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The Schrödinger Equation for Central Power Law Potentials and the Classical Theory of Ordinary Linear Differential Equations of the Second Order

MÁRCIA L. LIMA§

Departamento de Física Teórica, Instituto de Física, UFRJ, Caixa Postal 68528, Rio de Janeiro, 21944, RJ, Brasil

and

JUAN A. MIGNACO

Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, Rio de Janeiro, 22290, RJ, Brasil

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Abstract We show that the rational power law potentials in the two-body radial Schrödinger equations admit a systematic treatment available from the classical theory of ordinary linear differential equations of the second order. The resulting potentials come into families evolved from equations having a fixed number of elementary regular singularities. As a consequence, relations are found and discussed among the several potentials in a family.

1. INTRODUCTION

Historically the development of the theory of differential equations contributed to several fields in classical physics. Conversely, the problems of mechanics, continuous media and electromagnetism were sources of advances and stimulated the work of outstanding mathematicians.

An approach to differential equations from the point of view of analysis and the theory of functions of a complex variable was developed to a high degree during the preceeding century. The classical treatises of Forsythe[1] and Ince[2] witness the level of qualitative and quantitative understanding of the problem, in particular the existence, uniqueness and characterization of the possible solution.

The advent of quantum mechanics with the Schrödinger equation as the differential equation for the eigenvalue problem of the stationary states in two body problems opened some new applications for differential equations. The simplest atomic problem, the hydrogen atom, was

§Deceased on 11th April 1985.

promptly solved exactly with the help of the classical knowledge previously accumulated on the confluent hypergeometric equation and its solutions. Other problems of two body forces were later solved, like the S-wave for the exponential potential. But in fact no important development in the theory of the Schrödinger equation was made underthe stimulus of the pursuit of new solutions until recently.

It is true also that even mathematicians since 1920 abandoned almost completely the analytical approach to differential equations orienting themselves to the study of global geometric or algebraic properties of differential equations.

On the other hand, electronic calculators made increasingly easier to obtain approximate solutions with better precision. In all, everything concurred to a situation where the known solutions of the Schrödinger equation for two body problems are presented essentially as the application of *ad hoc* procedures of almost heuristic value and references are made to some good text in mathematical physics[3-5]. Notice that several texts are not much more modern that Forsythe or Ince.

Two decades ago Bose[6] took anew the developements of Ince in his attempt to obtain a general classification of linear ordinary differential equations of second order. The point made by Ince is that, starting from an equation with a given number of elementary singularities (i.e., singularities where the possible two solutions are given each one by a power series times a power, this power differing by one half between the two solutions) one is able to generate families from the gradual confluence of these singularities. Bose applied this to the Schrödinger equation and looked for the possible Schrudinger equations resulting from an original equation having six elementary singularities. Since the confluent hypergeometric equation is obtained by letting four singularities coalesce at thepoint at infinity and the other two at the origin, he was able to obtain the known cases of the harmonic isotropic oscillator and the Coulomb potential, and the exponential potential as well, with suitable changes of variables. Other interesting cases resulted, and he later extended this study with Lemieux[7] to the case of 8 elementary singularities. They started with the singularities already coalesced pairwise, which produces a differential equation with four regular singularities studied by Heun[8] as an extension of the Riemann problem for the hypergeometric equation (which has only three regular singular points). They considered the consequences of letting these singularities coalesce in several ways. The analogous to the confluent hypergeometric case is here obtained when three regular singularities join at infinity and the remaining is at the origin; it produces a family of potentials including

$$V(x) = V_1 x^{-2} + V_2 x^{-1} + V_3 x + V_4 x^2 \qquad (1)$$

and

$$V(x) = V_1 x^{-2} + V_2 x^2 + V_3 x^4 + V_4 x^6$$

which can be related through a transformation of variables which is by now well known (we shall analyse it in the following sections). The first of these potentials was solved exactly by Singh, Biswas and Datta[9]. Both have been considered in a seris of articles by Znojil[10]. In fact, most of the work done in the area of obtaining exact solutions for the Schrödinger equation ignores the pioneering work by Bose and Lemieux, which was only recalled by Johnson[11].

The physics of elementary particles gave new stimulus to the potential theory of potentials with a positive power law, which are presumed adequate to describe the confining forces between quarks and antiquarks[12].

The study of one dimensional anharmonic oscillators like the one in (1') was also of interest since they present some features which seem to be common to relativistic field theory: the energy levels are not analytic functions of the anharmonic coupling constant, and this prevents for instance the perturbative expansion of energy levels to converge[13]. Padé approximants have been shown to provide the right answer[14] for the x^4 potential. Recently, the solution via a continued fraction for the Green's function has been given by the indian group[9].

Znojil has produced a real leap forward studying in all generality (rational) power law potentials starting from normal and subnormal solutions[10]. He then proposed an extended continued fraction for the Green function in the general case, studied the transformations between families of potentials and gave some systematic classification for them.

In an almost similar approach Rampal and Datta[15] performed a study which cared among other things (as Znojil also did) upon the existence of polynomial solutions and gave some criteria for them to exist.

In this work we show how the analytic procedures developed by Ince apply in the general case for (rational) power law potentials. The essential feature to allow a systematic study is the number, N of elementary singularities of an ordinary linear differential equation of second order (OLDESO). In contrast with Bose and Lemieux, we consider all possible ways of making these singularities to coalesce in higher order ones. This of course allows us to obtain a larger number of families, among which there are some of undoubtedly physical interest.

We further study the set of transformations in the independent variable which transforms members of a given family into one another and how they reflect these properties in the solutions.

With a careful notation we study also the regular solutions of the equations so obtained. We believe that this systematic procedure allows for a unified treatment of all cases known in the literature. Needless to say this work could have been done half a century ago with little changes, if any.

A short version of this work has appeared already[16]. We propose however that this article be as self consistent as possible. The terminology is restated as can be found in the classical treatises[1,2]. The proofs of mathematical statements are not always easy and we omit most of them and refer to the classics.

In section 2 we develop the classical theory of OLDESO in a reduced form, just to allow for the understanding of the basic technology. This, as well as a lot of terminology has phased away with time, and is rarely used in books of physics. The coalescene of singularities is considered with some detail for the cases N = 4,6. This allows us to give a general account of Ince's classification criteria for OLDESO.

In section 3 we deal with the general case and analyse the relationship between potentials of a given family, generalizing results obtained by Johnson[11]. This is the extension of the relationship between the harmonic isotropic three-dimensional-oscillator and the Coulomb problem known in the classic[17] and quantum[18] domains.

Section 4 presents a systematic analysis of the analytic part of the normal and subnormal solutions. We show explicitly how the singularity completely determines the kind of possible solution available and its series expansion.

Finally, section 5 contains some conclusions and points to directions of further research.

2. CLASSICAL THEORY AND INCE'S CLASSIFICATION

2.1 - Singularities of a diferential equation and the forms of the solutions

An OLDESO can be written in general as:

$$\frac{d^2y(z)}{dz^2} + P(a)\ \frac{dy(z)}{dz} + Q(a)\,y(z) = 0 \qquad (2)$$

It is well known that in general it may exist two solutions to this equation: y_1 and y_2. A point z_0 in an *ordinary* one if P and Q have no singularity in it, i.e., if they are analytic functions around a,. The solutions around this point will be analytic functions of z, and their Taylor expansions will be convergent:

$$y(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad (3)$$

If z_0 is a singularity for P and/or Q, it is assumed that in its neighbourhood a seemingly similar development is valid:

$$y(z) = (z-z_0)^{\sigma} \sum_{n=0}^{\infty} d_n (z-z_0)^n \qquad (4)$$

Substituting (4) in (2) and equating powers, two possibilities are open: (i) A 2^{nd} degree equation for a. (ii) A linear equation for a or even no equation for a.

In the first case, z_0 is a *regular* singularity and the indicial equation has one or two solutions, $\sigma_{\pm}$. For $|\sigma_+ - \sigma_-| = 1/2$, the singularity is *elementary*. In the second case, z_0 is an *irregular* singularity.

In the second case, z_0 is an *irregular* singularity.

The necessary and sufficient condition for z_0 to be a regular singularity is:

$$P(z) = (z-z_0)^{-\lambda}\, g(z) \qquad \lambda = 0,1$$
$$Q(z) = (z-z_0)^{-\mu}\, h(z) \qquad \mu = 0,1,2 \tag{5}$$

where the functions g and h are analytic around z_0 and the case $\lambda=\mu=0$ is excluded (z_0 is then an ordinary point).

The transformation:

$$z = \frac{1}{\omega} \tag{6}$$

exchanges the points at the origin and at infinity. Eq. (2) then looks:

$$\frac{d^2y(\omega^{-1})}{d\omega^2} + \left[\frac{2}{\omega} - \frac{P(\omega^{-1})}{\omega^2}\right]\frac{dy(\omega^{-1})}{d\omega} + \frac{Q(\omega^{-1})}{\omega^4}\, y(\omega^{-1}) = 0 \tag{7}$$

Analysing its behaviour at $\omega=0$, infinity is:

(i) an ordinary point if

$$p(\omega) = \frac{2}{\omega} - \frac{P(\omega^{-1})}{\omega^2} = 0(1)$$
$$q(\omega) = \frac{Q(\omega^{-1})}{\omega^4} = 0(1) \tag{8}$$

(ii) a regular singularity if

$$p(\omega) = 0(\omega^{-1})$$
$$q(\omega) = 0(\omega^{-2}) \tag{9}$$

For what regards the solutions to eq. (2), let us first consider the case that z_0 is a regular singularity. The numbers $\sigma_\pm$ resulting for the roots of the indicial equation are the exponents of the singularity. The two solutions will be:

$$y_1(z) = (z-z_0)^{\sigma+}\, u_+(z)$$

$$y_2(z) = (z-z_0)^{\sigma-}\, u_-(z)$$

if $|\sigma_+ - \sigma_-|$ is not an integer. If $\sigma_+ = \sigma_- = 0$, then the two solutions *will* differ by a logarithm. If $|\sigma_+ - \sigma_-|$ is an integer, then the solutions *may* differ by a logarithm. For z_0 a regular singularity, the general solution of (2) will have a pole or a branch point at z_0.

