$

In this expression the purely longitudinal contribution as given by eq. (2.9) is separated from 5, . The new variable $\zeta_\perp$ satisfies a boundary condition

$$\zeta_\perp = 0 \quad \text{on} \quad r = R - c_0\tau \ , \qquad (4.2)$$

which replaces the second of eqs. (3.9).

We how define the following combinations of $\zeta_\perp$ and ξ

$$\psi = \zeta_\perp + c_0\,\xi \quad \text{and} \quad \phi = \zeta_\perp - c_0\,\xi \ , \qquad (4.3)$$

so that

$$\zeta_\perp = \frac{1}{2}(\psi+\phi) \quad \text{and} \quad \xi = \frac{1}{2c_0}(\psi-\phi) \qquad (4.4)$$

With the help of eqs. (4.1) through (4.4) and after an appropriate recombination of terms, we may rewrite the system of eqs. (3.4) in the canonical form as

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} + \dfrac{v_\perp + c_0}{1 + c_0 v_\perp}\dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2 v_\perp}{1 + c_0 v_\perp}\left[\dfrac{1}{r} - \dfrac{c_0}{\tau}\right] = 0 \ , \\[2ex] \dfrac{\partial\phi}{\partial\tau} + \dfrac{v_\perp - c_0}{1 - c_0 v_\perp}\dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2 v_\perp}{1 - c_0 v_\perp}\left[\dfrac{1}{r} + \dfrac{c_0}{\tau}\right] = 0 \ , \end{cases} \qquad (4.5)$$

where

$$v_{\perp} = \text{th}\,\frac{\psi-\phi}{2c_0} \quad . \tag{4.6}$$

This is a hyperbolic system of quasi-linear equations. It has the following family of characteristics, which we illustrate in fig. 2:

$$\begin{cases} \text{(a)} \quad \dfrac{dr}{d\tau} = \dfrac{v_{\perp} + c_0}{1 + c_0 v_{\perp}} \,, \\ \\ \text{(b)} \quad \dfrac{dr}{d\tau} = \dfrac{v_{\perp} - c_0}{1 - c_0 v_{\perp}} \end{cases} \tag{4.7}$$

These equations are precisely (A.4) of appendix A and may be obtained directly from (4.5) using the same procedure (note that the changes of variables, (4.1) and (4.4) do not affect the results). From (4.5), it follows that, along each family (a) and (b) given by (4.7), we have

$$\text{along (a)} \; : \; d\psi = \frac{c_0^2 v_{\perp}}{1 + c_0 v_{\perp}} \left[\frac{c_0}{\tau} - \frac{1}{r}\right] d\tau \,, \tag{4.8}$$

$$\text{along (b)} \; : \; d\phi = \frac{-c_0^2 v_{\perp}}{1 - c_0 v_{\perp}} \left[\frac{c_0}{\tau} + \frac{1}{r}\right] d\tau \,. \tag{4.9}$$

Therefore, our procedure in solving the transverse part of the hydrodynamical equations is to integrate (4.8,9) along the characteristics (4.7) using the boundary conditions (3.8-10) and (4.2). In principle this is possible to be done numerically. However, as it will be explained below, this procedure will present some difficulties associated with the boundary conditions, requiring a special care.

Ultrarelastivístic Approximation

Let us first consider the integration in the trivial region or region II of fig. 2. As far as the ψ-integration is concerned, everything goes as indicated above since its boundary value is well defined on the curve (3.7), where (4.8) is regular except at $\tau = 0$. On the contrary, the ϕ-integration is troublesome because, as shown in Fig. 2, all the characteristics (b) start at ($\tau = 0$, $r = R$), where the corresponding differential $d\phi$ is singular. They continue beyond the region II and end at (or reflect from) the straight line $r = 0$, $\tau > \frac{R}{c_0}$, where we do not have the boundary value of ϕ. Thus, we cannot integrate in ϕ neither starting from $\tau=0$, nor backward starting from $r=0$. Some other procedure is required to treat it. In so doing, we make use of the circumstances that the curves (b) pass through the ultrarelativistic region at the beginning of the expansion as expressed by (4.7) and illustrated in fig. 2. So we try to find the ultrarelativistic solution of (4.5) to describe the initial flow. By putting $v_r \to 1$, we may rewrite (4.5) as

