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The Blume-Emery-Griffiths Model as a Mapping Problem

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Abstract We give an exact formulation of the spin-1 Blume-Emery-Griffiths model on a Cayley tree of coordination z as a two-dimensional mapping problem. We calculate the regions of stability of the paramagnetic and the ferromagnetic fixed points. Below a tricritical temperature, there is an overlap of these regions of stability. In the infinite coordination limit, the mapping becomes one-dimensional, and we regain the mean-field results of Blume, Emery and Griffiths.

1. INTRODUCTION

In this paper we use an iterative scheme to obtain the magnetization of a spin-one Ising model deep in the interior of a Cayley tree. We have two purposes: first, we give an example of a mapping associated with a simple system showing a tricritical point; second, we emphasize the mechanism and the advantages of the so-called infinite coordination limit of a Cayley tree.

Ordinary spin-1/2 Ising models on a Cayley tree of coordination z have been studied by a number of authors[1,2,3]. In particular, Thompson[3] has shown that it is possible to write an iterative scheme of the form

$$m_j = f(m_{j-1}) \quad , \quad j = 1,2,\ldots, N , \tag{1.1}$$

for computing the magnetization per spin, m_j, in the j^{th} shell of a Cayley tree, in terms of m_{j-1}. In the ferromagnetic case, the m_j's converge to a stable fixed point m^*, such that $m^* = f(m^*)$, which may be interpreted as the local magnetization per spin deep in the interior of the tree. Indeed, this solution corresponds to the well known Bethe-Peierls approximation for the Ising model on a Bravais lattice of coordination z. Moreover, in the infinite coordination limit, $z\to\infty, J\to 0$, with zJ = constant, where J is the exchange parameter, one recovers the usual mean-field approximation for the Ising model. In the antiferromagnetic case, below certain critical values of temperature and magnetic field, the fixed point bifurcates to a stable two cycle (m_+, m_-), given by $m_+ = f(m_-)$ and $m_- = f(m_+)$.

In a very recent publication[4], Yokoi and the present authors have used this procedure to study an Ising model with competing interactions J_1 and J_2 between first and second neighbors restricted to the branches of a Cayley tree of coordination z. As had been pointed out by Vannimenus[5], who considered this model on a tree of coordination $z=3$, the iterative scheme consists of a set of three coupled non-linear first-order difference equations. However, in the infinite coordination limit, $z\to\infty$, $J_1\to 0$, $J_2\to 0$, with zJ_1 = constant and z^2J_2 = constant, the problem is considerably simplified. Under these circunstances, the set of equations is reduced to a system of only two first-order non-linear difference equations which lend themselves to a detailed numerical analysis. The phase diagram presents a variety of modulated phases, in addition to the usual ferromagnetic and paramagnetic phases. Also, it is shown that, given some initial conditions, the coupled equations may lead to a strange attractor with a fractal character[4].

A spin-one model is perhaps the simplest generalization of the ordinary spin-1/2 Ising model to present several multicritical points. In this paper, we consider a simple version of the Blume-Emery-Griffiths (BEG) model[6], given by the hamiltonian

$$H = -J \sum_{(\vec{\ell},\vec{\ell}')} S_{\vec{\ell}} S_{\vec{\ell}'} + D \sum_{\vec{\ell}} S_{\vec{\ell}}^2 - H \sum_{\vec{\ell}} S_{\vec{\ell}} , \qquad (1.2)$$

where $S_{\vec{R}} = +1,0,-1$, for all sites $\vec{\ell}$ of a crystal lattice, the first sum is over nearest neighbors, and J, D, and H, are positive parameters. On a Cayley tree of arbitrary coordination z, we write the solution of this problem in terms of a set of two first-order difference equations (for the magnetization per site $\langle S_i\rangle$, and the non-critical density $\langle S_i^2\rangle$). In the parameter space $p\times T$, where $p = D/zJ$ and T is the temperature in units of k_B/zJ, there is a critical line ending at a tricritical point. Below the tricritical temperature there is a region where both the paramagnetic and the ferromagnetic fixed points are stable. This should correspond to the Bethe-Peierls approximation for the BEG model. In the infinite coordination limit, however, the problem is reduced to a single first-order difference equation, and the numerical analysis is much simpler. In this limit, we draw the $p \times T$ phase diagram and recover the main results of the mean-field calculations of Blume, Emery, and Griffiths[6].