For irregular singular points, two situations may be present: a) the indicial equation is independent of σ; b) the indicial equation is linear in σ. There is no loss of generality by taking z_0 to be the origin. We shall take

$$p(z) = \sum_{n=0}^{\infty} p_n z^{n+\beta}$$

$$q(z) = \sum_{n=0}^{\infty} q_n z^{n+\gamma}$$

$$w(z) = \sum_{n=0}^{\infty} w_n z^{n+\alpha}$$

We see then that if

$$\beta - 1 = \gamma < -1$$

we have one index:

$$\alpha = \frac{q_0}{p_0}$$

For $\beta \geq -1$, $\gamma \geq -2$ we always have two indices. If these conditions are not fulfilled, we have no indicial equation. On the other hand, β and γ must be integers, otherwise the series expansions would not match coefficients.

In general, it can be shown[2] that if the series for w ends (i.e., is a polynomial) we have a regular solution. However, in most cases the series expansion is infinite and there is no regular solution at the origin.

If no regular solution exists, a normal solution is tried. A normal solution is of the form:

$$w(z) = \exp(-\mathbb{P}(z))\, v(z) \tag{10}$$

where

$$\mathbb{P}(z) = z^{-\delta} \sum_{n=0}^{N} \pi_n z^n \tag{11}$$

and $v(z)$ has a series expansion around the origin. The normal solution has an essential singularity at the origin. The problem is up to what point the normal solution may work. There is no a priori knowledgeabout the answer to this question, and in fact what is done is to try a solution and test that the new differential equation for $v(z)$ adrnits a regular solution.

A current example allows one to see how to determine $\mathbb{P}(z)$. Let's consider the one dimensional Schrödinger equation for the linear oscillator:

$$\left[-\frac{d^2}{dx^2}+\omega^2x^2\right]\psi(x) = E\psi(x)$$

That the point at infinity is an irregular point is seen by transforming it into the origin:

$$x = \frac{1}{\rho}$$

Then we have:

$$\frac{d^2\psi(\rho)}{d\rho^2}+\frac{2}{\rho}\frac{d\psi(\rho)}{d\rho}+\left[\frac{E}{\rho^4}-\frac{\omega^2}{\rho^6}\right]\psi(\rho) = 0$$

Trying a normal solution:

$$\psi(\rho) = \exp(-\mathbb{P}(\rho)\,v(\rho)$$

we have:

$$\frac{d^2v(\rho)}{d\rho^2}+\left[\frac{2}{\rho}-2\frac{d(\rho)}{d\rho}\right]\frac{dv(\rho)}{\rho}+\left[\left(\frac{d\mathbb{P}}{d\rho}\right)^2-\frac{d^2\mathbb{P}}{d\rho^2}-\frac{2}{\rho}\frac{d\mathbb{P}}{d\rho}+\frac{E}{\rho^4}-\frac{\omega^2}{\rho^6}\right]v(\rho)=0$$

There isone possible choice to eliminate the sixth order pole:

$$\frac{\mathbb{P}}{\rho}\,\alpha\,\frac{\omega}{\rho^3}+0(\rho^{-2}) \quad .$$

With this,

$$\mathbb{P} = -\frac{1}{2}\frac{\omega}{\rho^2}+\frac{a_1}{\rho}$$

and $a_1 = 0$ since it would give rise to a ρ^{-5} pole, whereas the coefficient in the derivative may have only a ρ^{-3} pole. So, at the end,

$$\psi(\rho) = \exp(-\psi/2\rho^2)\, v(\rho)$$

and the equation for $v(\rho)$ is:

$$\frac{d^2 v(\rho)}{d\rho^2} + \frac{2}{\rho}\left(1 + \frac{\omega}{\rho^2}\right)\frac{dv(\rho)}{d\rho} + \left(\frac{E-\omega}{\rho}\right) v(\rho) = 0 \tag{12}$$

The regular solutions of this equation are the Hermite polynomials. This can be seen going back to the original variable.

It may also happen that instead of a normal solution like (10), a similar one, but in a transformed dependent variable

$$w(z) = \exp(-\mathbb{P}(z^{1/k}))\, v(z^{1/k}) \quad ,$$

may originate an equation with a regular solution for v. In this case, we speak of the *subnormal* solutions[1,2].

2.2 - Normal form of the equation and the Schrödinger equation

An unfortunate use of the word (normal) is the one which tells that equation (2) is put in the following form:

$$\frac{d^2\chi(z)}{dz^2} + I(a)\chi(z) = 0 \tag{13}$$

where the "invariant"

$$I(z) = q(z) - \frac{1}{2}\frac{dp(z)}{dz} - \frac{1}{4}[p(z)]^2 \tag{14}$$

is obtained by eliminating the coefficient of the first derivative, $p(z)$, via:

$$w(z) = \chi(z)\exp\left(-\frac{1}{2}\int p(z')\,dz'\right) \tag{15}$$

There is an important property of an equation in normal form iike (13). Performing the transformation $z=\rho^{-1}$, it is easily shown that the new invariant has the following property:

$$I'(\rho) = \rho^{-4} I(\rho^{-1})$$

In the radial Schrödinger equation one has the normal form after separation of variables in spherical coordinates, $R(r)$, going to the radial wave function $u(r)$:

$$u(r) = r\,R(r) \tag{16}$$

Notice that there is often an interplay between extracting a singularity via a normal (or subnormal) solution like (10) (or (12)) and including the term with first derivatives.

It is essential for our study the relationship between the normal form of a given differential equation and the normal form of the Schrodinger equation where two particles interact through central forces. Following Bose[6], given

$$\frac{d^2\chi(z)}{dz^2} + I(z)\chi(z) = 0 \ ,$$

writing

$$z = z(r) \tag{17a}$$

$$\chi = \left(\frac{dz}{dr}\right)^{1/2} u(r) \ , \tag{17b}$$

we shall have

$$\frac{d^2u(r)}{dr^2} + I^{(s)}(r)u(r) = 0 \tag{18a}$$

with

$$I^{(s)}(r) = \left(\frac{dz}{dr}\right)^2 I[z(r)] + \frac{1}{2}\{z(r), r\} \tag{18b}$$

$$\{z(r), r\} = \frac{\left(\frac{d^3z}{dr^3}\right)}{\left(\frac{dz}{dr}\right)} - \frac{3}{2}\left[\frac{\left(\frac{d^2z}{dr^2}\right)}{\left(\frac{dz}{dr}\right)}\right]^2 \qquad \text{(Schwarz derivative)}$$

To be useful as a representation of a Schrödinger equation, $I^{(s)}$ should be of the form

$$I^{(s)} \propto k^2 - \lambda r^{-2} - U(r)$$

with λ a real number. If we know the regular solution of the original equation (13), we may expect that this transformation may lead to a solution of the Schrödinger equation for a given potential.

As an example of the procedure, involving also the treatment we shall give in the following to similar problems, let us take the case of the Whittaker equation for the confluent hypergeometric equation (dealt with also in Bose's article[6]):

$$\frac{d^2W(z)}{dz^2} + \left\{ -\frac{1}{4} + \frac{\lambda}{z} + \frac{1/4-\mu^2}{z^2} \right\} W(z) = 0$$

The linearly independent solutions are:

$$\text{(i)} \quad M_{\lambda,\mu}(z) = z^{\mu+1/2}\, e^{-\frac{1}{2}z}\, \Phi(\mu-\lambda+\tfrac{1}{2},\ 2\mu+1;\ z)$$

which is regular at the origin (assuming $\mu \geq -1/2$) and valid for $2\mu + 1 \neq 0,-1,-2, \ldots$;

$$\text{(ii)} \quad M_{\lambda,-\mu}(z) = z^{-\mu+1/2}\, e^{-\frac{1}{2}z}\, \Phi(-\mu-\lambda-\tfrac{1}{2},\ -2\mu+1;\ z) ,$$

which is irregular at the origin ($-\mu \geq -1/2$) and valid for $-2\mu + 1 \neq 0,-1,-2, \ldots$. The function $\phi(\alpha,\beta,\gamma)$ is given by:

$$\Phi(\alpha,\gamma;z) = {}_1F_1(\alpha,\gamma;z) = 1 + \frac{\alpha}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \ldots$$

Let us restrict ourselves to transformations of the independent variable using a power (more general transformations are given in Bose[6] and

Lemieux and Bose[7] but they are of no interest for the matters considered in what follows):

$$z = ar^p.$$

Then,

$$W = (apr^{p-1})^{1/2}\, u(r)$$

$$\{z(r), r\} = \frac{1-p^2}{2r^2}$$

and

$$I_w[z(r)] = -\frac{1}{4} + \frac{\lambda}{ar^p} + \frac{1/4-\mu^2}{a^2 r^{2p}}$$

The corresponding candidate for an invariant of a Schrödinger equation is:

$$I_p^{(s)}(r) = -\frac{p^2a^2r^{2p-2}}{4} + p^2a\lambda r^{p-2} + \frac{1/4-\mu^2p^2}{r^2}$$

To make it an invariant for a Schrödinger equation we have two possible choices:

A) $p = 1$,

$$I_1^{(s)}(r) = -\frac{a^2}{4} + \frac{a\lambda}{r} + \frac{1/4-\mu^2}{r^2}$$

$$\equiv -E + \frac{g}{r} + \frac{\ell(\ell+1)}{r^2} \quad \text{(the Coulomb potential)}$$

Then

$$u_1(r) = (2\kappa r)^{\ell+1} e^{-\kappa r}\, \Phi(\ell+1-\tfrac{g}{2}, 2(\ell+1); 2\kappa r) \tag{19}$$

with $\kappa = \sqrt{-E}$. The series (19) terminates if $\ell + 1 - g/2\kappa = N$ which gives for κ (and E) the values for the Coulomb potential well.

B) $p = 2$

$$I_2^{(s)}(r) = -a^2r^2 + 4a\lambda + \frac{1/4-4\mu^2}{r^2}$$

$$= -\Gamma^2r^2 + k^2 - \frac{\ell(\ell+1)}{r^2}\,,\quad \text{the isotropic harmonic oscillato}$$

Then

$$u_2(r) = B_2 r^{\ell+1}\, r^{-\frac{1}{2}\Gamma r^2}\, \Phi\left(\frac{\ell+3/2}{2} - \frac{k^2}{4\Gamma}\,,\ \ell+\frac{3}{2}\,;\ \Gamma r^2\right) \tag{20}$$

The series (20) terminates whenever

$$k^2 = 4\Gamma\left(N + \frac{\ell+3/2}{2}\right)$$

which are the energy levels of the harmonic oscillator.