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} + \dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} - \dfrac{c_0}{\tau}\right] \cong 0 \quad , \\ \dfrac{\partial\phi}{\partial\tau} + \dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2}{1-c_0}\left[\dfrac{1}{r} + \dfrac{c_0}{\tau}\right] \cong 0 \quad , \end{cases} \tag{4.10}$$

that is, in this limit the equations become decoupled in ψ and ϕ, and so are easily integrated. The solution which satisfies the boundary conditions

$$\psi = \phi = 0 \quad \text{on} \quad r = R - c_0\tau \tag{4.11}$$

which follow from (3.9), (4.2) and (4.3) are, as will be computed in appendix B,

$$\begin{cases} \psi \cong \dfrac{c_0^2}{1+c_0} \log\left[\dfrac{R-c_0(\tau-r)}{(1+c_0)r}\left(\dfrac{(1+c_0)\tau}{R+\tau-r}\right)^{c_0}\right] \quad , \\ \phi \cong \dfrac{c_0^2}{1-c_0} \log\left[\dfrac{R-c_0(\tau-r)}{(1+c_0)r}\left(\dfrac{R+\tau-r}{(1+c_0)\tau}\right)^{c_0}\right] \quad . \end{cases} \tag{4.12}$$

Strictly speaking, since eqs. (4.10) are ultrarelativistic, they are not valid close to the curve (3.7) where $\xi \cong 0$. So imposing the condition (4.11) to their solution is actually not correct. It is however a good approximation which we will take as the boundary conditions close to (3.6), replacing (3.8). Anyhow, from the physical point of view, it is intuitive that in solving the system (4.5), influences coming from the entire boundary (3.7) is much more important than those coming from the $\tau=0$ region, or in other words coming from the vacuum side boundary, thus justifying our approximation.

Boundary Condition on $r=0$, $\tau > R/c_0$

In the last subsection, we have explained how to solve eqs. (4.5) in the trivial region. Now, we shall turn our attention to the region III, where the characteristics (b) arriving at $r=0$ (τ axis) suffer a reflection and leave $r=0$ as characteristics (a). This domain has two boundaries, namely, one which separates it from region II (curve A), where ψ and ϕ are in principle known and the other which is the straight line $r=0$, $\tau > R/c_0$, where $\xi = 0$, according to (3.10), but we do not know which is the value of $\zeta_\perp$. In terms of ψ and ϕ, this means that we know a particular combination of these functions there, but not ψ itself whose value is needed there in order to carry the ψ-integration out.

For treating this problem, we rewrite (4.5) for the neighbouring points of the τ-axis ($r \cong 0$), where $V_r \cong \xi \cong 0$:

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} + c_0 \dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2\xi}{r} \cong 0 \quad , \\[2ex] \dfrac{\partial\phi}{\partial\tau} - c_0 \dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2\xi}{r} \cong 0 \quad . \end{cases} \tag{4.13}$$

Now, along τ-axis ($r = 0$, $\tau > R/c_0$), it follows from (3.10) that

$$\frac{\partial\xi}{\partial\tau} = 0 \quad , \tag{4.14}$$

or with the help of (4.4), this is rewritten as

$$\begin{cases} \psi = \phi \quad , \\[1ex] \dfrac{\partial\psi}{\partial\tau} = \dfrac{\partial\phi}{\partial\tau} \qquad (\text{for } r = 0,\ \tau > R/c_0) \quad . \end{cases} \tag{4.15}$$

By subtracting eqs. (4.13) from each other and using (4.15) we also have

$$\frac{\partial\psi}{\partial r} = - \frac{\partial\phi}{\partial r} \quad , \quad (r = 0,\ \tau > R/c_0) \tag{4.16}$$