In section 2 we define the model, obtain the recursion relations, and analyze the stability of the paramagnetic and the ferromagnetic fixed points. The results in the infinite coordination limit are discussed in section 3. In section 2 the recursion relations are obtained according to the procedures of Eggarter[1] and Vannimenus[5], while in the appendix we present an alternative derivation, according to a scheme used by Thompson[3].

2. THE RECURSION RELATIONS AND THE STABILITY CRITERIA

The Cayley tree is a cycle-free graph, that is, a lattice with no closed paths. In fig. 1 we depict a Cayley tree with coordination $z=3$ and three generations. Consider a spin-1 Ising model, given by the hamiltonian (1.2), on a Cayley tree of arbitrary coordination z.

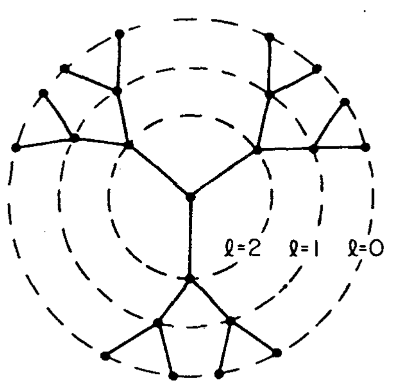


Fig.1 - Cayley tree with coordination $z=3$ and three generations of sites.

Let $Z_\ell^{(+)}$, $Z_\ell^{(0)}$, and $Z_\ell^{(-)}$ denote the partial partition functions for a tree of R generations, with the central spin fixed at the values +1, 0, and -1, respectively. Then, it is straightforward to write the relations[1,2]

$$Z_{\ell+1}^{(+)} = e^{-\beta D+\beta H}\left[e^{\beta J} Z_\ell^{(+)} + Z_\ell^{(0)} + e^{-\beta J} Z_\ell^{(-)}\right]^r \tag{2.1}$$

$$Z_{\ell+1}^{(0)} = \left[Z_{\ell}^{(+)} + Z_{\ell}^{(0)} + Z_{\ell}^{(-)} \right]^{r} , \tag{2.2}$$

and

$$Z_{\ell+1}^{(-)} = e^{-\beta D-\beta H} \left[e^{-\beta J}\, Z_{\ell}^{(+)} + Z_{\ell}^{(0)} + e^{\beta J}\, Z_{\ell}^{(-)} \right]^{r} , \tag{2.3}$$

where r = z-1. At this point it is convenient to introducethevariables

$$R_{\ell} = \frac{Z_{\ell}^{(+)}}{Z_{\ell}^{(0)}} \tag{2.4}$$

and

$$Q_{R} = \frac{Z_{\ell}^{(-)}}{Z_{\ell}^{(0)}} \tag{2.5}$$

in terms of which we have the following set of two non-linear first--order difference equations,

$$R_{\ell+1} = e^{-\beta D+\beta H} \left(\frac{e^{\beta J} R_{\ell} + 1 + e^{-\beta J} Q_{\ell}}{R_{\ell} + 1 + Q_{\ell}} \right)^{r} , \tag{2.6}$$

and

$$Q_{\ell+1} = e^{-\beta D-\beta H} \left(\frac{e^{-\beta J} R_{\ell} + 1 + e^{\beta J} Q_{\ell}}{R_{\ell} + 1 + Q_{\ell}} \right)^{r} \tag{2.7}$$

Alternatively, we may define the variables

$$m_{\ell} = \frac{Z_{\ell}^{(+)} - Z_{\ell}^{(-)}}{Z_{\ell}^{(+)} + Z_{\ell}^{(0)} + Z_{\ell}^{(-)}} , \tag{2.8}$$

and

$$q_{\ell} = \frac{Z_{\ell}^{(+)} + Z_{\ell}^{(-)}}{Z_{\ell}^{(+)} + Z_{\ell}^{(0)} + Z_{\ell}^{(-)}} , \tag{2.9}$$

which correspond to the magnetization per spin, $<S_{R}>$, on the R^{th} generation of the tree, and to the non-critica1 density, $<S_{R}^{2}>$, respectively. Also, it is easy to see that

