Revista Brasileira de Física, Vol. 15, nº 3, 1985

Perturbative Analysis of the Ising Model on the Koch Carpet

R.F.S. ANDRADE

Instituto de Física, Universidade Federal da Bahia, 40000, Salvador, BA, Brasil

and

S.R. SALINAS

Instituto de Física, Universidade de São Paulo, Caixa Postal 20516, São Paulo, 01498, SP, Brasil

Recebido em 28 de outubro de 1985

Abstract We use a perturbative scheme to consider an Ising model on a lattice defined as the cartesian product of a Koch curve by a linear chain. To leading order, we write the singular part of the free energy in terms of certain spin-spin correlations on the Ising square lattice. There is no change in the critical exponent associated with the specific heat.

1. INTRODUCTION

There are many questions regarding the effect of the Hausdorff dimensionality D on the critical behavior of spin models defined on fractal lattices. It was first conjectured that it plays the same role as the Euclidean dimension d, as in the case of E = 4-d expansions used in the renormalization-group scheme. This has been checked by the application of approximate and also exact decimation procedures to several models. The results obtained so far indicate that the critical behavior may depend on D, but also depends on the so-called spectral dimensionality, D_S, and on other properties of the fractal lattices, as the branching character and the connectivity[1-4].

Despite the elegance of the decimation procedures, which are quite adequate for taking care of the self-similarity of the fractal structures, we believe there is still room for additional exact as well as approximate analyses of the critical behavior. We have recently used the transfer matrix formalism to study the Ising model on a Koch curve[5]. From the exact results in zero field, we show that there is no transition, and the fractal dimensionality plays a trivial role. In the present paper we consider an Ising model on a lattice defined as a cartesian product of a Koch curve by a linear chain. The fractal dimensionality of this anisotropic Koch carpet is $D = 1 + \log 4/\log 3$. If we walk along the x direction we have a Koch curve. Along the y direction, however, there is just a simple linear chain of spins. This model is non-trivial because there are two kinds of nearest-neighbor interactions: (i) nearest-neighbor pairs of spins along x and y chains, $S_{i,j}$ $S_{\ell,k}$, with $|i+j-\ell-k| = 1$, interact with an exchange parameter J; (ii) nearest-neighbor pairs of spins introduced by the Koch curve, that is, of the form $S_{i,j}S_{i+2,j}$, interact with an exchange parameter J_F.

It is easy to formulate the partition function of the Ising model on the anisotropic Koch carpet according to the standard procedures to consider two-dimensional statistical problems. In particular, we write a transfer matrix, but the analysis of the largest eigenvalue becomes a non-trivial problem, probably without an analytical solution. We then resort to a perturbative scheme, introduced by Barber to study the Ising model on a square lattice with nearest and next-nearest neighbor interactions, which gives the leading term of the free energy for $|\epsilon| \equiv |J_F/J| << 1$. It should be mentioned that it is equally easy to apply this scheme directly, without reference to the transfer matrix method. The free energy is written, as in the work of Barber, in terms of the free energy of the corresponding Ising model on the underlying simple quadratic lattice, that is, with $J_F = 0$, plus a correction term involving the spin-spin correlation functions related to the terms depending on J_F.

Unlike the case of the super-antiferromagnetic (SAF) model considered by Barber, in the present case the critical temperature is shifted but the critical index α associated with the specific heat remains unchanged. As we are not aware of other applications of Baber's scheme, we also checked this procedure in the case of the exactly soluble Ising model on the Union-Jack lattice. We then have strong indications that the fractal dimensionality of the anisotropic Koch carpet plays a trivial role, which is restricted to the scaling of the correlation length along the x direction.

This paper is organized as follows. In section 2 we define the model and discuss, on the basis of the transfer matrix formulation, the main difficulties to obtain an exact solution. In section 3 we obtain the leading order expression of the free energy in the limiting case

$|\epsilon| = |J_F/J| << 1$. In section 4 we use Barber's procedure to analyze the critical behavior of the model. Finally, some concluding remarks are presented in section 5.

2. THE FORMULATION OF THE PROBLEM

In zero field the Ising model on the anisotropic Koch carpet is given by the Hamiltonian

$$H_N = -J \sum_{i,j=1}^{N} S_{i,j}(S_{i,j+1} + S_{i+1,j}) - J_F \sum_{L=1}^{N-1} \sum_{i=1}^{N/4L} S_{g,j} S_{g+2,j} \qquad (2.1)$$

where $N = 4^N$, $L = 4^L$, $g = g(L,i) = L(4i-2)$, and N indicates the N-th step in the construction of the carpet. The first sum in eq. (2.1) describes the nearest neighbor Ising interactions which are homeomorphic to the Ising square lattice with N^2 sites. The second sum takes care of the non-periodic interactions allong the x-direction of the model, which are responsible for the fractal character.

