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Exact Kantowski-Sachs and Bianchi Types I and III Cosmological Models with a Conformally Invariant Scalar Field

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Abstract Exact solutions of the Einstein-Conformally Invariant Scalar Field Equations are obtained for Kantowski Sachs and Bianchi types I and III cosmologies. The presence of the conformally invariant scalar field is responsible for some interesting features of the solutions. In particular it is found that the Bianchi I model is consistent with the big-bang theory of cosmology.

1. INTRODUCTION

In the last three decades a respectable effort was devoted to the investigation of the field equations of the general theory of relativity foi- spatially homogeneous but anisotropic space-times. These space-times belong either to the Bianchi types I-IX or to the Kantowski Sachs class and are generally interpreted as cosmological models. Recently, the study of spatially homogeneous anisotropic cosmological models with a conformally invariant scalar field as the matter field has received some attention[1-3]. The properties of the energy-momentum tensor of the conformally invariant scalar field are quite distinct from those concerning the ordinary scalar field[4-6]. Besides, the predlctions of the theories involving the two tensors are rather different in strong gravitational fields[7].

In this paper exact solutions, given in a unique parametrisation, for Kantowski-Sachs (KS) and Bianchi types I and III cosmological models with a conformally invariant scalar field are obtained. An interesting feature of the former models is that one can recover from them the corresponding vacuum solutions in a straightforward way. The cosmological implications of the Bianchi I model are discussed in a systematic way.

2. FIELD EQUATIONS AND SOLUTIONS

In choosing local orthonormal bases $\sigma^{\mu}{}_{(a)}$, the KS metric $(a=1)$, the Bianchi type III metric $(a=2)$, and the Bianchi type I metric with two anisotropic directions $(a=3)$ can be put in the form .

$$ds^2_{(a)} = \eta_{\mu\nu} \, \sigma^{\mu}_{(a)} \, \sigma^{\nu}_{(a)} , \quad \eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1) \;, \tag{1}$$

where

$$\sigma^0_{(a)} = dt, \quad \sigma^1_{(a)} = x(t) \begin{cases} dr \\ dr \\ dx \end{cases}, \quad \sigma^2_{(a)} = y(t) \begin{cases} \sin\theta d\phi \\ \sinh\theta d\phi \\ dy \end{cases}$$

$$\sigma^3_{(a)} = y(t) \begin{cases} d\theta \\ d\theta \\ dz \end{cases} .$$

From the field equations'

$$R^{\mu}{}_{\nu} \, f(S) = \delta^{\mu}{}_{\nu} \, \partial_{\alpha} S \partial^{\alpha} S - 4\partial^{\mu} S \, \partial_{\nu} S + 2s \, \nabla^{\mu}\nabla_{\nu} S \;, \tag{2a}$$

$$R = 0 \;, \tag{2b}$$

$$\Box S = 0 \;, \tag{2c}$$

where ∇ denotes covariant differentiation and

$$f(S) = 1 - S^2, \; S = (n/6)^{1/2} \, \Phi \;, \tag{2d}$$

Φ being the massless scalar field, we obtain the following set of equations:

$$R^0{}_0 \rightarrow \left(\frac{\ddot{x}}{x} + 2\,\frac{\ddot{y}}{y}\right) f = -\,3\dot{S}^2 - 2S\dot{S}\,\left(\frac{\dot{x}}{x} + 2\,\frac{\dot{y}}{y}\right) \;, \tag{3a}$$

$$R^1{}_1 \rightarrow \left(\frac{\ddot{x}}{x} + 2\,\frac{\dot{x}\dot{y}}{xy}\right) f = \dot{S}^2 + 2S\dot{S}\,\frac{\dot{x}}{x} \;, \tag{3b}$$

$$R^2{}_2 = R^3{}_3 \rightarrow \left(\frac{\ddot{y}}{y} + \frac{\dot{x}\dot{y}}{xy} + \frac{\varepsilon + \dot{y}^2}{y^2}\right) f = \dot{S}^2 + 2S\dot{S}\,\frac{\dot{y}}{y} \;, \tag{3c}$$

$$S = K_1/xy^2 \;, \tag{3d}$$

where dots, as usual, denote derivative with respect to t, &=+l,-1, or

0, according to whether a=1, a=2 or a=3 is chosen, and K_1 is an integration constant.