Now, making use of (3.10) and (4.14), we expand ξ in (4.13) in a power series around a point $(\tau,0)$:

$$\xi(\tau,r) \cong \frac{\partial\xi}{\partial r}(\tau,0)\, r \cong \frac{1}{c_0} \frac{\partial\psi}{\partial r}(\tau,0)\, r \quad , \tag{4.17}$$

valid up to the first order. In the last step, we have used (4.4) and

and (4.16). We add eqs. (4.13) each other and with the help of (4.15) and (4.16) we finally obtain

$$\frac{\partial\psi}{\partial\tau} = -2c_0 \frac{\partial\psi}{\partial r} = 2c_0 \frac{\partial\phi}{\partial r} \qquad (4.18)$$

(for $r = 0$, $\tau > R/c_0$). This is a relation between the time variation of ψ and the space variation of ψ or ϕ, so once ψ or ϕ is given for a fixed τ and in a small neighbourhood of $r = 0$, it allows us to compute its value in the vicinity along τ-axis. Thus, (4.18) is our boundary condition to be used in ψ-integration.

5. TRANSVERSE RAPIDITY DISTRIBUTION OF THE HADRONIC FLUID AND THE INCLUSIVE PARTICLE DISTRIBUTIONS

In the last section, we have shown how ,to solve the hydrodynamical equations for transverse flows and to obtain ψ and ϕ, and so ζ and ξ by means of eqs. (4.4) and (4.1), as functions of τ and r (in our approximation, the solution is independent of a, and of the azimutal angle $). Now, let us obtain the momentum distribution of particles to allow for a comparison with the experimental data. Allthough it is not indispensable for this end, first we will derive the transverse rapidity distribution of the hadronic fluid and then the inclusive particle distributions.

Transverse Rapidity Distributions of the Hadronic Fluid

In the original version of hydrodynamical model, the initially hot, pancake-shaped blob expands until each part of it reaches some critical temperature, called dissociation temperature T_d, after which particles appear as independent, non-interacting objects. In terns of a current view, we would initially have a hot quark-gluon plasma which would suffer a phase transition as the fluid expands and the temperature decreases. In any case, the final particle distribution as well as the rapidity distribution* of the hadronic fluid must be calculated on

* We observe, however, that in the expansion of the quark-gluon plasma, an additional complication appears which is related to what happenswith the system during the phase transition. It will surely continue to expand but, as far as we know, this problem has not yet been treated in the literature.

the hypersurface where $T = T_d$. We have, thus

$$dN = n\, u^\mu \sqrt{-g}\, d\sigma_\mu \Big|_{T=T_d} , \qquad (5.1)$$

where n is the particle density and $d\sigma_\mu$ are the components of the surface element (including the normal direction). The meaning of eq. (5.1) is clear, especially if one uses the Cartesian coordinates when $\sqrt{-g}$ will be reduced to 1. In our coordinate system, u^μ are given by (2.15) and, with the approximation (3.3), they become

$$u^\mu = (\mathrm{ch}\xi, 0, \mathrm{sh}\xi, 0) . \qquad (5.2)$$

By using the usual procedure*, we obtain for $d\sigma_\mu$ in the same approximation

$$d\sigma_\mu = da_\| \, d\phi\, (-dr, 0, d\tau, 0) . \qquad (5.3)$$

So, by putting (5.2), (5.3) and (2.14) into (5.1) and considering the axial symmetry of the problem, we have

$$dN = 2\pi\, n\, \tau r\, (\mathrm{sh}\xi d\tau - \mathrm{ch}\xi dr)\, d\alpha_0 \Big|_{T=T_d} , \qquad (5.4)$$

where the signs have been chosen so that the (τ, r) integration starts from the point $(0, R)$. It is convenient to rewrite (5.4) in terms of $d\xi$ instead of $d\tau$ and dr, because we are interested in the ξ distribution. Then,