The model described by eq. (2.1) is homeomorphic to the Ising model on a square lattice *plus* non-periodic third-neighbor interactions. We can then use the well known techniques developed to analyze two-dimensional spin problems, However, this homeomorphism, which relates planar lattices with higher order neighbor interactions and fractal models, already indicates the kind of difficulties in the solution of the problem. Indeed, the presence of crossing bonds in planar lattices precludes the establishment of exact solutions by all known methods. In the present case this is easily seen if we discuss the main steps towards the solution in terms of the transfer matrix method. We refer to the classical treatments[7-9] for further discussions and details of the calculations.

Let P be the transfer matrix which takes into account all interactions in a row along the x direction and also between spins in adjacent rows (y and $y+1$). Then the free energy per spin, f, is given by

$$f = -\frac{k_B T}{N^2} \log(\mathrm{Tr}\, P) \xrightarrow[N \to \infty]{} -\frac{k_B T}{N} \log \lambda_{max} , \qquad (2.2)$$

where λ_{max} is the largest eigenvalue of P. In order to find λ_{max} it is convenient to write P in terms of a set of $2N$ matrices Γ_μ of order

$2^N \times 2^N$ (see ref. 8 for a characterization of this set and is properties),

$$P = \left[2\,\mathrm{senh}\,(2\beta J)\right]^{2N/2} \left[\prod_{\alpha=1}^{N} \exp\left(-i\beta J \Gamma_{2\alpha+1} \Gamma_{2\alpha}\right)\right]$$

$$\times \left[\prod_{L=0}^{N-1} \prod_{\alpha=1}^{N/4L} \exp\left(-\beta J_F \Gamma_{2g+3} \Gamma_{2g+2} \Gamma_{2g+1} \Gamma_{2g}\right)\right]$$

$$\times \left[\prod_{\alpha=1}^{N} \exp\left(-i\theta \Gamma_{2\alpha} \Gamma_{2\alpha-1}\right)\right] , \tag{2.3}$$

where $\theta = \tanh^{-1} (\exp(-2\beta J))$

If we call P_I the transfer matrix of the simple Ising square lattice, it is easy to see that it differs from P by the presence of terms with a four-fold product of Γ matrices. The diagonalization of P_I is based on the fact that each term $U_{\mu\nu} = \exp(-i\,\theta\Gamma_\mu\Gamma_\nu)$ describes a similarity transformation between two well-defined representations of the matrices Γ_μ. This transformation may also be represented by a rotation $u_{\mu\nu}$ in the $2N$-dimensional space of the matrices Γ_μ. Thus, if P_I isgiven by the product of several terms of the type $U_{\mu\nu}$, it is possible to define p_I as the product of the corresponding terms $u_{\mu\nu}$. The diagonalization of p_I is then an easy problem which leads to the eigenvalues of P_I. However, the presence of four-fold products in eq. (2.5) makesit impossible to relate P to a $2N\times 2N$ matrix p. Indeed, the four-fold products do not describe rotations among the Γ_μ, since the similarity transformation gives rise to the presence of three-fold products of the Γ's which are linearly independent of the Γ_μ's.

Due to this specific difficulty (which will also appsar if we choose another method) , solutions of planar models with crossirig bonds, and hence of fractal lattices with Hausdorff dimensions D > 2, still remain to be found. So, in the next section we restrict our treatment to a formulation in the limit $|\epsilon| \equiv |J_F/J| << 1$. Also, we are forced to resort to the perturbative scheme introduced by Barber.

3. THE LIMIT OF SMALL J_F

When $|\varepsilon| << 1$, we may write to leading order

$$\prod_{L=0}^{N-1} \prod_{\alpha=1}^{N/4L} \exp\left(-\beta \epsilon J\, \Gamma_{2g+3}\, \Gamma_{2g+2}\, \Gamma_{2g+1}\, \Gamma_{2g}\right)$$

$$\cong (\cosh \beta J \epsilon)^{N/3} \left[I - Q_N \tanh \beta J \epsilon \right] , \qquad (3.1)$$

where

$$Q_N = \sum_{L=0}^{N=1} \sum_{\alpha=1}^{N/4N} \Gamma_{2g+3}\, \Gamma_{2g+2}\, \Gamma_{2g+1}\, \Gamma_{2g} . \qquad (3.2)$$