Introducing the new time variable T by $dt = y dT$, the linear combination of (3b) and (3c) gives

$$(xyf)'' + K_1 xyf = 0, \tag{4a}$$

where $(\ )' = \frac{d}{dT}$. From (3b) one gets

$$(x'yf - K_1 S)' = 0 \tag{4b}$$

In addition one has the constraint equation

$$(xy)''/xy + (y'/y)' - x'y'/xy + \varepsilon = 0 . \tag{4c}$$

The solutions of the set of eqs. (4) can be presented in the form

$\varepsilon = +1$:

$$x = \left[(\tan \tfrac{T}{2})^{\beta-\alpha} + (\tan \tfrac{T}{2})^{\beta+\alpha}\right],$$

$$y = (\sin T) \left[(\tan \tfrac{T}{2})^{-(\alpha+\beta)} + (\tan \tfrac{T}{2})^{\alpha-\beta}\right],$$

$$\frac{1+S}{1-S} = (\tan \tfrac{T}{2})^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1 ; \tag{5a}$$

$\varepsilon = -1$:

$$x = \left[(\tanh \tfrac{T}{2})^{\beta-\alpha} + (\tanh \tfrac{T}{2})^{\beta+\alpha}\right],$$

$$y = [\sinh T] \left[(\tanh \tfrac{T}{2})^{-(\alpha+\beta)} + (\tanh \tfrac{T}{2})^{\alpha-\beta}\right],$$

$$\frac{1+S}{1-S} = (\tanh \tfrac{T}{2})^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1 ; \tag{5b}$$

$\varepsilon = 0$:

$$x = T^{\beta-\alpha} + T^{\beta+\alpha},$$

$$y = T^{1-\alpha-\beta} + T^{1+\alpha-\beta},$$

$$\frac{1+S}{1-S} = T^{2\alpha}, \quad 3\alpha^2 + \beta^2 = 1 . \tag{5c}$$

For convenience, the multiplicative constants, as well as an additive integration constant concerning the variable T have been eliminated.

3. DISCUSSION

Solutions (5a) and (5b) are new. The first one represents a closed spatially homogeneous anisotropic cosmological model, whereas the second belongs to an open universe. A universe is said to be closed or open, according to whether the space-time presents spatially homogeneous sections with compact or noncompact manifolds structures, respectively. Traditionally, the open or closed feature of a cosmological model is related to the sign of the 3-scalar of curvature $^{(3)}R$, but, as was pointed out by Assad and soares[8], the scalar of curvature is not a quantity that can in general be associated with the topological properties of the models.

A specially prominent characteristic of the previous solutions is that from them the corresponding vacuum solutions ($S\to 0$, as $\alpha\to 0$) can be recovered in a trivial way:

$$\varepsilon = +1: \quad x = \left(\tan \frac{T}{2}\right)^{\beta}, \quad y = \left(\tan \frac{T}{2}\right)^{-\beta} \sin T, \quad \beta^2 = 1; \qquad (5a')$$

$$\varepsilon = -1: \quad x = \left(\tanh \frac{T}{2}\right)^{\beta}, \quad y = \left(\tanh \frac{T}{2}\right)^{-\beta} \sinh T, \quad \beta^2 = 1. \qquad (5b')$$

These solutions were first obtained by Kantowski and Sachs and rediscovered by Vajk and Eltgroth[10], and Lorenz[11,12] using different parametrizations. The present coincide with those of Lorenz[11]. To get an idea of the influence of the scalar field on the models, eqs. (5a) and (5b) are rewritten in the form

$$\varepsilon = +1: \quad x = \frac{1}{\sqrt{f}} \left(\tan \frac{T}{2}\right)^{\beta}, \quad y = \frac{1}{\sqrt{f}} (\sin T) \left(\tan \frac{T}{2}\right)^{-\beta}$$

$$3\alpha^2 + \beta^2 = 1; \qquad (5a'')$$

$$\varepsilon = -1: \quad x = \frac{1}{\sqrt{f}} \left(\tanh \frac{T}{2}\right)^{\beta}, \quad y = \frac{1}{\sqrt{f}} (\sinh T) \left(\tanh \frac{T}{2}\right)^{-\beta}$$

$$3\alpha^2 + \beta^2 = 1. \qquad (5b'')$$

Then, one observes that the effect of the conformally invariant scalar field manifests itself through the factor $1/\sqrt{f}$ and the constraint $3\alpha^2 + \beta^2 = 1$. (It is worth noticing that $f \to 1$, as $S \to 0$.)