We have obtained the laws of force and their solutions from the differential equation. This is the main content of the article, namely, to show that a family of differential equations that can be suitably characterized produce a corresponding interesting family of laws of force.

2.3 - The classification of OLDESO's made by Ince

The number of elementary regular singularities can be taken as a starting point, a basic set of data, to begin considering systematically OLDESO's.

The most general equation having N elementary regular sing. larities, one of them being at infinity is

$$\frac{d^2w(z)}{dz^2} + \left[\sum_{r=1}^{N-1} \frac{1/2-2\alpha_r}{z-z_r}\right]\frac{dw(z)}{dz} + \left[\sum_{r=1}^{N-1} \frac{\alpha_r(\alpha_r+1/2)}{(z-z_r)^2}\right.$$

$$\left. + \frac{A_0+A_1z+\ldots+A_{N-3}z^{N-3}}{(z-z_1)(z-z_2)\ldots(z-z_{N-1})}\right] w(z) = 0 \tag{21}$$

The N-l singularities at finite distances are located at z_r $(r=1,\ldots,N-1)$ and their exponents are α_r and $\alpha_r+1/2$.

The constants $A_1,\ldots,A_{N-4}$ are arbitrary. Since the point at infinity is also an elementary regular singularity we must have:

$$\alpha_\infty = \frac{N}{4} - 1 - \sum_{r=1}^{N-1} \alpha_r$$

and this fixes the value of A_{N-3} to be

$$A_{N-3} = \left(\sum_{r=1}^{N-1} \alpha_r\right)^2 - \sum_{r=1}^{N-1} \alpha_r^2 - \frac{N-2}{2} \sum_{r=1}^{N-1} \alpha_r + \frac{(N-2)(N-4)}{16} \tag{22}$$

The sum of the exponents is not arbitrary:

$$2\alpha_\infty + \frac{1}{2} + \sum_{r=1}^{N-1} (2\alpha_r + \frac{1}{2}) = N-2$$

Let now two singularities coalesce, for instance, by making $z_2 \to z_1$. Then, the pole term in $p(z)$ picks a new residue:

$$\frac{1}{2} - 2\alpha_1 \to 1-2(\alpha_1+\alpha_2)$$

The double pole in (21) which contributes to the indicial equation, at z_1 has a residue now equals to:

$$\alpha_1(\alpha_1 + \frac{1}{2}) + \alpha_2(\alpha_2 + \frac{1}{2}) + \frac{A_0+A_1z_1+\ldots+A_{N-3}z_1^{N-3}}{(z_1-z_3)(z_1-z_4)\ldots(z_1-z_{N-1})}$$

Since the coefficients A_n $(n=0,1,\ldots N-4)$ are fully arbitrary the indicial equation has two roots that in general will not differ by 1/2.

In the notation by Ince, the inicial equation was of the kind $[N,0,0]$ and became $[N-2,1,0]$. In general, by making singularities to coincide we can get

$$[N,0,0] \to [N-K,K_0,K_1,K_2,\ldots,K_s]$$

such that K, describes regular singularities, K_1 irregular singularities coming out of the coalescence of three elementary singularities, K_2 irregular singularities coming out of the coalescence of four elementary regular ones and, in general, K_J will indicate the number of singularities produced by coalescence of J+2 elementary regular ones. That is:

$$K = \sum_{r=0}^{s} (r+2) K_r$$

In this notation, the Riemann problem[2], which is to write the solution of an OLDESO having three regular singularities, is obtained from $[6,0,0]$ into $[0,3,0]$.

In this work we shall talk almost exclusively of the coalescence of elementary regular singularities to the origin and/or infinity.

Let us try a simple application. Consider $N=4$. We then find:

$$A_1 = 2(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3) - (\alpha_1+\alpha_2+\alpha_3)$$

Making $z_1 \to z_2 \to z_3 \to \infty$, and imposing

$$\lim_{z_1\to z_2\to z_3\to\infty} \frac{A_0}{z_1 z_2 z_3} = -k^2 \quad , \quad \text{real}$$

the equation (21) becomes

$$\frac{d^2w(z)}{dz^2} + k^2 w(z) = 0 \tag{23}$$

This is a well known equation: it is the classical harmonic oscillator in one dimension (with z the time parameter), or the quantum mechanical Schrodinger equation for the free particle in one dimension (stationary states).

It is at this point reasonable to make some remarks about the physical relevance of the elementary regular singularities of the equations. It seems that the equations of interest in Physics are precisely those having irregular singularities. The original elementary regular singularities might be of some value for physical situations if one could, knowing the solutions of the corresponding equations, keep some control of the parameters as the singularities coalesce. To our knowledge, this problem has not been solved in all generality.

A comment at this point on the free particle case is perhaps in order. The link between the quantum mechanical case (23) and the classicalcaseforthe free particle in one dimension is provided by the superposition of solutions which give a solution of the Schrödinger equation depending on time. Mean values in these states are related to the classical observables. For instance,

$$<x>(t) = x_0 + <v>t$$

This is the solution of the classical equation, which is linearly divergent for $t \to \pm\infty$. It is a milder behaviour than the original solution of the stationary quantum case (23). This could have been expected since the beginning, and if singularities of the differential equations govern the dynamical behaviour of physical systems, it is plausible to expect that classical equations will be less singular than the original quantum equations.

In fact, the classical free one dimensional equation comes out of the equation with only two elementary regular singularities. We are considering two different variables, but it may be apparent that the singularities in space are somewhat reflected and washed out in the superposition procedure giving rise to singularities less fierce in time.

Let us continue with the analysis by increasing N. For $N=5$, the equations resulting from the confluences of singularities are

(i) $$[0,0,1_3] : \quad \frac{d^2w(z)}{dz^2} + [k^2 - gz]w = 0$$

which is obtained by imposing

$$\lim_{z_1 \to z_2 \to z_3 \to z_4 \to \infty} \frac{A_0}{z_1 z_2 z_3 z_4} = k^2 , \quad \lim_{z_1 \to z_2 \to z_3 \to z_4 \to \infty} \frac{A_1}{z_1 z_2 z_3 z_4} = -g$$

This is the Schrödinger equation for the linear potential whose solutions are Airy functions[19].

(ii) $[0,1,1]$:

By taking

$$\lim_{z_3 \to z_4 \to \infty} \frac{A_2}{z_3 z_4} = B_0 , \quad -\lim_{z_3 \to z_4 \to \infty} \frac{A_1}{z_3 z_4} = B_1 ,$$

we have:

$$\frac{d^2w}{dz^2} + \frac{1-2(\alpha_1+\alpha_2)}{z}\frac{dw}{dz} + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2})+\alpha_2(\alpha_2+\frac{1}{2})}{z^2} + \frac{B_0}{z^2} + \frac{B_1}{z}\right]w = 0$$

Now, putting:

$$w(z) = z^{\alpha_1+\alpha_2-1/2} f(z)$$

we get the normal form:

$$\frac{d^2f}{dz^2} + \left[\frac{B_2}{z^2} + \frac{B_1}{z}\right]f = 0$$

with

$$B_2 = \alpha_1(\alpha_1+1/2)+\alpha_2(\alpha_2+1/2)+1-2(\alpha_1+\alpha_2) - \frac{1}{4}(1-2(\alpha_1+\alpha_2))^2 + B_0$$

The invariant of this equation goes into the Schrödinger form by writing

$$z = \alpha r^2 \quad , \quad f = (2\alpha r)^{1/2} u(r)$$

and, finally

$$\frac{d^2u}{dr^2} + \left[\frac{4B_2-3/4}{r^2} + 4\alpha B_1\right]u = 0$$

which is the Schrödinger equation for three-dimensional free motion provided $4B_2-3/4 = -\ell(\ell+1)$ and $4\alpha B_1 = k^2$. Other confluences can be found in the book by Ince[2], but most of them do not produce equations of interest to us here.

We shall now consider the case of $N=6$. This is quite important both theoretically and practically. Starting from $[6,0,0]$, going to $[0,3,0]$ one gets the hypergeometric equation which was well studied during the last century, and which provided several useful sets of functions largely used in mathematical physics. We shall skip most of the intermediate steps, going straight to the hypergeometric equation. Letting $z_5 \to \infty$, $z_1, z_2 \to 0$ and $z_3, z_4 \to a$

$$\lim_{z_5\to\infty} \frac{A_0}{z_5} = -B_0 \; ; \quad \lim_{z_5\to\infty} \frac{}{z_5} = B_1 \; ; \quad \lim_{z_5\to\infty} \frac{A_2}{z_5} = B_2$$

one gets:

$$z(z-a)^2 \frac{d^2w}{dz^2} + \{[1-2(\alpha_1+\alpha_2)]z(z-a)^2 + [1-2(\alpha_3+\alpha_4)]z^2(z-a)\} \frac{dw}{dz}$$

$$+ \{[\alpha_1(\alpha_1+1/2) + \alpha_2(\alpha_2+\tfrac{1}{2}](z-a)^2 + [\alpha_3(\alpha_3+\tfrac{1}{2}) + \alpha_4(\alpha_4+\tfrac{1}{2})]z^2$$

$$+ B_0 + B_1 z + B_2 z^2\}w = 0$$

Writing now the indicial equation for $z=0$, a and ∞, calling the two indices at each point (α,α'), (β,β') and (γ,γ'), respectively one has the famous relation:

$$\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$$

and the differential equation will look now

$$z^2(z-a)^2 \frac{d^2w}{dz^2} + z(z-a)[(1-\alpha-\alpha')(z-a) + (1-\beta-\beta')z] \frac{dw}{dz}$$

$$+ [\gamma\gamma'(z-a)z + a\beta\beta' - a\alpha\alpha'(z-a)]\psi = 0$$

The solutions of this equation are denoted by the famous Riemann P symbol

$$P\begin{Bmatrix} 0 & a & \infty & \\ \alpha & \beta & \gamma & ; z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

which indicates an equation with three regular-singular points

Setting $\alpha'=\beta'=0$, $a=1$, one gets the usual form

$$(z-1)z\frac{d^2w}{dz^2} + [(2-\alpha-\beta)z - (1-\alpha)]\frac{dw}{dz} + \gamma\gamma' w = 0$$

The confluent form of this equation is obtained by letting

$$z_2 \to z_1 \to 0 \quad , \quad z_3 \to z_4 \to z_5 \to \infty$$

Calling again a and a' the indices at the origin:

$$\frac{d^2w}{dz^2} + \frac{1-\alpha-\alpha'}{z}\frac{dw}{dz} + \left[\frac{\alpha\alpha'}{z} + \frac{B_1}{z} + B_2\right]w = 0$$

or, writing

$$\alpha = \frac{1}{2} + m\ , \quad \alpha' = \frac{1}{2} - m$$

$$\frac{d^2w}{dz^2} + \left[\frac{1/4-m^2}{z} + \frac{B_1}{z} + B_2\right]w = 0$$

This is the equation $[0,1,1_2]$. We have already analyzed, following Bose, some of the laws of force obtained from this equation (Eqs. (19) and (20).