$$m_{\ell} = \frac{R_{\ell} - Q_{\ell}}{R_{\ell} + 1 + Q_{\ell}} \tag{2.10}$$

and

$$q_\ell = \frac{R_\ell + Q_\ell}{R_\ell + 1 + Q_\ell} \quad . \tag{2.11}$$

Then we have the recursion relations

$$m_{\ell+1} = \frac{R(m_\ell, q_\ell) - Q(m_\ell, q_\ell)}{R(m_\ell, q_\ell) + 1 + Q(m_\ell, q_\ell)} \tag{2.12}$$

and

$$q_{\ell+1} = \frac{R(m_\ell, q_\ell) + Q(m_\ell, q_\ell)}{R(m_\ell, q_\ell) + 1 + Q(m_\ell, q_\ell)} \quad , \tag{2.13}$$

where

$$R(m,q) = e^{-\beta D + \beta H} \left[q \cosh \beta J + m \sinh \beta J + 1 - q \right]^r \tag{2.14}$$

and

$$Q(m,q) = e^{-\beta D - \beta H} \left[q \cosh \beta J - m \text{ iinh } \beta J + 1 - q \right]^r \tag{2.15}$$

In zero field, equations (2.12) and (2.13) have a trivial fixed point given by $m^*=0$ and $q^* \neq 0$ such that

$$q^* = \frac{2A^r}{2A^r + e^{\beta D}} \tag{2.16}$$

where

$$A = q^*(\cosh \beta J - 1) + 1 \quad . \tag{2.17}$$

To leading order, the linearization about this trivial paramagnetic fixed point yields the relations

$$m_{\ell+1} = \Lambda_1 m_\ell \tag{2.18}$$

and

$$(q_{\ell+1} - q^*) = \Lambda_2 (q_\ell - q^*) \tag{2.19}$$

where

$$\Lambda_1 = \frac{2rA^{r-1}}{2A^r + e^{\beta D}} \sinh \beta J \tag{2.20}$$

and

$$\Lambda_2 = \Lambda_1 \frac{e^{\beta D}}{2A^r + e^{\beta D}} \tanh \frac{\beta J}{2} . \tag{2.21}$$

Clearly $\Lambda_2 < \Lambda_1$, so that the trivial fixed point is stable when $\Lambda_1 < 1$. This defines a region of the phase diagram which we call the region of paramagnetic stability. Besides the trivial paramagnetic fixed point, there is a ferromagnetic fixed point, given by the solutions of eqs. (2.12) and (2.13), such that $m^* = m_{\ell+1} = m_\ell \neq 0$, and $q^* = q_{\ell+1} = q_\ell \neq 0$. We have used numerical methods to analyse the stability of this fixed point. The region of the phase diagram where the ferromagnetic fixed point is stable is called the region of ferromagnetic stability.

In figs. 2-4 we show some $(p = D/zJ) \times (k_B T/zJ)$ phase diagrams for $z=3$, $z=6$, and $z=500$. The solid line, which corresponds to the limit of stability of the paramagnetic region $(\Lambda_1=1)$, is given bethe analytic expression

$$D = 1 \left\{ \log 2 + \log \left[r \sinh \beta J - \cosh \beta J \right]_r + r \log \left[\frac{r \sinh \beta J}{r \sinh \beta J - \cosh \beta J - 1} \right] \right\} \tag{2.22}$$

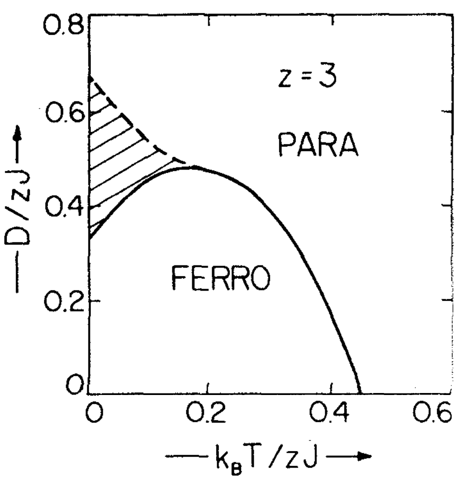


Fig. 2 - Phase diagram in zero field for a Cayley tree with $z=3$. The solid line represents the limit of stability of the paramagnetic fixed point. The dashed line is the limit of stability of the ferromagnetic fixed point. Both fixed points are stable in the hatched region. The tricritical point is at the smooth junction of the stability lines.

