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Variational Principles for Collisional Transport in Magnetoplasmas

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Abstract Different variational approaches for the problem of collisional transport in a magnetically confined magnetoplasma are discussed, and approximate methods of solution are pointed out. In particular, an approach analogous to the well known Rayleigh-Ritz method is proposed, yielding an energy-convergent approximate solution for the relevant drift Fokker-Planck equation. Selected applications to the investigation of magnetoplasmas of arbitrary degree of collisionality are briefly discussed.

1. INTRODUCTION

Variational approaches for the study of the Boltzmann and Fokker-Planck kinetic equations corresponding to prescribed initial and boundary conditions have been investigated by several authors both in the context of rarified gas dynamics[1-3] and plasma dynamics[4-10]. Their main interest lies in the possibility of adopting simple numerical methods which enable the direct determination of relevant macroscopic physical quantities in terms of appropriate approximate solutions or even of the so-called trial-functions (i.e., polynomial functions containing undetermined constants which are then chosen in sych a way as to extremize the relevant variational functional). In fact, even in linear problems (i.e., for which it sufficies to consider linearized approximations of the previous kinetic equations), the application of direct solution methods as, for example, expansions in a complete basis of orthonormal polynomials leads to unsatisfactory results due to the slow convergence of the approximate solution and the consequentpoor accuracy in estimating the physically relevant dynamical variables. In addition, it should be noted that in many cases such as the macroscopic description of a dynamical system, which is the case of a magnetoplasma,

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we need much less than the precise knowledge of the solutions of the kinetic equation (distribution function). For example, in the case of a quiescent magnetoplasma, i.e. a plasma magnetically confined in which turbulence is negligible, the relevant macroscopic quantities are the "radial" material fluxes (i.e., the particle and kinetic energy fluxes across an isobaric surface).

It is well-known that variational methods are particularly convenient when the variational functional itself can be expressed in terms of such macroscopic dynamical variables. In such cases, in fact, an error of order $O(\delta)$ in estimating the solution of the kinetic equation leads to an error of higher-order ($O(\delta^2)$) for the variational functional.

Approaches of this kind have been previously developed by several authors in kinetic theory[1-10]. As far as what concerns specific applications to plasma dynamics, variational methods have been adopted to investigate both collisional transport problems in quiescent systems[4-8] and linear stability problems in weakly turbulent systems[9-10].

Up to now, for the first class of problems, variational approaches have been developed systematically only for the so-called weakly-collisional plasmas, that is, subject to the assumption $\nu_{s,eff}/\omega_{b,s} \ll 1$ (where $\nu_{s,eff}$ is an effective collision frequency defined in Appendix A, and $\omega_{b,s} = (\oint_C ds/|v_{\parallel}|)^{-1}$ is the bounce or transit frequency which characterizes the unperturbed particle motion along a magnetic flux line; C is a closed unperturbed particle orbit) and for "symmetric hydromagnetic equilibria" (i.e., assuming that the confining magnetic field $\vec{B}$ is spatially symmetric)[4-8]. Such theories fall essentially in two classes: a) *asymptotic theories*, based on an asymptotic expansion for the trial function obtained in the limit 6-10 where 6 is given by:

$$\delta = \langle (1-B/B_{max})^{1/2} \rangle_S \ll 1$$

(B_{max} is the absolute maximum of B on a given isobaric surface S; the brackets "$\langle \ \rangle_S$" denote an appropriately weighted average taken on the same surface), corrections of order $O(\delta^3)$ or higher to the variational functional[5,6] were neglected; b) *perturbative theories*, based on a perturbative expansion for the determination of an approximate solution[7,8] which allows for the investigation of hydromagnetic equilibria with "arbitrary" magnetic field. For this case, in particular, the previous asymptotic condition $\delta \ll 1$ does not need to be invoked (here a perturbative expansion is performed in terms of the adimensional parameters $A = \delta|1-\delta|^2)$.

Similar techniques, on the other hand, have been developed for weak-turbulence problems, and in particular for the investigation of linear stability of magnetoplasmas subject to linear dissipative perturbations of the drift type[9-10]. In this case, a variational formulation was obtained only by adopting a simplified collision operator model for the Fokker-Planck kinetic equation (the so-called pitch-angle -scattering approximation for the Fokker-Planck collision operator).