There is another possible confluence that can be made. This happens when both the origin and the infinity are made irregular singular points of the first kind by making three elementary singularities confluence at each of them. Defining

$$F \equiv \frac{3}{2} - 2\sum_{i=1}^{3}\alpha_i \quad ; \quad G = \sum_{r=1}^{3}\alpha_r(\alpha_r+1/2)$$

$$\lim_{z_4\to z_3\to\infty}\frac{A_r}{z_4 z_3} = B_r \quad (r=0,1,2)$$

We get:

$$\frac{d^2w}{dz^2} + \frac{F}{2}\frac{dw}{dz} + \left\{\frac{B_0}{z^3} + \frac{G+B_1}{z^2} + \frac{B_2}{z}\right\}w = 0$$

With the transformation:

$$w = v(z)\, z^{-F/2}$$

we get

$$\frac{d^2v}{dz^2} + \left[\frac{B_0}{z^3} + \frac{B_1 + G + \frac{F}{2}\left(1 - \frac{1}{2}F\right)}{z^2} + \frac{B_2}{z}\right]v = 0 \qquad (24)$$

which may be written as:

$$z^2\frac{dv}{dz^2} + z\frac{dv}{dz} - \frac{1}{4}\left[\alpha + \frac{1}{2}k^2 + \frac{1}{4}k^2\left(z + \frac{1}{2}\right)\right]v = 0$$

and, under a transformation

$$z = e^{2ix}$$

becomes the Mathieu equation

$$\frac{d^2v}{dx^2} + (a+k^2\cos^2 x)\,v = 0$$

Notice that with the invariant of (24), setting

$$z = \alpha r^2$$

and transforming v correspondingly (eq.(17b)) we obtain a Schrödinger invariant of the form

$$I^{(s)}(r) = \gamma_1 r^{-4} + \gamma_2 r^{-2} + k^2$$

which corresponds to a potential studied by Spector[20] (*) for $\gamma_1 < 0$. For the validity of our procedures however, $\gamma_1 > 0$. This lifts any restriction on the value of γ_2, which could be $\gamma_2 < -1/4$.

The main point here is that the classification made by Ince allows one to handle appropriately the OLDESO's of several types.

From the point of view of Physics, it is clear that as the number of elementary regular singularities, N, increases, the same happens to the difficulty of some physical problems. We have just seen

$N = 2$, Classical free particle motion in one dimension.

$N = 3$, Classical motion on a linear potential in one dimension.

$N = 4$, Classical one dimensional linear oscillator; Schrödinger equation for the free particle motion in one dimension.

$N = 5$, Schrodinger equation for the linear potential in one dimension. Schrodinger equation for the three dimensional free-particle motion.

$N = 6$, Schrodinger equation for the three dimensional Coulomb potential and for the harmonic oscillator. Repulsive r^{-4} central potential.

(*) In our previous letter[16] it could be interpreted that we were in same case as Spector.

Notice also that it is always comented in textbooks that the S-waves in two body central forces are easier to solve. It is attributed to the fact that the problem of S-waves has the structure of an one dimensional problem; this translates in Ince's classification by the elimination of the regular singularity at the origin, i.e. to the elimination of the contribution of two elementary regular singularities which were made to coincide.

3. POWER LAW CENTRAL POTENTIALS IN THE SCHRÖDINGER EQUATION

3.1 - Application of Ince's classification

This section is the crux of the article. We are interested in differential equations in normal form (remind (13)) with singularities at the origin and infinity. They will then be transformed through the procedures applied in section 2.2 into families of radial Schrödinger equations for power law potentials, i.e., with potentials of the form:

$$V(r) = \sum_{i=-L}^{M} \gamma_i \, r^{\alpha_i} \tag{25}$$

where the numbers γ_i are in principle arbitrary couplings and α_i are members of a set of rational numbers specific to the transformations used.

Let us just look to what was done in the study of the descendents of the original $N=6$ elementary singularities. The possible confluences and invariants were:

a) $r = 0$ regular singularity, $r = \infty$ irregular singularity: $[0,1,1_2]$;

$$I(z) = Az^{-2} + Bz^{-1} + C$$

b) $r = 0$ irregular singularity, $r = \infty$ irregular singularity: $[0,0,2_1]$;

$$I(z) = A'z^{-3} + B'z^{-2} + C'z^{-1}$$

These are the only possibilities. In order to have $r=\infty$ as a regular singularity one should just exchange the origin and infinity in the first case.

Note that the second case is invariant also as a result of the exchange of the origin and infinity. This will always happen for equations represented by the symbol $[0,0,2_{\frac{N}{2}-2}]$, N = 6,8,10,... .

Let us now consider the general case starting from eqs.(21) and (22) ; make:

$$z_1 = z_2 = 0$$

$$z_3 \to z_4 \to \dots \to z_{N-1} \to \infty$$

We then have the case represented by $[0, 1, 1_{N-4}]$ with one regular singular point at the origin and an irregular singularity of order $N-4$ at infinity. The corresponding differential equation is:

$$\frac{d^2w(z)}{dz^2} + \frac{1-2(\alpha_1+\alpha_2)}{z}\frac{dw(z)}{dz} + \left[\frac{\alpha_1(\alpha_1+\frac{1}{2}) + \alpha_2(\alpha_2+\frac{1}{2}) + B_0}{z^2} + \sum_{r=1}^{N-4} B_r\, z^{r-2}\right] w(z) = 0$$

where

$$B_n = \lim_{z_3\to z_4\to\dots\to z_{N-1}\to\infty} \frac{A_n}{\prod_{r=3}^{N-1}(-z_r)} \qquad n = 0,1\dots,N-4$$

Writing:

$$w(z) = z^{(\alpha_1+\alpha_2)-1/2}\, v(z)$$

then the differential equation appears in normal form:

$$\frac{d^2v}{dz^2} + \left[\frac{B'_0}{z^2} + \sum_{r=1}^{N-1} B_r\, z^{r-2}\right] v(z) = 0 \tag{26}$$

Now only one constant changes:

$$B'_0 = B_0 + \frac{1}{2}(\alpha_1+\alpha_2) + \frac{1}{4} - 2\alpha_1\alpha_2$$

The invariant in eq. (26) may be in the form of a Schrödinger equation provided that $B_2 \neq 0$.

An analogous procedure for $[N,0,0]$, but making now $z_1\to z_2\to z_3\to 0$ and $z_4\to z_5\to\dots\to z_{N-1}\to\infty$, results in

$$\frac{d^2 v(z)}{dz^2} + \left[\sum_{r=0}^{N-4} C_r \; z^{r-3}\right] v(z) = 0$$

The equation is of the type $|0,0,1_1,1_{N-5}|$, and is of the Schrodinger kind provided C_1 and C, are not zero.

We can continue this procedure to obtain all possible invariants with regular and irregular singularities at the origin and infinity. They are listed in Table 1 for $5 \leqslant N \leqslant 10$.

3.2 - The construction of the Schrodinger invariant

We shall extend the procedures applied before to show that a given invariant in an equation with a given kind of maximum irregular singularity can be related to invariants of the radial Schrödinger equation with two body central forces.

As we have seen the invariants obtained from a given equation with N elementary regular singularities are in general of the form

$$I(z) = \sum_{\substack{k=-L \\ k\neq -2}}^{M} A_k z^k + A_{-2} z^{-2} \tag{27}$$

If the origin is a regular singularity, then L=l and $M=N-6$. If it is not, it can only be an irregular singularity of order $L-2$, and then, $M=N-L-4$. The other point of interest is infinity; the singularity there is of order $M+2$. All interesting invariants for $5 \leqslant N \leqslant 10$ are found in Table 1.

Let us now transform the variable:

$$z = a_j \, r_j^{\frac{2}{j+2}} \tag{28}$$

The index j must not be *-2* and may have any other value between -L and M. For the dependent variable, we have:

$$w(z(r_j)) = N_j r_j^{-\frac{j}{2(j+2)}} \; u_j(r_j) \tag{29}$$

and the corresponding Schrödinger invariant is:

$$I^{(s)}(r_j) = \frac{4}{(j+2)^2} \, a_j^2 r_j^{-\frac{2j}{j+2}} \, I(z(r_j)) + \frac{j(j+4)}{4(j+2)^2} \, r_j^{-2} \tag{30}$$

Table 1

Invariants obtained from the ordinary linear differential equation of the second order with N elementary singularities through confluence to the origin and infinity.