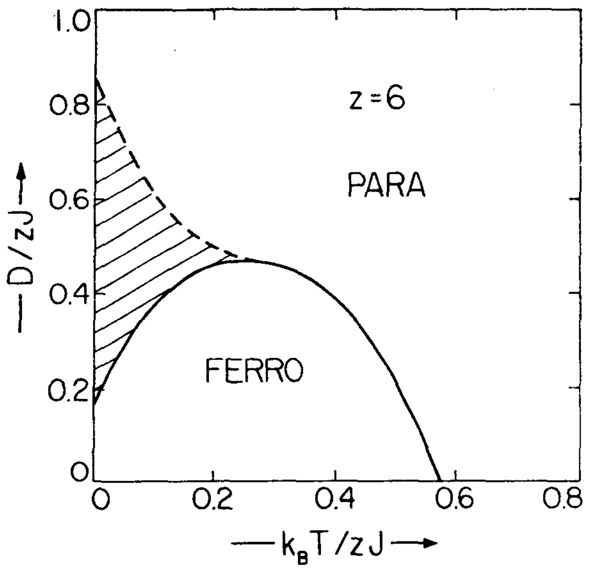


Fig.3 - Phase diagram in zero field for a Cayley tree with $z=6$. The meaning of the lines is explained in the captionof fig. 2.

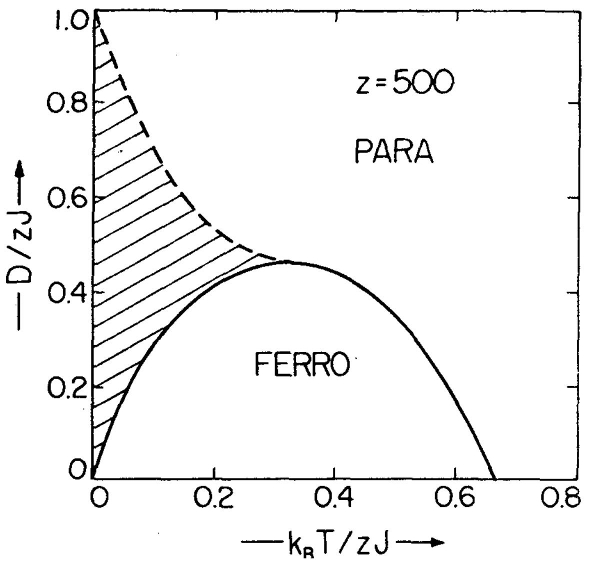


Fig.4 - Phase diagram in zero field for a Cayley tree with $z=500$. Within the precision of the calculations, this figure already corresponds to the infinite coordination limit, $z\to\infty$, $J\to 0$, with zJ = constant. The lines have thesame meaning as in figs. 2 and 3.

The dashed line corresponds to the limit of stability of the ferromagnetic fixed point. Numerical calculations indicate that there is a flow to the ferromagnetic fixed point whenever the initial conditions are given by $m_0 = q_0 = 1$, with D/zJ and $k_B T/zJ$ below this dashed line. Only in the hatched region of the phase diagram we may have flows either to the paramagnetic or to the ferromagnetic fixed point, depending on the initial conditions. This hatched region thus corresponds to the overlapping of the paramagnetic and the ferromagnetic stability regions. To decide whether a point in this region belongs to the thermodynamic paramagnetic phase or to the ferromagnetic phase it would be necessary to analyze the free energy. However, it is not our purpose to perform this analysis in the present paper. Anyhow, the presence of an overlapping region where two fixed points are stable indicates the existence of a first-order transition line. This region of coexistence ends at the tricritical point, which is located by the smooth junction of the stability lines. Above the tricritical temperature both stability lines coincide and then we have a critical line. The tricritical point is present for all physical values of the coordination z.