Calculating the curvature invariants for such metrics one finds that they diverge at $T = 2n\pi$ $(\varepsilon=+1)$ and $T = 0$ $(\varepsilon=-1)$, if the parameters α and β are such that

$$(\alpha,\beta) \in A \;,\; A = \{(a,b) \in R^2 \,|\, 3a^2 + b^2 = 1, \quad 0 < a < \frac{1}{2} \,,\; -\frac{1}{2} < b < 0\} \;. \tag{6}$$

On the other hand, the vacuum solutions (5a') and (5b') are singular at $T = 2n\pi$ $(\varepsilon=+1)$ and $T = 0$ $(\varepsilon=-1)$, respectively, if $\beta = -1$. So, if (6) holds, the conformally invariant scalar field cannot prevent the singularities. In the scalar-tensor theory proposed by Schmidt et *al*[13], a scalar field is used to avoid the problem of a singularity. Banerjee and Santos[14] showed that, in general, this is not possible for a Friedmann-Robertson-Walker model. The scalar-tensor theory being dealt with here is a particular case of that presented by Schmidt et *al*.

The metric corresponding to solution (5c) is given by

$$ds^2 = \left[T^{1-\alpha-\beta} + T^{1+\alpha-\beta}\right]^2 (dT^2 - dy^2) - \left[T^{\beta-\alpha} + T^{\beta+\alpha}\right]^2 dx^2 - \left[T^{1-\alpha-\beta} + T^{1+\alpha-\beta}\right]^2 dz^2 \tag{7}$$

When the conformally invariant scalar field is switched off ($S \to 0$, as $\alpha \to 0$) this metric reduces to

$$ds^2 \to dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \tag{8}$$

where the parameters p_1, p_2 and p_3 are given by*

$$p_1 = \frac{\beta}{2-\beta} \;,\; p_2 = p_3 = \frac{1-\beta}{2-\beta} \;, \tag{9}$$

and satisfy the relantionships

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1 \;. \tag{10}$$

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* From (5c) one has that $\beta^2 \to 1$, as $\alpha \to 0$, so the possibility $\beta = 2$ is ruled out.

which is the Kasner empty universe[15].

The conformally flat metric is found to be a very special case when $\alpha = \beta = \frac{1}{2}$. The metric (7) takes then the very simple form:

$$ds^2 = (1+T)^2 (dT^2 - dx^2 - dy^2 - dz^2) \quad . \tag{11}$$

The cosmological model represented by solution (7) is such that its proper volume is given by

$$V = [1+T^{2\alpha}]^3 \; T^{2-3\alpha-\beta} \quad . \tag{12}$$

It is clear from the previous result that V vanishes at T=O if

$$(\alpha,\beta) \in B \; , \; B = \{ (a,b) \in R^2 | 3a^2 + b^2 = 1 , \; a \in (0,\tfrac{1}{2}) \cup (\tfrac{1}{2}, 1/\sqrt{3}) \; , \; b \in (-1,0) \cup (0,\tfrac{1}{2}) \cup (\tfrac{1}{2}, 1) \} \tag{13}$$

Otherwise, if this relation is valid, V becomes infinite as $T\to\infty$. Thus, the model starts from a point singularity and expands to an infinite volume*. The Hubble parameters** concerning the present model are given by

$$H_1 = \frac{\dot{x}}{x} = \frac{T^{\alpha+\beta-2}[\beta - \alpha + (\beta+\alpha)\,T^{2\alpha}]}{[1+T^{2\alpha}]^2}$$

$$H_2 = H_3 = \frac{\dot{y}}{y} = \frac{T^{\alpha+\beta-2}[1-\alpha-\beta+(1+\alpha-\beta)\,T^{2\alpha}]}{[1+T^{2\alpha}]^2} \tag{14}$$

and it can be concluded that the model admits anisotropic expansions** for $(\alpha,\beta) \in B$, where B is given by (13)***. Besides Phis, it is regular in the range $0 < T < \infty$.

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* The curvature invariants regarding solutions (5c) are found to diverge at $T=0$, if $(\alpha,\beta) \in C$, $C = \{ (a,b) \in R^2 | 3a^2+b^2=1 , \; a \in (0,\tfrac{1}{2}) \cup (\tfrac{1}{2}, 1/\sqrt{3}) , \; b \in (-1,0) \cup (0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1) \}$. Hence, $B=C$, and, as a consequence, the proper volume equals 0, where the curvature invariants of the model diverge.

** These are kinematical parameters associated to the 4-velocities $\partial/\partial t$ (in the coordinate system used in (1)).

*** The possibility $\alpha = \beta = \frac{1}{2}$ is forbidden by (13).

The model in view is consistent with the big-bang theory and is different frorn the solutions obtained by Accioly et *al.*[1] and by Ram[3].

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Resumo

Se obtém soluções exatas das equações de Einstein-Campo- Estalar Conformemente Invariante referentes a cosmologias de Kantowski-Sachs e Bianchi dos tipos I e 111. A presença do campo escalar conformemente irivariante é responsável por alguns aspectos interessantes das soluções. Encontra-se, em particular, que o modelo de Bianchi I é compatível com a teoria cosmológica do big-bang.