N	$I(z)$	Classification
5	$Az^{-2}+Bz^{-1}$	$[0,1,1_1]$
6	$Az^{-2}+Bz^{-1}+C$	$[0,1,1_2]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}$	$[0,0,2_1]$
7	$Az^{-2}+Bz^{-1}+C+Dz$	$[0,1,1_3]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D$	$[0,0,1,1_2]$
8	$Az^{-2}+Bz^{-1}+C+Dz+Ez^{2}$	$[0,1,1_4]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D+Ez$	$[0,0,1_1,1_3]$
	$Az^{-4}+Bz^{-3}+Cz^{-2}+Dz^{-1}+E$	$[0,0,2_2]$
9	$Az^{-2}+Bz^{-1}+C+Dz+Ez^{2}+Fz^{3}$	$[0,1,1_5]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D+Ez+Fz^{2}$	$[0,0,1_1,1_4]$
	$Az^{-4}+Bz^{-3}+Cz^{-2}+Dz^{-1}+E+Fz$	$[0,0,1_2,1_3]$
10	$Az^{-2}+Bz^{-1}+C+Dz+Ez^{2}+Fz^{3}+Gz^{4}$	$[0,1,1_6]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D+Ez+Fz^{2}+Gz^{3}$	$[0,0,1_1,1_5]$
	$Az^{-4}+Bz^{-3}+Cz^{-2}+Dz^{-1}+E+Fz+Gz^{2}$	$[0,0,1_2,1_4]$
	$Az^{-5}+Bz^{-4}+Cz^{-3}+Dz^{-2}+Ez^{-1}+F+Gz$	$[0,0,2_3]$
11	$Az^{-2}+Bz^{-1}+C+Dz+Ez^{2}+Fz^{3}+Gz^{4}+Hz^{5}$	$[0,1,1_7]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D+Ez+Fz^{2}+Gz^{3}+Hz^{4}$	$[0,0,1_1,1_6]$
	$Az^{-4}+Bz^{-3}+Cz^{-2}+Dz^{-1}+E+Fz+Gz^{2}+Hz^{3}$	$[0,0,1_2,1_5]$
	$Az^{-5}+Bz^{-4}+Cz^{-3}+Dz^{-2}+Ez^{-1}+F+Gz+Hz^{2}$	$[0,0,1_3,1_4]$
12	$Az^{-2}+Bz^{-1}+C+Dz+Ez^{2}+Fz^{3}+Gz^{4}+Hz^{5}+Iz^{6}$	$[0,1,1_8]$
	$Az^{-3}+Bz^{-2}+Cz^{-1}+D+Ez+Fz^{2}+Gz^{3}+Hz^{4}+Iz^{5}$	$[0,0,1_1,1_7]$
	$Az^{-4}+Bz^{-3}+Cz^{-2}+Dz^{-1}+E+Fz+Gz^{2}+Hz^{3}+Iz^{4}$	$[0,0,1_2,1_6]$
	$Az^{-5}+Bz^{-4}+Cz^{-3}+Dz^{-2}+Ez^{-1}+F+Gz+Hz^{2}+Iz^{3}$	$[0,0,1_3,1_5]$
	$Az^{-6}+Bz^{-5}+Cz^{-4}+Dz^{-3}+Ez^{-2}+Fz^{-1}+G+Hz+Iz^{2}$	$[0,0,2_4]$

Substituting (27) in (30) it is rewritten as:

$$I^{(s)}(r_j) = \frac{4}{(j+2)^2}\left[\sum_{\substack{k=-L\\k\neq-2}}^{M} A_k a_j^{k+2}\, r_j^{\frac{2(k-j)}{j+2}} + \frac{A_{-2}+j(j+4)/16}{r_j^2}\right] \tag{31}$$

Obviously, the term in the sum with $k=j$, $A_k\neq0$, plays the role of the energy. We can also write:

$$I^{(s)}(r_j) = \sum_{\substack{k=-L\\k\neq-2}}^{M} \gamma_{j,k}\, r_j^{\alpha_{j,k}} + \gamma_{j,-2}\cdot r^{-2}\ , \tag{32}$$

to use a more popular form, with

$$\gamma_{j,k} = \frac{4}{(j+2)^2} A_k\ a_j^{k+2}\quad ,\ k\neq-2\ ; \tag{33a}$$

$$\gamma_{j,-2} = \frac{4}{(j+2)^2}\left[A_{-2} + \frac{j(j+4)}{16}\right]\ , \tag{33b}$$

$$\alpha_{j,k} = \frac{2(k-j)}{j+2}\quad . \tag{33c}$$

In Table 2 we represent all the Schrodinger invariants obtained foreach invariant of Table 1, once we make a transformation, Other possibilities can not appear, except the trivial ones obtained by exchanging the singular points at the origin and infinity.

How many transformations are available from a given invariant in Table 1 to generate Schrodinger invariants? In principle, the index j in (32) can take $N-4$ values. A glance at Table 2 shows that this is not always the case: whenthe two singularities are of the same kind, not all the transformations will be different, but only half of them (for N = even).

3.3 - Transformations between Schrodinger invariants

As we studied the obtention of Schrodinger invariarits, we showed that we could get several of these out of the same invariant. Now

Table 2

Schrödinger invariants coming from Table 1 by power transformation of the independent variable.

$$\underline{N=5}\ ;\ [0,1,1]\ ;\ L=1,\quad M=-1$$

$$j=1\quad z=a_{-1}r_{-1}^{2}\qquad I^{(s)}(r_{-1})=\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}$$

$$\underline{N=6}\ ;\ [0,1,1_2]\ ;\ L=1,\quad M=0$$

$$j=-1,z=a_{-1}r_{-1}^{2}\ ;\ I^{(s)}(r_{-1})=\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}$$

$$=0,z=a_0r_0\ ;\ I^{(s)}(r_0)=\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}$$

$$[0,0,2_1]\ ;\ L=3,\quad M=-1$$

$$j=-1,z=a_{-1}r_{-1}^{2}\ ;\ I^{(s)}(r_{-1})=\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}$$

$$\underline{N=7}\ ;\ [0,1,1_3]\ ;\ L=1,\quad M=1$$

$$j=-1,z=a_{-1}r_{-1}^{2}\ ;\ I^{(s)}(r_{-1})=\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}$$

$$=0,z=a_0r_0\ ;\ I^{(s)}(r_0)=\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0$$

$$=1,z=a_1r_1\ ;\ I^{(s)}(r_1)=\gamma_{1,-2}r_1^{-2}+\gamma_{1,-1}r_1^{-4/3}+\gamma_{1,0}r_1^{-2/3}+\gamma_{1,1}$$

$$[0,0,1_1,1_2]\ ;\ L=3,\quad M=0$$

$$j=3,z=a_{-3}r_{-3}^{-2}\ ;\ I^{(s)}(r_{-3})=\gamma_{-3,-3}+\gamma_{-3,-2}r_{-3}^{-2}+\gamma_{-3,-1}r_{-3}^{-4}+\gamma_{-3,0}r_{-3}^{-6}$$

$$=-1\quad z=a_{-1}r_{-1}^{2}\ ;\ I^{(s)}(r_{-1})=\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}$$

$$=0,z=a_0r_0\ ;\ I^{(s)}(r_0)=\gamma_{0,-3}r_0^{-3}+\gamma_{0,-3}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}$$

$$\underline{N=8}\ ;\ [0,1,1_4]\ ;\ L=1,\quad M=2$$

$$j=-1,z=a_{-1}r_{-1}^{2}\ ;\ I^{(s)}(r_{-1})=\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}+\gamma_{-1,2}r_{-1}^{6}$$

$$=0,z=a_0r_0\ ;\ I^{(s)}(r_0)=\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0+\gamma_{0,2}r_0^{2}$$

$$=1,z=a_1r_1^{2/3}\ ;\ I^{(s)}(r_1)=\gamma_{1,-2}r_1^{-2}+\gamma_{1,-1}r_1^{-4/3}+\gamma_{1,0}r_1^{-2/3}+\gamma_{1,1}+\gamma_{1,2}r_1^{2/3}$$

$$=2,z=a_2r_2^{1/2}\ ;\ I^{(s)}(r_2)=\gamma_{2,-2}r^{-2}+\gamma_{2,-1}r_2^{-3/2}+\gamma_{2,0}r_2^{-1}+\gamma_{2,1}r_2^{-1/2}+\gamma_{2,2}$$

$$[0,0,1_1,1_3] \quad ; \quad L=3, \quad M=1$$

$$j=-3, z=a_{-3}r_{-3}^{-2} \quad ; \quad I^{(s)}(r_{-3})=\gamma_{-3,-3}+\gamma_{-3,-2}r_{-3}^{-2}+\gamma_{-3,-1}r_{-3}^{-4}+\gamma_{-3,0}r_{-3}^{-6}+\gamma_{-3,1}r_{-3}^{-8}$$

$$=-1, z=a_{-1}r_{-1}^{2} \quad ; \quad I^{(s)}(r_{-1})=\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}$$

$$=0, z=a_0 r_0 \quad ; \quad I^{(s)}(r_0)=\gamma_{0,-3}r_0^{-3}+\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0$$

$$=1, z=a_1 r_1^{2/3} \quad ; \quad I^{(s)}(r_1)=\gamma_{1,-3}r_1^{-8/3}+\gamma_{1,-2}r_1^{-2}+\gamma_{1,-1}r_1^{-4/3}+\gamma_{1,0}r_1^{-2/3}+\gamma_{1,1}$$

$$[0,0,2_2] \quad ; \quad L=4, \quad M=0$$

$$j=-1, z=a_{-1}r_{-1}^{2} \quad ; \quad I^{(s)}(r_1)=\gamma_{-1,-4}r_{-1}^{-6}+\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}$$

$$=0, z=a_0\, r_0 \quad ; \quad I^{(s)}(r_0)=\gamma_{0,-4}r_0^{-4}+\gamma_{0,-3}r_0^{-3}+\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}$$

$$\underline{N=9} \quad ; \quad [0,1,1_5] \quad ; \quad L=1, \quad M=3$$

$$j=-1, z=a_{-1}r_{-1}^{2} \quad ; \quad I^{(s)}(r_{-1})=\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}+\gamma_{-1,2}r_{-1}^{6}+$$
$$+\ \gamma_{-1,3}r_{-1}^{8}$$

$$=0, z=a_0 r_0 \quad ; \quad I^{(s)}(r_0)=\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0+\gamma_{0,2}r_0^{2}+\gamma_{0,3}r_0^{3}$$

$$=1, z=a_1 r_1^{2/3} \quad ; \quad I^{(s)}(r_1)=\gamma_{1,-2}r_1^{-2}+\gamma_{1,-1}r_1^{-4/3}+\gamma_{1,0}r_1^{-2/3}+\gamma_{1,1}+\gamma_{1,2}r_1^{2/3}$$
$$+\ \gamma_{1,3}r_1^{4/3}$$

$$=2, z=a_2 r_2^{1/2} \quad ; \quad I^{(s)}(r_2)=\gamma_{2,-2}r_2^{-2}+\gamma_{2,-1}r_2^{-3/2}+\gamma_{2,0}r_2^{-1}+\gamma_{2,1}r_2^{-1/2}+\gamma_{2,2}$$
$$+\ \gamma_{2,3}r_2^{1/2}$$