Let us assume that it is not necessary to consider higher order terms in eq. (3.1) to detect possible changes in the critical behavior of the model with respect to the simple square lattice. If we call

$$P' = P(\cosh \beta J \epsilon)^{-N/3} , \qquad (3.3)$$

and use eqs. (3.1) and (3.2), we have to leading order

$$P' \cong P_I - Q_N (\tanh \beta J \epsilon) P_I , \qquad (3.4)$$

since

$$(2 \sinh \beta J)^{N/2} \left[\prod_{\alpha=1}^{N} \exp\left(-i\beta J\, \Gamma_{2\alpha+1}\, \Gamma_{2\alpha}\right)\right] Q_N \left[\prod_{\alpha=1}^{N} \exp\left(-i\theta\, \Gamma_{2\alpha}\, \Gamma_{2\alpha-1}\right)\right] = Q_N P_I \qquad (3.5)$$

To first order in $\omega = \tanh \beta J \epsilon$, the eigenvalues of P' aregiven

$$\lambda_{P'} = \lambda_I - \omega \langle \chi_I | Q_N P_I | \chi_I \rangle = \lambda_I \left[1 - \omega \langle \chi_I | Q_N | \chi_I \rangle \right], \qquad (3.6)$$

where λ_I and $|\chi_I|$ are the eigenvalues and eigenvectors of the transfer matrix P_I of the simple Ising square lattice. The free energy per spin is given by

$$f = \lim_{N\to\infty} - \frac{k_B T}{N^2} \log\left[\mathrm{Tr}\, P^N \right] = - \frac{k_B T}{N^2} \log\left[\lambda_I^{\max} (1 - \omega \langle \lambda_I^{\max} | Q_\infty | \lambda_I^{\max} \rangle) \right] N , \qquad (3.7)$$

where the superscript max refers to the largest eigenvalue and its corresponding eigenvector. In the limit $N \to \infty$, the expectation value of Q_N is given by

$$\lim_{N\to\infty} \frac{1}{N} \langle\chi_I^{max}|Q_N|\chi_I^{max}\rangle = \langle\chi_I^{max}|\frac{1}{3}\Gamma_7\ \Gamma_6\ \Gamma_5\ \Gamma_4|\chi_I^{max}\rangle = \frac{1}{3}\langle S_{2,1}\ S_{4,1}\rangle_I \tag{3.8}$$

where $\langle S_{i,j}\ S_{k,\ell}\rangle_I$ is a spin-spin correlation, on the underlying Ising square lattice, between sites (i,j) and (k,ℓ). Thus we have

$$f = f_I - \frac{k_B T\omega}{3}\langle S_{1,1} S_{3,1}\rangle_I \quad , \tag{3.9}$$

where f_I is the free energy of the Ising square lattice. If we define

$$k \equiv \frac{2\ \sinh\ 2\beta J}{\cosh\ \ 2\beta J} \quad , \tag{3.10}$$

and call K and E the complete elliptic integrals of the first andsecond kind respectively, with modulus k, it is easy to obtain the expression

$$\langle S_{1,1} S_{3,1}\rangle_I = \frac{1 - (1-k^2)^{1/2}}{k^2} - \frac{4}{\pi^2 k^2}\left[E - K^2(1-k^2)^{3/2} - 2EK(1-k^2)^{1/2}\right] \tag{3.11}$$

This correlation is continuous as a function of T, with a derivative which becomes singular at the critical temperature, given by $k=1$, of the Ising model on the square lattice. In the following section we use eqs. (3.9) and (3.11) to perform a detailed analysis in the neighborhood of the critical temperature.

4. THE CRITICAL BEHAVIOR

In the neighborhood of the critical temperature, T_c, we can write the following asymptotic expressions for the singular parts of the free energy and the spin-spin correlation function of the Ising model on a simple square lattice,

$$\beta f_I = \frac{4}{\pi}(\Delta j)^2\ \log\ |\Delta j| \quad , \tag{4.1}$$

and

$$\langle S_{1,1} S_{3,1}\rangle_I = -\frac{16}{\pi^2}\sqrt{2}\ \Delta j\ \log|\Delta j| + \frac{48}{\pi^2}(\Delta j)^2\ \log\ |\Delta j|, \tag{4.2}$$

where $j = J/k_BT$, and $\Delta j = j - j_c$, with $j_c = J/k_BT_c$. Inserting these expressions into eq. (3.9), we have to leading order the singular part of the free energy

$$\beta f = \frac{4}{\pi} (\Delta j)^2 \log|\Delta j| + \frac{1}{3}\omega_c \left[\frac{16\sqrt{2}}{\pi^2} \Delta j \log|\Delta j| - \frac{48}{\pi^2} (\Delta j)^2 \log|\Delta j|\right] \qquad (4.3)$$

where $\omega_c = \tanh(\epsilon j_c)$.