It is the purpose of this paper to address the problem of a variational formulation for the linearized Fokker-Planck equation which occurs in problems of collisional transport under standard assumptions (i.e., a so-called "small-Larmor-radius" ordering[11,12]) and which, in general, also refers to "collisional" or "strongly collisional" plasmas, in the sense that $O(\rho_s) \sim O(1)$ or $O(\rho_s) \gg 1$ (with $\rho_s = \nu_{s,eff} / \omega_{b,s}$) respectively.

A basic feature of the relevant boundary-value problem associated with such an equation is that the equation itself is non-self-adjoint while the (linear) operator therein defined is *not* positive definite.

In the sequel we intend to discuss various possible variational formulations for the given problem, and, in particular, a so-called "constrained variational formulation" recently proposed by the author[13], where the class of admissible variations is constrained appropriately 'a priori'.

We intend to show that the variational functional may indeed be chosen in such a way as to be a physically meaningful quantity (related, in fact, to the surface-average of the local entropy production rate) which turns out to be bilinear with respect to the material fluxes and the thermodynamic forces to be later defined.

Finally, an approximate solution method which is, in a sense, a generalization of the well-known Rayleigh-Ritz direct variational solution method is discussed. We are able to prove that the approximate solution obtained in the form of an expansion in terms of a basis of coordinate functions which are both orthonormal and complete in energy, i.e., with respect to the symmetric and positive-definite part of the linear operator appearing in the kinetic equation, converges weakly in the sense of energy convergence.

2. BASIC EQUATIONS

Let us briefiy recall the matheniatical model[8,10]. We shall consider a magnetoplasma embedded in a spatially symmetric magnetic field (in the sequel we shall limit our analysis to the case of toroidal axisymmetry as in ref. 8), and subject to the so-called "small -Larmor-radius" ordering[10] (known as "neoclassical" ordering for such a type of hydromagnetic equilibria[14]), i.e. a perturbative scheme in terms of an adimensional parameter $\varepsilon_s \ll 1$ (where $\varepsilon_s = r_s/L$ isthe ratio between the Larmor radius $r_s = v_{th,s}/\Omega_s$, with $v_{th,s}$ the thermal velocity and Ω_s the Larmor frequency $\Omega_s = e_s B/m_s c$, and L is the smallest characteristic scale length of the "equilibrium" dynamical variables to be defined appropriately). Expanding all the physically relevant quantities in power series of ε_s, and in particular the one-particle distribution function $f_s(\vec{r},\vec{v},t)$ (the index s denotes the particle species, and the leading-order contribution with respect to ε_s for a given dynamical variable is denote as its "equilibrium" part), one is left with a hierarchy of perturbative kinetic equations which are coupled to Maxwell's equations for the electromagnetic field (which is obviously dependent on the plasma dynamics).

For the investigation of most of the transport problems it is sufficient to solve only the first two perturbative equations, determining both the "equilibrium" distribution functions $f_{0,s}(\vec{r},\vec{v},t)$ and their first order perturbation $f_{1,s}(\vec{r},\vec{v},t)$ (here $f_{i,s}(\vec{r},\vec{v},t)$ denotes the perturbation of order $O(\varepsilon_s^i)$ for $i = 0,1,\ldots)$[8]. Assuming for $f_{0,s}(\vec{r},\vec{v},t)$ a local maxwellian distribution constant on a givenisobaric surface (i.e., such that $\hat{n}.\nabla f_{0,s} = 0$ being $\hat{n} = \mathrm{vers}\{\vec{B}_0\}$) with vanishing mass velocity and subject to the so-called condition of temperature equilibrium ($T_{0,k} = T_{0,k}$ forallspeciesandgivenby:

$$T_{0,s} = \int d^3v m_s v^2 f_{0,s}(\vec{r},\vec{v},t)/3N_{0,s}$$

and

$$N_{0,s} = \int d^3v f_{0,s}(\vec{r},\vec{v},t) \;),$$

one obtains for $f_{1,s}$:

$$f_{1,s} = \tilde{f}_{1,s} + \bar{f}_{1,s} \tag{1}$$

where

$$\tilde{f}_{1,s} = \vec{v}\Lambda\hat{n}.\nabla f_{0,s}/\Omega_{0,s} \tag{2}$$

(with $\Omega_{0,s} = e_s B_0/m_s c$) ánd $\bar{f}_{1,s}$ is the solution of the so-called drift Fokker-Planck equation:

$$L_s(\bar{f}_1) = F_s \tag{3}$$

where L_s is the linear operator given by:

$$L_s(\bar{f}_1) = v_{\shortparallel}\hat{n}.\nabla\bar{f}_{1,s} - C_s(f_0|\bar{f}_1) \tag{4}$$

and F_s is the source term:

$$F_s = -\vec{v}_{D,s}.\nabla f_{0,s} + \frac{e_s}{T_{0,s}} v_{\shortparallel}E_{\shortparallel}^{rot} f_{0,s} \tag{5}$$