$$=3, z=a_3 r_3^{4/5} \quad ; \quad I^{(s)}(r_3)=\gamma_{3,-2}r_2^{-2}+\gamma_{3,-1}r_3^{-8/5}+\gamma_{3,0}r_3^{-6/5}+\gamma_{3,1}r_3^{-4/5}$$
$$+\gamma_{3,2}r_3^{-2/5}+\gamma_{3,3}$$

$$[0,0,1_1,1_4] \quad ; \quad L=3, \quad M=2$$

$$j=-3, z=a_{-3}r_{-3}^{-2} \quad ; \quad I^{(s)}(r_{-3})=\gamma_{-3,-3}+\gamma_{-3,-2}r_{-3}^{-2}+\gamma_{-3,-1}r_{-3}^{-4}+\gamma_{-3,0}r_{-3}^{-6}+\gamma_{-3,1}r_{-3}^{-8}$$
$$+\ \gamma_{-3,2}r_{-3}^{-10}$$

$$=-1, z=a_{-1}r_{-1}^{2} \quad ; \quad I^{(s)}(r_{-1})=\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}$$
$$+\ \gamma_{-1,2}r_{-1}^{6}$$

$$=0, z=a_0 r_0 \quad ; \quad I^{(s)}(r_0)=\gamma_{0,-3}r_0^{-3}+\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0+\gamma_{0,2}r_0^{2}$$

$$j=1, z=a_1 r_1^{2/3} \;;\; I^{(s)}(r_1) = \gamma_{1,-3} r_1^{-8/3} + \gamma_{1,-2} r_1^{-2} + \gamma_{1,-1} r_1^{-4/3} + \gamma_{1,0} r_1^{-2/3} + \gamma_{1,1} + \gamma_{1,2} r_1^{2/3}$$

$$= 2, z=a_2 r_2^{1/2} \;;\; I^{(s)}(r_2) = \gamma_{2,-3} r_2^{-5/2} + \gamma_{2,-2} r_2^{-2} + \gamma_{2,-1} r_2^{-3/2} + \gamma_{2,0} r_2^{-1} + \gamma_{2,1} r_2^{-1/2} + \gamma_{2,2}$$

$[0,0,1_2,1_3]$; $L=4$, $M=1$

$$j=-4, z=a_{-4} r_{-4}^{-1} \;;\; I^{(s)}(r_{-4}) = \gamma_{-4,-4} + \gamma_{-4,-3} r_{-4}^{-1} + \gamma_{-4,-2} r_{-4}^{-2} + \gamma_{-4,-1} r_{-4}^{-3} + \gamma_{-4,0} r_{-4}^{-4} + \gamma_{-4,1} r_{-4}^{-5}$$

$$=-3, z=a_{-3} r_{-3}^{-2} \;;\; I^{(s)}(r_{-3}) = \gamma_{-3,-4} r_{-3}^{2} + \gamma_{-3,-3} + \gamma_{-3,-2} r_{-3}^{-2} + \gamma_{-3,-1} r_{-3}^{-4} + \gamma_{-3,0} r_{-3}^{-6} + \gamma_{-3,1} r_{-3}^{-8}$$

$$=-1, z=a_{-1} r_{-1}^{2} \;;\; I^{(s)}(r_{-1}) = \gamma_{-1,-4} r_{-1}^{-6} + \gamma_{-1,-3} r_{-1}^{-4} + \gamma_{-1,-2} r_{-1}^{-2} + \gamma_{-1,-1} + \gamma_{-1,0} r_{-1}^{2} + \gamma_{-1,1} r_{-1}^{4}$$

$$= 0, z=a_0 r_0 \;;\; I^{(s)}(r_0) = \gamma_{0,-4} r_0^{-4} + \gamma_{0,-3} r_0^{-3} + \gamma_{0,-2} r_0^{-2} + \gamma_{0,-1} r_0^{-1} + \gamma_{0,0} + \gamma_{0,1} r_0$$

$$= 1, z=a_1 r_1^{2/3} \;;\; I^{(s)}(r_1) = \gamma_{1,-4} r_1^{-10/3} + \gamma_{1,-3} r_1^{-8/3} + \gamma_{1,-2} r_1^{-2} + \gamma_{1,-1} r_1^{-4/3} + \gamma_{1,0} r_1^{-2/3} + \gamma_{1,1}$$

<u>$N=10$</u> ; $[0,1,1_6]$; $L=1$, $M=4$

$$j=-1, z=a_{-1} r_{-1}^{2} \;;\; I^{(s)}(r_{-1}) = \gamma_{-1,-2} r_{-1}^{-2} + \gamma_{-1,-1} + \gamma_{-1,0} r_{-1}^{2} + \gamma_{-1,1} r_{-1}^{4} + \gamma_{-1,2} r_{-1}^{6} + \gamma_{-1,3} r_{-1}^{8} + \gamma_{-1,4} r_{-1}^{10}$$

$$= 0, z=a_0 r_0 \;;\; I^{(s)}(r_0) = \gamma_{0,-2} r_0^{-2} + \gamma_{0,-1} r_0^{-1} + \gamma_{0,0} + \gamma_{0,1} r_0 + \gamma_{0,2} r_0^{2} + \gamma_{0,3} r_0^{3} + \gamma_{0,4} r_0^{4}$$

$$= 1, z=a_1 r_1^{2/3} \;;\; I^{(s)}(r_1) = \gamma_{1,-2} r_1^{-2} + \gamma_{1,-1} r_1^{-4/3} + \gamma_{1,0} r_1^{-2/3} + \gamma_{1,1} + \gamma_{1,2} r_1^{2/3} + \gamma_{1,3} r_1^{4/3} + \gamma_{1,4} r_1^{2}$$

$$= 2, z=a_2 r_2^{1/2} \;;\; I^{(s)}(r_2) = \gamma_{2,-2} r^{-2} + \gamma_{2,-1} r^{-3/2} + \gamma_{2,0} r_2^{-1} + \gamma_{2,1} r_2^{-1/2} + \gamma_{2,2} + \gamma_{2,3} r_2^{1/2} + \gamma_{2,4} r_2$$

$$j=3,\, z=a_3 r_3^{2/5} \;;\quad I^{(s)}(r_3)=\gamma_{3,-2}r_3^{-2}+\gamma_{3,-1}r_3^{-8/5}+\gamma_{3,0}r_3^{-6/5}+\gamma_{3,1}r_3^{-4/5}+\gamma_{3,2}r_3^{-2/5}+\gamma_{3,3}+\gamma_{3,4}r_3^{2/5}$$

$$=4,\, z=a_4 r_4^{1/3} \;;\quad I^{(s)}(r_4)=\gamma_{4,-2}r_4^{-2}+\gamma_{4,-1}r_4^{-5/3}+\gamma_{4,0}r_4^{-4/3}+\gamma_{4,1}r_4^{-1}+\gamma_{4,2}r_4^{-2/3}+\gamma_{4,3}r_4^{-1/3}+\gamma_{4,4}$$

$$[0,0,1_1,1_5] \;;\; L=3,\quad M=3$$

$$j=-3,\, z=a_{-3} r_{-3}^{-2} \;;\quad I^{(s)}(r_{-3})=\gamma_{-3,-3}+\gamma_{-3,-2}r_{-3}^{-2}+\gamma_{-3,-1}r_{-3}^{-4}+\gamma_{-3,0}r_{-3}^{-6}+\gamma_{-3,1}r_{-3}^{-8}+\gamma_{-3,2}r_{-3}^{-10}+\gamma_{-3,3}r_{-3}^{-12}$$

$$=-1,\, z=a_{-1} r_{-1}^{2} \;;\quad I^{(s)}(r_{-1})=\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}+\gamma_{-1,2}r_{-1}^{6}+\gamma_{-1,3}r_{-1}^{8}$$

$$=0,\, z=a_0 r_0 \;;\quad I^{(s)}(r_0)=\gamma_{0,-3}r_0^{-3}+\gamma_{0,-2}r_0^{-2}+\gamma_{0,-1}r_0^{-1}+\gamma_{0,0}+\gamma_{0,1}r_0+\gamma_{0,2}r_0^{2}+\gamma_{0,3}r_0^{3}$$

$$=1,\, z=a_1 r_1^{2/3} \;;\quad I^{(s)}(r_1)=\gamma_{1,-3}r_1^{-8/3}+\gamma_{1,-2}r_1^{-2}+\gamma_{1,-1}r_1^{-4/3}+\gamma_{1,0}r_1^{-2/3}+\gamma_{1,1}+\gamma_{1,2}r_1^{2/3}+\gamma_{1,3}r_1^{4/3}$$

$$=2,\, z=a_2 r_2^{1/2} \;;\quad I^{(s)}(r_2)=\gamma_{2,-3}r_2^{-5/2}+\gamma_{2,-2}r_2^{-2}+\gamma_{2,-1}r_2^{-3/2}+\gamma_{2,0}r_2^{-1}+\gamma_{2,1}r_2^{-1/2}+\gamma_{2,2}+\gamma_{2,3}r_2^{1/2}$$

$$=3,\, z=a_3 r_3^{2/5} \;;\quad I^{(s)}(r_3)=\gamma_{3,-3}r_2^{-12/5}+\gamma_{3,-2}r_3^{-2}+\gamma_{3,-1}r_3^{-8/5}+\gamma_{3,0}r_3^{-6/5}+\gamma_{3,1}r_3^{-4/5}+\gamma_{3,2}r_3^{-2/5}+\gamma_{3,3}$$

$$[0,0,1_2,1_4] \;;\; L=4,\quad M=2$$

$$j=-4,\, z=a_{-4} r_{-4}^{-1} \;;\quad I^{(s)}(r_{-4})=\gamma_{-4,-4}+\gamma_{-4,-3}r_{-4}^{-1}+\gamma_{-4,-2}r_{-4}^{-2}+\gamma_{-4,-1}r_{-4}^{-3}+\gamma_{-4,0}r_{-4}^{-4}+\gamma_{-4,1}r_{-4}^{-5}+\gamma_{-4,2}r_{-4}^{-6}$$

$$j=-3, z=a_{-3}r_{-3}^{-2} \;;\; I^{(s)}(r_{-3}) = \gamma_{-3,-4}r_{-3}^{2}+\gamma_{-3,-3}+\gamma_{-3,-2}r_{-3}^{-2}+\gamma_{-3,-1}r_{-3}^{-4}$$