3. THE INFINITE COORDINATION LIMIT

In the infinite coordination limit, given by $J \to 0$, $z \to \infty$, with zJ = constant, the recursion relations are drastically simplified. In zero field, from eqs. (2.12) and (2.13), we have

$$m_{\ell+1} = \frac{2 \sinh \frac{m_\ell}{T}}{2 \cosh \frac{m_\ell}{T} + e^{p/T}} \tag{3.1}$$

and

$$q_{\ell+1} = \frac{2 \cosh \frac{m_\ell}{T}}{2 \cosh \frac{m_\ell}{T} + e^{p/T}} , \tag{3.2}$$

where T is the temperature in units of k_B/zJ, and $p = D/zJ$. Since $q_{\ell+1}$ does not depend on q_ℓ, the problem is reduced to a one-dimensional mapping. This kind of reduction of the dimensionality of the mapping, which occurs for several models defined on a Cayley tree[4,7], seems to be a peculiar effect brought about by the infinite coordination limit.

The linearization in a neighbourhood of the trivial paramagnetic fixed point, $m^*=0$, yields the relation

$$m_{\ell+1} = \frac{2/T}{2 + e^{p/T}} m_\ell . \qquad (3.3)$$

From eq. (3.3), we obtain the line of stability of the paramagnetic phase,

$$p = T \log \left(\frac{2}{T} - 2\right) , \qquad (3.4)$$

which corresponds to the infinite coordination limit of eq. (2.22). For $p=0$, we regain the mean-field critical temperature, $T_c = 2/3$, for the ordinary spin-1 Ising model.

The ferromagnetic fixed point, $m^*\neq 0$, is given by the numerical solution of eq. (3.1), with $m_{\ell+1} = m_\ell = m^*$. Whenever there is a solution $m^*\neq 0$, the ferromagnetic fixed point is stable. From eq. (3.1), it is easy to see that there is a tricritical point at $T_t = 1/3$ and $p_t = 2/3 \log 2$, which almost coincides with the maximum of the $p\times T$ curve. Above T_t, the critical line indicates the limit of stability of both the ferromagnetic and the paramagnetic fixed points (that is, $m^* = 0$ at the critical line). Moreover, it is possible to show that the critical line and the line of stability of the ferromagnetic fixed point meet tangentially at the tricritical point.

Fig. 4, which was drawn for $z = 500$, already corresponds to the infinite coordination limit. In the hatched region, both the ferromagnetic and the paramagnetic fixed points can be reached, depencling upon the initial conditions. From eq. (3.4), we see on the line of paramagnetic stability that $p(T) \to 0$ as $T \to 0$ with an infinite. slope. On the other nand, for finite coordinations it goesto the limit $1/z$ for $T = 0$. Also, it is easy to show that the line of stability ofthe ferromagnetic fixed pointsgoes to the limit $p = 1$ as $T \to 0$, with a negative infinite slope. Finally, it should be pointed out that the expression for the critical line as well as the iocation of the tricritical point agree with the mean-field results of Blume, Emery and Griffths[6].

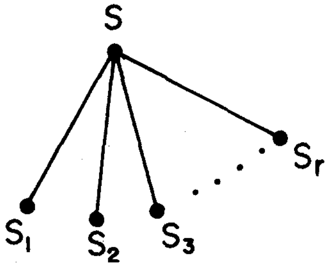
4. CONCLUSIONS

We have presented an exact formulation of the Blume-Emery-Griffiths model, on a Cayley tree of arbitrary coordination z, as a two-dimensional mapping problem. We have found analytical expressions for the boundary of a region of paramagnetic stability in the $D/zJ \times k_B T/zJ$ phase diagram. Also, we have performed numerical calculations to find the line of stability of the ferromagnetic fixed point. Below the tricritical temperature, there is a region where both the paramagnetic and the ferromagnetic fixed points are stable. In the infinite coordination limit, $z \to \infty, J \to 0$, with zJ = constant, the problem is reduced to a one-dimensional mapping, and we have regained the mean-field results of Blume, Emery and Griffths.