$$dN = -2\pi\, n \frac{\tau r}{\left| \frac{\partial(\xi,\zeta)}{\partial(\tau,r)} \right|} \left(\mathrm{sh}\xi \frac{\partial\zeta}{\partial r} + \mathrm{ch}\xi \frac{\partial\zeta}{\partial\tau}\right) \Big|_{T=T_d} d\xi d\alpha , \qquad (5.5)$$

where $\frac{\partial(\xi,\zeta)}{\partial(\tau,r)}$ is Jacobian of the transformation $(\xi,\zeta) \to (\tau, r)$ and we have also replaced $a_\|$ by α with the help of eq. (3.3). It is clear that the distribution given above is independent of the longitudinal rapidity α because of our approximation (3.3). A more correct α-distribution would indeed be an approximate Gaussian as it follows from eq. (2.6) and 2.7).

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* See for instance ref.17.

Inclusive Particle Distributions (for Fixed M)

In order to obtain the (semi-) inclusive particle distribution, one must further consider the thermal fluctuation at $T = T_d$. The correct receipt for this, consistent with energy conservation, has been obtained by Cooper and Frye[18] starting from the transport theory of a relativistic gas and it reads

$$E \frac{dN}{d\vec{p}} = \frac{w}{(2\pi)^3} \int \frac{p^\mu \sqrt{-g}\, d\sigma_\mu}{\exp(\bar{E}/T_d) \pm 1}, \tag{5.6}$$

where the integration is taken over the hypersurface $T=T_d$, w is the statistical weight and p^μ is the four momentum of the particle to be observed, which may be written in our coordinate system (with $\alpha=\alpha_0$ and using the usual rapidity variables $y_{\shortparallel}$, $y_\perp$) as

$$p^\mu = m\,(\mathrm{ch}(y_{\shortparallel}-\alpha)\,\mathrm{ch}y_\perp\ ,\ \frac{1}{\tau}\,\mathrm{sh}(y_{\shortparallel}-\alpha)\,\mathrm{ch}y_\perp,\ \mathrm{sh}y_\perp,\ 0)\ . \tag{5.7}$$

The proper-frame energy $\bar{E}$ is expressed in terms of the variable at the center-of-mass frame (of M) as

$$\bar{E} = m\left[\mathrm{ch}(y_{\shortparallel}-\alpha)\,\mathrm{ch}y_\perp\,\mathrm{ch}\xi - \mathrm{sh}y_\perp\,\mathrm{sh}\xi\,\cos\phi\right]\ . \tag{5.8}$$

By putting (5.3), (5.7) and (5.8) into (5.6), one obtains

$$E \frac{dN}{d\vec{p}} = \frac{wm}{(2\pi)} \iiint_{T=T_d} \frac{\tau r\left[\mathrm{sh}y_\perp d\tau - \mathrm{ch}(y_{\shortparallel}-\alpha)\,\mathrm{ch}y_\perp\, dr\right] d\alpha d\phi}{\exp\{\frac{m}{T_d}\left[\mathrm{ch}(y_{\shortparallel}-\alpha)\,\mathrm{ch}y_\perp\,\mathrm{ch}\xi - \mathrm{sh}y_\perp\,\mathrm{sh}\xi\,\cos\phi\right]\} \pm 1} \tag{5.9}$$

Despite the a-independence of $d^2N/d\alpha d\xi$ as given by eq. (5.5), the integrand above contains an α-dependence which predominantes over the actual a-dependence of $d^2N/d\alpha d\xi$ as discussed below (5.5) dueto its sharp form.

6. CONCLUDING REMARKS

In the present paper, we have developped an algorithm for solving the hydrodynamical transverse expansion of an initially flat ($A << R$) and hot ($T_0 >> m_\pi$) disc of large mass M and obtained both the rapidity distribution of the fluid and the inclusive particle distributions which emerge from the expansion. Although our aim is applying this prescription first to studying hadron-hadron collisions, it may evidently be used also for nucleus-nucleus collisions.