$$+\ \gamma_{-3,0}\ r_{-3}^{-6}+\gamma_{-3,1}r_{-3}^{-8}+\gamma_{-3,2}r_{-3}^{-10}$$

$$=-1, z=a_{-1}r_{-1}^{2} \;;\; I^{(s)}(r_{-1}) = \gamma_{-1,-4}r_{-1}^{-6}+\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}+\gamma_{-1,-1}+\gamma_{-1,0}r_{-1}^{2}$$

$$+\ \gamma_{-1,1}r_{-1}^{4}+\gamma_{-1,2}r_{-1}^{6}$$

$$=0, z=a_{0}r_{0} \;;\; I^{(s)}(r_{0}) = \gamma_{0,-4}r_{0}^{-4}+\gamma_{0,-3}r_{0}^{-3}+\gamma_{0,-2}r_{0}^{-2}+\gamma_{0,-1}r_{0}^{-1}+\gamma_{0,0}$$

$$+\ \gamma_{0,1}r_{0}+\gamma_{0,2}r_{0}^{2}$$

$$=1, z=a_{1}r^{2/3} \;;\; I^{(s)}(r_{1}) = \gamma_{1,-4}r_{1}^{-10/3}+\gamma_{1,-3}r_{1}^{-8/3}+\gamma_{1,-2}r_{1}^{-2}+\gamma_{1,-1}r_{1}^{-4/3}$$

$$+\ \gamma_{1,0}r_{1}^{-2/3}+\gamma_{1,1}$$

$$=2, z=a_{2}r_{2}^{1/2} \;;\; I^{(s)}(r_{2}) = \gamma_{2,-4}r_{2}^{-3}+\gamma_{2,-3}r_{2}^{-5/2}+\gamma_{2,-2}r_{2}^{-2}+\gamma_{2,-1}r^{-3/2}$$

$$+\ \gamma_{2,0}r_{2}^{-1}+\gamma_{2,1}r_{2}^{-1/2}+\gamma_{2,2}$$

$$[0,0,2_{3}] \;;\; L=5,\quad M=1$$

$$j=-1, z=a_{-1}r_{-1}^{2} \;;\; I^{(s)}(r_{-1}) = \gamma_{-1,-5}r_{-1}^{-8}+\gamma_{-1,-4}r_{-1}^{-6}+\gamma_{-1,-3}r_{-1}^{-4}+\gamma_{-1,-2}r_{-1}^{-2}$$

$$+\ \gamma_{-1,-1}+\ \gamma_{-1,0}r_{-1}^{2}+\gamma_{-1,1}r_{-1}^{4}$$

$$=0, z=a_{0}r_{0} \;;\; I^{(s)}(r_{0}) = \gamma_{0,5}r_{0}^{-5}+\gamma_{0,-4}r_{0}^{-4}+\gamma_{0,3}r_{0}^{-3}+\gamma_{0,2}r_{0}^{-2}+\gamma_{0,-1}r_{0}^{-1}$$

$$+\ \gamma_{0,0}+\gamma_{0,1}r_{0}$$

$$=1, z=a_{1}r_{1}^{2/3} \;;\; I^{(s)}(r_{1}) = \gamma_{1,-5}r_{1}^{-4}+\gamma_{1,-4}r_{1}^{-10/3}+\gamma_{1,-3}r_{1}^{-8/3}+\gamma_{1,-2}r_{1}^{-2}$$

$$+\ \gamma_{1,-1}r_{1}^{-4/3}+\gamma_{1,0}r_{1}^{-2/3}+\gamma_{1,1}$$

let us take two transformations:

$$z = a_m r_m^{\frac{2}{m+2}} \tag{34a}$$

$$z = a_n r_n^{\frac{2}{n+2}} \tag{34b}$$

The indices $\alpha_{j,k}$ in (33c) take the same values for $j=m$ and for $j=n$, but they are related to different constants A_k in the original invariant. With this in mind, we find:

$$\alpha_{n,k} = \frac{m+2}{n+2}\,\alpha_{m,k} + \frac{2(m-n)}{n+2} \quad , \qquad k\neq -2 \;; \tag{35a}$$

$$u_n(r_n) = N_{m,n}\, r^{\frac{m-n}{(m+2)(n+2)}}\, u_m(r_m) \quad , \tag{35b}$$

$$\gamma_{n,k} = \left(\frac{a_n}{a_m}\right)^{k+2} \left(\frac{m+2}{n+2}\right)^2 \gamma_{m,k} \quad , \qquad k\neq -2 \;; \tag{35c}$$

$$\gamma_{n,-2} = \left(\frac{m+2}{n+2}\right)^2 \gamma_{m,-2} + \frac{n(n+4)-m(m+4)}{4(n+2)^2} \tag{35d}$$

It is easy to see that any "diagonal" term $\gamma_{k\,k}$ is just the energy of the relevant problem.

It is also easy to see that the term $\gamma_{n,n}$ is related to the coupling $\gamma_{m,n}$ for r_m.

Applications of these transformations have been made to the case $N=6$ for the Coulomb and spherical oscillator problems. Since n=-1 is the harmonic potential, calling

$$\gamma_{-1,-2} = -\ell_{-1}(\ell_{-1}+1) \quad , \quad \gamma_{-1,-1} = k_{-1}^2 \quad , \quad \gamma_{-1,0} = -\,\omega^2$$

the angular momentum, energy and force constant, the corresponding Schrödinger equation is

$$\frac{d^2u_{-1}}{dr_{-1}^2}(r_{-1}) + \left[\ell_{-1}(\ell_{-1}+1)\,r_{-1}^{-2} + k_{-1}^2 - \omega^2 r_{-1}^2\right]u_{-1}(r_{-1}) = 0 \tag{36}$$

For $j=0$ we have the Coulomb potential, writing

$$\gamma_{0,-2} = -\ell_0(\ell_0+1) \quad , \quad \gamma_{0,-1} = -g_1 \quad \text{and} \quad \gamma^{0,0} = k_0^2 \ ;$$

the corresponding Schrödinger equation is

$$\frac{d^2u_0(r_0)}{dr_0^2} + \left[-\ell_0(\ell_0+1)r_0^{-2} - g_1r_0^{-1} + k_0^2\right]u_0(r_0) = 0 \ . \tag{37}$$

We obtain the well known relations[11,16,21]

$$\ell_{-1} + \frac{1}{2} = 2\left(\ell_0 + \frac{1}{2}\right) \tag{38}$$

$$k_{-1}^2 = -4\left(\frac{a_{-1}}{a_0}\right) g_1 \tag{39}$$

Notice that the signs are such that for positive k_{-1}^2 and ω^2 only negative k_0^2 and positive g_1 are related, showing that only bound states enter the game.

Bose[6] and Gazeau[21] studied these transformations for the rather special case of potentials with a single term (apart from the energy centrifugal terms). Johnson[11] has developed this work to cover multi-term potentials. All this work represents a set of special cases of the transformations we present here. In our case we profit from the knowledge of a common foundation for all cases to understand better their properties; in other terms, we take advantage of the fact that all potentials in a family are obtained out of the same invariant.

A remark is appropriate here: since we intend to discuss normal solutions (eq. (10)), which decrease strongly at infinity, the signs of the several couplings are important. If the origin is a regular singularity for the corresponding equation, for potentials with positive maximum power the corresponding coefficient must be positive, as well as the energy term. If, however, the highest power is the energy term, it has to have a negative coefficient, as well as the highest negative power. If the singularity at the origin is not regular, the smallest (more negative) power must have a positive coefficient to avoid the "fall into the center" catastrophe[22].

We shall now concentrate on the particular case of potentials with a single term. For these, we have

$$\gamma_{j,-2} = -\ell_j(\ell_j+1) \quad , \quad \gamma_{j,j} = k_j^2$$

and the only term in the potential we choose to be $\gamma_{j,n}$ $(n \neq j,\ n \neq -2)$. There are three cases possible, corresponding to the range of values of α_{jn}. They are

1) $-\infty < \alpha_{jn} < -2$
2) $-2 < \alpha_{jn} < 0$
3) $0 < \alpha_{jn} < \infty$

Let us consider the second case. Then, as the centrifugal term controls the behaviour at the origin and the energy term controls the behavíour at infinity, we have the following conditions:

a) The lowest power, r_1^{-2}, imposes $L=1$.
b) The highest power being $r_3^?$ we must have

$$j = M$$

So

$$I^{(s)}(r_M) = \gamma_{M,-2}\, r_M^{-2} + \gamma_{M,n}\, r_M^{\alpha_{M,n}} + \gamma_{M,M}$$

To allow $\alpha_{M,n}$ to take values in the interval $(-2,0)$, we must have

$$-1 \leqslant n \leqslant M-1 \ .$$

Choosing one particular value for n, say m, using the transformations (35):

$$\gamma_{M,M} = \left(\frac{a_M}{a_m}\right)^{M+2} \left(\frac{m+2}{M+2}\right)^2 \gamma_{m,M}$$

$$\gamma_{M,m} = \left(\frac{a_M}{a_m}\right)^{m+2} \left(\frac{m+2}{M+2}\right)^2 \gamma_{m,m}$$

the invariants will be:

$$I^{(s)}(r_M) = \gamma_{M,-2}\, r_M^{-2} + \gamma_{M,m}\, r_M^{\alpha_{M,m}} + \gamma_{M,M}$$

$$I^{(s)}(r_m) = \gamma_{m,-2}\, r_m^{-2} + \gamma_{m,m} + \gamma_{m,M}\, r^{\alpha_{m,M}}$$

We see that the potential coupling $\gamma_{M,m}$ becomes the energy of the new invariant, and the respective energy now becomes the highest potential term whose exponent is, now:

$$\alpha_{m,M} = \frac{2\,(M-m)}{m+2}$$

which is positive, since $m \leqslant M-1$. That is

$$\frac{2}{(M+1)} \leqslant \alpha_{m,M} \leqslant 2\,(M+1)$$

A study of Table 2 shows several examples, among them the relation $-1 \rightarrow 2$, Coulomb to oscillator, which has already been considered (eqs. (36) to (39)).