The central idea of Barber's method is the assumption that the dependence of the singular part of f with respect to $\Delta j' = \Delta j + \eta$ is the same as the dependence of the singular part of f_I with respect to Δj, where η is a function of the small parameter ω. For small deviations it is enough to suppose a linear dependence of η, as well as of the critical exponent α, with respect to ω. If we write $\eta = A\omega$ and $\alpha = B\omega$, we should have

$$\beta f = \frac{4}{\pi} (\Delta j + A\omega)^2 \frac{1 - (\Delta j + A\omega)^{-B\omega}}{B\omega}, \qquad (4.4)$$

which can be written as

$$\beta f = \frac{4}{\pi} \{(\Delta j)^2 \log|\Delta j| + \omega_c[2A \Delta j \log|\Delta j| - \frac{B}{2} (\Delta j \log|\Delta j|)^2]\} + 0(\omega^2). \qquad (4.5)$$

Comparing with eq. (4.3) we have

$$A = \frac{2\sqrt{2}}{3\pi} \quad \text{and} \quad B = 0, \qquad (4.6)$$

which indicates that the critical exponent α keeps the same value as in the Ising model on the square lattice. There is just a shift in the critical temperature, which is given by

$$T_c = T_{c,0} \left[1 + \frac{2\sqrt{2}}{3\pi} \epsilon\right], \qquad (4.7)$$

if we make $\omega_c \approx \epsilon j_c$, and $T_{c,0}$ is the critical temperature of the Ising square lattice.

It is now appropriate to make some comments concerning the difference between Barber's and our own results. To this purpose, let us consider an Ising model on a square lattice, with first and second neighbor interactions, as in the case of the SAF model of Barber. Due to the absence of a term proportional to $(\Delta_j \log|\Delta_j|)^2$ in eq. (4.3), we have not found any change in the value of the critical exponerit. Terms of this type, however, do appear in the SAF model because in this case the dominant correction is a four-spin correlation function which splits into a square of two-spin correlations, each one of them with a term proportional to $\Delta_j \log|\Delta_j|$. It shoud be noticed that these terms donot appear in the case of the present model due to the form of eq. (3.11). Indeed, terms of the type $(1-k^2)K^2$, which, for $T \sim T_c$, are proportional to $(\Delta_j \log|\Delta_j|)^2$, cancel out in this equation indentically. We are not aware whether this situation also occurs for all correlations at longer distances. If this happens as described above, the models belonging to this category will have the same critical behavior as the simple square Ising lattice.

5. CONCLUSIONS

We have studied an Ising model on a lattice defined by a cartesian product of a Koch curve and a linear chain (which we call the anisotropic Koch carpet). Our main interest resides on the application of different techniques to investigate the influence of the fractal dimensionality on the critical properties of Ising models on fractal lattices. The exact transfer matrix formalism for the anisotropíc Koch carpet has been discussed and we have pointed out that the presence of four-fold products of Γ matrices makes it impossible to obtain an explicit expression for the largest eigenvalue. We then restrict our treatment to the region $|J_F/J| << 1$, and follow a procedure introduced by Barber to analyze the critical behavior of the SAF Ising model. The singular part of the free energy can be written in terms of the free energy and a particular spin-spin correlation function, which is not difficult to calculate, for the Ising model on a simple quadratic lattice. These results, within the framework of Barber's procedure, could also have been obtained without any connection with the transfer matrix formalism. Unlike the case of the SAF model, we detect a shift in the critical temperature but no change in the critical exponent associated with the specific heat. We thus conclude that thefractal dimensionality of the anisotropic Koch carpet has a trivial effect on the critical behavior. This conclusion also holds for other models wiht the same form of the spin-spin correlations functions. Incidentally, we checked that Barber's scheme really works in the case of the exactly soluble Ising model on the Union Jack lattice.

REFERENCES

1. Y. Gefen, B.B. Mandelbrot, and A. Aharony, Phys. Rev. Lett. 45, 8556 (1980).
2. Y. Gefen, A. Aharony, and B.B. Mandelbrot, J. Phys. A16, 1267 (1983).
3. R. Rammal and G. Tolouse, J. Phys. Lett. 44, 13 (1983).
4. M. Suzuki, Prog. Theor. Phys. 69, 65 (1983).
5. R.F.S. Andrade and S.R. Salinas, J. Phys. A17, 1665 (1984).
6. M.N. Barber, J. Phys. A12, 679 (1979).
7. B. Kaufman, Phys. Rev. 76, 1232 (1949).
8. K. Huang, *Statistical Mechanics*, John Wiley, New York, 1983.
9. R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London, 1982.

Resumo

Usamos um esquema perturbativo para considerar o modelo de Ising numa rede definida pelo produto cartesiano de uma curva de Koch por uma cadeia linear. Em ordem dominante, escrevemos a parte singular da energia livre em termos de determinadas correlações spin-spin para o modelo de Ising definido numa rede quadrada. Não há mudanças no expoente crítico associado ao calor específico.