APPENDIX - Alternative Formulation of the Problem

An aiternative way to set up the iterative scheme consists of summing successively overspins in the outermost shell[3], as shown schematically in fig. 5. We may write

$$\sum_{S_1,\dots,S_r} \exp\Big[\beta J S (S_1 + \dots + S_r) - \beta D (S_1^2 + \dots + S_r^2) + \beta H (S_1 + \dots + S_r)\Big] \equiv A \exp(BS - CS^2) \,, \qquad (A.1)$$



Çig.5 - The interacting spins of the outermost shell of a Cayley tree with coordination $z = r + 1$.

from which we have

$$B = \frac{r}{2} \log \frac{1 + 2\, e^{-\beta D} \cosh(\beta J + \beta H)}{1 + 2\, e^{-\beta D} \cosh(\beta J - \beta H)} \,, \qquad (A.2)$$

and

$$C = -\frac{r}{2}\log\frac{\left[1+2e^{-\beta D}\cosh(\beta J+\beta H)\right]\left[1+2e^{-\beta D}\cosh(\beta J-\beta H)\right]}{\left[1+2e^{-\beta D}\cosh\beta H\right]^2} \tag{A.3}$$

Thus, we can write the recursion relations in terms of the field B and the anisotropy C,

$$B_{\ell+1} = \beta H + \frac{r}{2}\log\frac{1+2e^{-C_\ell}\cosh(\beta J+B_\ell)}{1+2e^{-C_\ell}\cosh(\beta J-B_\ell)}, \tag{A.4}$$

and

$$C_{\ell+1} = \beta D - \frac{r}{2}\log\frac{\left[1+2e^{-C_\ell}\cosh(\beta J+B_\ell)\right]\left[1+2e^{-C_\ell}\cosh(\beta J-B_\ell)\right]}{\left[1+2e^{-C_\ell}\cosh B_\ell\right]^2}. \tag{A.5}$$

If we introduce the definitions

$$m_\ell = \frac{\sum_\sigma \sigma\exp(B\sigma-C\sigma^2)}{\sum_\sigma \exp(B\sigma-C\sigma^2)} = \frac{2\sinh B_\ell}{e^{C_\ell}+2\cosh B_\ell}, \tag{A.6}$$

and

$$q_\ell = \frac{\sum_\sigma \sigma^2\exp(B\sigma-C\sigma^2)}{\sum_\sigma \exp(B\sigma-C\sigma^2)} = \frac{2\cosh B_\ell}{e^{C_\ell}+2\cosh B_\ell}, \tag{A.7}$$

it is straightforward to show that the recursion relation (A.4) and (A.5) are completelyequivalent to eqs. (2.12) and (2.13).

In the infinite coordination limit, eqs. (A.2) and (A.3) are reduced to the expression

$$B = \frac{2e^{-\beta D}\sinh\beta H}{1+2e^{-\beta D}\cosh\beta H}\,\beta Jz, \tag{A.8}$$

and

$$C \to -\frac{z}{2}\,\frac{2\,e^{-\beta D}\left[\cosh\beta H + 2\,e^{-\beta D}\right]}{\left[1 + 2\,e^{-\beta D}\cosh\beta H\right]^2}\,(\beta J)^2 \to 0 \quad . \qquad (A.9)$$

Since C vanishes identically, the mapping can be written as

$$B_{\ell+1} = \beta H + \frac{2\sinh B_\ell}{e^{\beta D} + 2\cosh B_\ell}\,\beta J z \quad , \qquad (A.10)$$

and

$$C_{\ell+1} = \beta D \; . \qquad (A.11)$$

Using eq. (A.6) we have

$$B_{\ell+1} = \beta H + \beta J z\, m_\ell , \qquad (A.12)$$

which gives the one-dimensional mapping

$$m_{\ell+1} = \frac{2\sinh(\beta H + \beta J z\, m_\ell)}{e^{\beta D} + 2\cosh(\beta H + \beta J z\, m_\ell)} \qquad (A.13)$$

In zero field, this expression is identical to eq. (3.1).

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Resumo

Apresentamos uma formulação exata do modelo de spin-1 de Blume, Emery e Griffiths numa árvore de Cayley de coordenação z como um problema de mapeamento em duas dimensões. Calculamos as regiões de estabilidade dos pontos fixos paramagnéticos e ferromagnético. Abaixo de uma determinada temperatura tricrítica, há uma justaposição destas regiões de estabilidade. No limite de coordenação infinita, o mapeamento se torna unidimensional e recuperamos os resultados de campo médio de Blume, Emery e Griffiths.