The relation between both exponents is

$$\alpha_M = -\frac{2\alpha_{m,M}}{\alpha_{m,M}+2}$$

Considering potentials in the first range of values of the exponent, they transform into potentials in the same interval. Since the energy term and the singular term at the origin must be present, the starting invariant is:

$$I^{(s)}\,(r_j) = \gamma_{j,k}\, r^{\alpha_{j,k}} + \gamma_{j,-2}\, r_j^{-2} + \gamma_{jj}$$

An analysis analogous to the one performed before shows that again the energy term in one invariant becomes the potential term in the other.

For potentials with several terms, or multiterm potentials, we can use again the transformation from a given invariant via the change of variable (34).

Looking at eqs. (35) we see that the energy term $\gamma_{k,k}$ in $I^{(s)}\,(r_k)$ is related to the coupling constant $\gamma_{j,k}$ for the potential in $I^{(s)}\,(r_j)$, whose exponent $\alpha_{j,k}$ is:

$$\alpha_{j,k} = \frac{2\,(k-j)}{j+2}$$

Conversely, the energy term $\gamma_{j,j}$ in $I^{(s)}\,(r_j)$ is related to the term with coupling $\gamma_{k,j}$ in $I^{(s)}\,(r_k)$ with exponent $\alpha_{k,j}$ such that

$$\alpha_{k,j} = \frac{2\,(j-k)}{k+2}$$

or, again:

$$\alpha_{k,j} = - \frac{2\alpha_{j,k}}{\alpha_{j,k}+2} .$$

The remaining potential terms are of the form:

$$\alpha_{k,m} = \frac{2(\alpha_{j,m} - \alpha_{j,k})}{\alpha_{j,k} + 2}$$

and these relations are extensions of the ones obtained by Johnson[11].

The fact that the coupling constants and exponents transform with the change of variables that carries from one member of a family of potentials to another member of the same family might create the illusion that some sort of transformation for a given N may take it toa simpler case, such as N=6. An examination of the relations involved (eqs. (28)-(30)) shows this to be impossible. The number N gennuinely represents a set of functions which are different from the ones that resulted for N=6. In other terms, unless one makes the coefficients to vanish in the original invariant different from A_{-2}, A_{-1} and A_0, in no way it is possible to relate any solution for a given $N(\neq 6)$ to the oscillator or Coulomb solutions.

4. ON THE SOLUTIONS FOR POWER LAW POTENTIALS

The problem of solving the radial Schrodinger equation can be faced by first looking at the solutions of

$$\frac{d^2w(z)}{dz^2} + I(z)w(z) = 0$$

where $I(z)$ is an invariant such that with appropriate changes of variable several Schrödinger invariants follow. The changes of variables and the corresponding changes in the dependent variable are eqs. (28) and (29), respectively.

The Schrödinger equations so obtained are of different nature according to whether the origin is a regular or irregular singular point. In the first case, we must have:

$$I(z) = \sum_{i=-2}^{M} A_i z^i$$

We exchange the points at the origin and infinity, as we did before, by the transformation $z = \rho^{-1}$:

$$\frac{d^2w(\rho)}{d\rho^2} + \frac{2}{\rho}\frac{dw(\rho^{-1})}{d\rho} + \frac{1}{\rho^4} I(z(\rho))w(\rho) = 0 \qquad (40)$$

In the notation of eq. (8):

$$p(\rho) = \frac{2}{\rho}$$

$$q(\rho) = \sum_{i=-2}^{M} A_i \rho^{-(i+4)}$$

There is no regular solution at infinity, since the smallest value for M is -1. If we now propose a normal solution:

$$w(\rho) = \exp[-f(\rho)]v(\rho)$$

with $f(\rho)$ a polynomial in $1/\rho$:

$$f(\rho) = \sum_{j=1}^{J} \frac{a_j}{j} \rho^{-j} ,$$

substituting it in (40) we have:

$$\frac{d^2v}{d\rho^2} + \left[\frac{2}{\rho} + \sum_{j=1}^{J} a_j \rho^{-(j+1)}\right] \frac{dv}{d\rho} + \left[\sum_{j=1}^{J}\sum_{k=1}^{J} a_j a_k \rho^{-(j+k+2)}\right.$$

$$\left. - \sum_{j=2}^{J} (j-1) a_j \rho^{-(j+2)} + \sum_{i=-2}^{M} A_i \rho^{-(i+4)}\right] v = 0$$

In order to obtain a possibly regular solution for v at infinity, since the strongest pole in the second bracket is of order $-J-1$, we must assure that the third bracket has no term with a pole of higher order than $-2J-2$. The only possible relation involving the original invariant is:

$$-(2J+2) = -(M+4)$$

$$a_J^2 = A_M$$

or

$$J = \frac{M}{2} + 1$$

The remaining coefficients result from *(40)* by demanding the following unwanted singularities in the' coefficients to cancel.

It is, however, an easy and interestlng exercise to calculate these terms in the exponential by another, physically more appealing, way. Let's look at the action-like integral

$$\int p_r dr = \int \left[2mE - \frac{\ell^2}{r^2} - V(r)\right]^{1/2} dr$$

If the potential is divergent at infinity, let's then extract the most divergent term and expand the square root keeping all theterms up to the convergent ones at infinity. This determines uniquely the coefficients of the exponential factor in the normal solution. For instance, let $V(r)$ be

$$\begin{aligned} V(r) &= V_1 r^2 + V_2 r^4 + V_3 r^6 \\ &= V_3 r^6 \left(1 + \frac{V_2}{V_3} r^{-2} + \frac{V_1}{V_3} r^{-4}\right) \quad . \end{aligned}$$

The square root in the above integral becomes

$$V_3^{1/2}\, r^3 \left(1 + \frac{1}{2}\frac{V_2}{V_3} r^{-2}\right) + 0(r^{-2})$$

which upon integration results in the same coefficients for the exponential as the substitution in *(40)* does.

The same holds for the origin. Exponentiating the result one obtainsthe r^{ℓ} term for the radial Schrödinger wave functions.

The amazing thing is perhaps related to the WKB approximation for the potential.

The only possible normal solution of eq. (40) is for even values of M (and of N, consequently). To obtain a solution for odd M one needs a subnormal[2] solution.

We pass now to the Schrödinger equation, through (28) and (29), one having $I^{(s)}(r_{-1})$ as invariant:

$$I^{(s)}(r_{-1}) = \sum_{k=-2}^{M} \gamma_{-1,k}\, r_{-1}^{\alpha_{-1,k}}$$

The cases (see Table 2) correspond to potentials of the confining type with maximum power

$$\alpha_{-1,M} = 2,6,10,14,\ldots$$

for N = 6,8,10,12,..., respectively and

$$\alpha_{-1,M} = 4,8,12,16,..$$

for N = 7,9,11,13,..., respectively.

The cases for N even are the ones related to normal solutionsof (40) ; for odd N, the solutions of the Schrodinger equation are normal but for (40) they are subnormal.

This result for odd M (and N) gives support to the choice made by Znojil in his original work, namely, for the potentials with even positive power the Schrödinger equation admits always a normal solution.

The other potentials in a given family for any N admit at most subnormal solutions. For N even, two kinds of potential admit a normal solution: the one obtained directly from the invariant $I(z)$ and the one with $\alpha_{-1,M} = 2,6,10,14....$ For N odd, just the potentials with $\alpha_{-1,M} = 4,8,12,\ldots$ admit a normal solution.

The normal or subnormal solutions may be expanded in power series. The relevant expressions may be found in the articles by Znojil[10] and Rampal and Datta[15].

It has been proved[1,2] that if the series in a normal or subnormal solution doesn't end, it must be divergent. That's why solutions are written[9,10] in terms of continued fractions. Znojil[10] has proved the convergence of the Green function of the extended continued fraction, and showed numerical examples of good convergence for the solutions.

When neither the origin nor infinity are regular singular points, the procedure just applied is extended to the case when normal solutions work for the origin.

5. CONCLUSION

We have presented what we believe to be the most systematic mathematical treatment for the two body Schrödinger equation with power law potentials. It is probably the most general admissible one in terms of the theory of analytic functions.

The method developed by Ince allowed us to establish relationships between families of potentials, as exemplified in Table 2. This may be quite useful when trying to devise a solution for a given problem if the solution for another member of the family is known.

The potentials displayed in Table 2 are all of rational exponent, and cover most of the examples found in the Physics literature. It is curious, at least, that a potential which was proposed almost entirely from heuristic arguments, the famous 6-12 of Lennard and Jones, is a member of one of these families. It is possible that other combinations which are useful in other domains of physics appear inthe list.

We have discussed shortly the solutíons for members of these families of potentials. Normal solutions should exist for several of them and, presumably, the relations among the members of a family should help in finding new solutions. An important question is: are these the *only* possible solutions? Apotential such as r^{π} doesn't fit in the scheme; does this mean it doesn't have a normal or subnormal solution? The question surely admits several roads to its answer.

The understanding of the properties of the possible solutions remains also far from complete. The case N=6 has been widely studied in the past and its applications continue. However, for $N \geqslant 7$, the possible solutions constitute in general gennuinely new classes offunctions with the exception of the generalizations of the hypergeometric case (N even, the origin as a regular singularity). It would be quite interesting to advance in the sense of substantially allowing for a classification of solutions and their properties in terms of the singularities at the origin and infinity.

Another interesting question is the apparent one to one relationship between the spectra of energy values of several potentials in a given family as was originally remarked by Johnson[11]. We conjecture that discrete spectra map into discrete spectra, and continuum into continuum. This can be seen in the oscillator-Coulomb relationship[23] and deserves further exploration.

Whereas we have made an analysis in Section 4 of the existence of normal or subnormal solutions, this analysis is obviously restricted to the discrete spectrum. Regarding scattering (or unbounded) solutions, almost nothing is known and the present treatment might be of relevance to them. This is of special interest for people working in atomic collision theory.

Another point to be considered is how could we generate solutions for non central potentials, such as multipole interactions for two body problems, by applying similar reasoning.

We hope to have shown a promising field of future research is open since we made evident a powerful analytic method to group problems in non relativistic quantum potential theory.

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Resumo

Mostramos que os potenciais do tipo das combinações de potencias com expoentes racionais admitem um tratamento sistemático disponível da teoria clássica das equações diferenciais lineares de segunda ordem. Os potenciais resultantes nesta análise se apresentam em familias obtidas a partir de equações diferenciais que possuem um número determinado de singularidades regulares elementares. Em decorrência, encontramos e discutimos relações entre os potenciais diferentes de uma mesma familia.