The basic assumption of the present receipt is the approximate validity of the one-dimensional solution as given by eq. (2.6) [and eq. (2.9)], which allows to separate the transverse from the longitudinal flows. The resulting couple of equations (3.4) for transverse flows are then put in the canonical form (4.5), which allows the integration along the characteristics, eq. (4.7), with the use of eqs. (4.8) and (4.9). This result is always valid as far as the assumption above remains valid.

As for the initial conditions, we have assumed the fluid begins to expand at a time $t \cong A \cong 0$, in accordance to the most orthodox view. If one assumes a model as discussed in ref. 11, we think natural to include also the transverse rapidity distribution of fragments or of ε at $\tau = \tau_0$. Such a distribution would probably be more os less constant over R, but with a surface with a finite thickness which would increase with τ_0 (if $\vec{v}$ = const., as assumed by those authors). Although there is no apparent reason to being so, the expansion for $\tau < \tau_0$ - ' fm, as calculated here may well simulate this thickness-widening effect. This is certainly the case in the one-dimensional approximation to treating nucleus-nucleus collisions in the central rapidity region, when $v = x/t$ is usually assumed and then the source is guessed by using $dN/dy_{\shortparallel}$ for nucleon-nucleon collision.

An explicit numerical computation will be reported in the forthcoming paper[20].

APPENDIX A

In this appendix, we calculate the surface of separation between regions of one-dimensional and three-dimensional flows. Such a surface is given as a characteristic surface of the system of equations (3.4), which describes three-dimensional flows. Following the standard method*, the equation of a characteristic surface is written as

* See for instance ref. 19.

$$F(\tau, r) = 0 \ , \tag{A.1}$$

where F is a function satisfying

$$\begin{vmatrix} \frac{\partial F}{\partial \tau} & -(1-c_0^2)\,\mathrm{sh}\xi\ \mathrm{ch}\xi\ \frac{\partial F}{\partial \tau} - (\mathrm{sh}^2\xi - c_0^2\ \mathrm{ch}^2\xi)\ \frac{\partial F}{\partial r} \\ \frac{\partial F}{\partial r} & (\mathrm{ch}^2\xi - c_0^2\ \mathrm{sh}^2\xi)\ \frac{\partial F}{\partial \tau} + (1-c_0^2)\,\mathrm{sh}\xi\ \mathrm{ch}\xi\ \frac{\partial F}{\partial r} \end{vmatrix} = 0 \ . \tag{A.2}$$

By developing this equation and factorizing it, we may rewrite it as

$$\left[(\mathrm{ch}\xi - c_0\ \mathrm{sh}\xi)\ \frac{\partial F}{\partial \tau} + (\mathrm{sh}\xi - c_0\ \mathrm{ch}\xi)\ \frac{\partial F}{\partial r}\right]$$

$$\times \left[(\mathrm{ch}\xi + c_0\ \mathrm{sh}\xi)\ \frac{\partial F}{\partial \tau} + (\mathrm{sh}\xi + c_0\ \mathrm{ch}\xi)\ \frac{\partial F}{\partial r}\right] = 0 \ ,$$

so

$$(\mathrm{ch}\xi \pm c_0\ \mathrm{sh}\xi)\ \frac{\partial F}{\partial \tau} \pm (c_0\ \mathrm{ch}\xi \pm \mathrm{sh}\xi)\ \frac{\partial F}{\partial r} = 0 \ . \tag{A.3}$$

By dividing each of these equations by $\partial F/\partial r$, we obtain

$$\frac{dr}{d\tau} = \pm \frac{c_0 \pm \mathrm{th}\xi}{1 \pm c_0\ \mathrm{th}\xi} \tag{A.4}$$

So, we have two characteristics of eqs. (3.4), which pass through each point. The first of these equations represents a surface which moves outward, whereas the second one corresponds to the one which moves inward (with respect to the fluid element). In sec. 4, the system of eqs. (4.5), which is just another form of (3.4), will be integrated following these characteristics. In the particular problem of surface of separation that we are presently considering, we choose the minus sign in (A.4) and, using the boundary condition we put $\xi = 0$. Taking also the initial condition into account, (A.4) may be easily integrated andgives

$$r = -c_0\tau + R \ , \tag{A.5}$$

which is our eq. (3.7) of sec. 3. Although it is clear enough that the other boundary of the three-dimensional-flow region is given by (3.6), note that it may be obtained in a similar way, by taking the limit $\xi \to \infty$ of (A.4) with the plus sign.

APPENDIX B

Consider the equations

$$\begin{cases} \dfrac{\partial\psi}{\partial\tau} + \dfrac{\partial\psi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} - \dfrac{c_0}{\tau}\right] = 0 \quad , & \text{(B.1)} \\ \dfrac{\partial\phi}{\partial\tau} + \dfrac{\partial\phi}{\partial r} + \dfrac{c_0^2}{1+c_0}\left[\dfrac{1}{r} + \dfrac{c_0}{\tau}\right] = 0 \quad , & \text{(B.2)} \end{cases}$$

wh:ch we would like to solve assuming

$$\psi = \phi = 0 \quad \text{on} \quad r = R - c_0\tau \quad . \qquad \text{(B.3)}$$

Take (5.1) first. Its characteristics as well as its solution ψ satisfy the following system of ordinary equations (in q) :

$$\frac{d\tau}{1} = \frac{dr}{1} = \frac{d\psi}{-\dfrac{c_0^2}{1+c_0}\left(\dfrac{1}{r} - \dfrac{c_0}{\tau}\right)} = dq. \qquad \text{(B.4)}$$

Upon integration, this will give

$$\begin{cases} \tau = q + \tau_0 \ , \\ r = q + r_0 \ , & \text{(B.5)} \\ \psi = - \dfrac{c_0^2}{1+c_0}\left[\log(q+r_0) - c_0 \log(q+\tau_0)\right] + \psi_0 , \end{cases}$$

or eliminating fhe auxiliary variable q and then taking τ as the independent variable

$$\begin{cases} r = \tau - \tau_0 + r_0 \quad , \\ \psi = - \dfrac{c_0^2}{1+c_0} \log \dfrac{r}{\tau^{c_0}} + \psi_0 \quad . \end{cases} \qquad \text{(B.6)}$$

Due to the boundary condition (B.3) , we have

$$\begin{cases} r_0 = R - c_0\tau_0 \quad , \\ \psi = 0 \quad , \quad \text{when} \quad \tau = \tau_0 \ . \end{cases} \qquad \text{(B.7)}$$

From (8.6) and (B.7), it follows

$$\begin{cases} \tau_0 = \dfrac{\tau + R - r}{1 + c_0} , \\ \psi_0 = \dfrac{c_0^2}{1+c_0} \log \dfrac{R - c_0\tau_0}{\tau_0^{c_0}} . \end{cases} \qquad (B.8)$$

Inserting these equations into (B.6) we finally obtain

$$\psi = \frac{c_0^2}{1+c_0} \log \left[\frac{R - c_0(\tau - r)}{(1 + c_0) r} \left(\frac{(1+c_0)\tau}{R + \tau - r} \right)^{c_0} \right] . \qquad (B.9)$$

The integration of (8.2) is entirely similar so we do not repeat it here. The result is

$$\phi = \frac{c_0^2}{1-c_0} \log \left[\frac{R - c_0(\tau - r)}{(1 + c_0) r} \left(\frac{R + \tau - r}{(1 + c_0)\tau} \right)^{c_0} \right] . \qquad (B.10)$$

(8.9) and (8.10) are precisely the ultrarelativistic solution given in section 4.

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Resumo

Trata-se a expansão hidrodinâmica tridimensional com um método semelhante ao de Milekhin, porém mais explícito. Embora no estágio final tenhamos que apelar a um cálculo numérico, as equações diferenciais a derivadas parciais que governam as expansões transversais são tratadas sem transformá-las em equações ordinárias com a introdução de quantidades médias. Trata-se no presente artigo do formalismo e os resultados numéricos serão relatados no próximo trabalho.