Here the notation is standard (see ref. 8,11). Thus, $C_s(f_0|\bar{f}_1)$ is the linearized Fokker-Planck collision operator in the Landau form recalled in Appendix A, $\vec{v}_{D,s}$ is the diamagnetic drift velocity and $\vec{E}^{rot}$ is the inductive part of the electric field. Furthermore, we have denoted $v_{\shortparallel} = v.\hat{n}$ and $E_{\shortparallel}^{rot} = \vec{E}^{rot}.\hat{n}$. Eq. (3) implies, upon averaging on an isobaric surface, the integral equation:

$$<\frac{B}{v_{\shortparallel}}\{-\vec{v}_{D,s}.\nabla f_{0,s} \_ \frac{e_s}{T_{0,s}} v_{\shortparallel}E_{\shortparallel}^{rot} f_{0,s} + C_s(f_0|\bar{f}_1)\}>_{S(\lambda)} = 0 \tag{6}$$

where $S(\lambda) \equiv S$ is the subdomain of S in which $v_{\shortparallel}$ is real.

We immediately notice that while Eq. (6) is self-adjoint, Eq. (3) is nat. Such equations are supplemented by standard boundary and regularity conditions[••]; the boundary conditions being obtained by requesting $f_{1,s}(\vec{r},\vec{v},t)$ to be periodic on a given isobaric surface.

In the asymptotic limit, where $\rho_s \to 0$, it has been previously shown for both symmetric[5,8] and non-symmetric hydromagnetic equilibria that it is actually sufficient to determine only the general solution of the integral equation (6) (in fact, denoting $\bar{f}_{1,s} = \langle f_{1,s} \rangle_S + f'_{1,s}$, one can verify that $f'_{1,s}/\langle f_{1,s}\rangle_S \sim O(\rho_s))$. However, if $O(\rho_s) \sim O(1)$ or even if $O(\rho_s) \gg 1$ (for the case of a collisional or a strongly collisional plasma respectively) then the solution of Eq. (3) is required. Thus, while previous variational theories[5-8] concerning weakly-collisional plasmas have dealt only with the self-adjoint Eq. (6), in general, one needs to obtain a variational formulation for a non-self-adjoint equation of type (3), supplemented with the appropriate boundary and regularity conditions.

3. VARIATIONAL FORMULATIONS FOR THE DRIFT FOKKERPLANCK EQUATION

Here, we wish to give the previous boundary-value problems a variational formulation, i.e. we look for a functional, $W(\bar{f}_1|\bar{f}_1)$, bi-linear in $\bar{f}_{1,s}$ ($s = 1,r$; is the number of particle species present in the system, $r > 2$ in the case of a multicomponent plasma), real and irreducible in the sense that if $W(f|g) = 0$ for an arbitrary choice of the function g_s in an appropriate functional class, then, $f_s = 0$ identically. Furthermore, we require $\delta W(\bar{f}_1|\bar{f}_1) = \langle L\bar{f}_1|\delta\bar{f}_1\rangle$, i.e., the operator $L = \{L_1,\ldots,L_r\}$ must coincide with the Euler operator associated to the functional $W(\bar{f}_1|\bar{f}_1)$, where "δ" denotes the usual variational differentiation and the brackets "$\langle\,\rangle$" an appropriate scalar product on a domain to be defined.

Variational formulations of such a type usually rely on the self-adjointness of the equation, i.e., in the present case on requiring the linear operator L to be symmetric in the usual sense, $\langle f|Lg\rangle = \langle Lf|g\rangle$. We recall, in fact, that, thanks to the Theorem of Volterra, a necessary condition for the existence of a functional $W(\bar{f}_1|\bar{f}_1)$ fulfilling the previous condition $(\delta W(\bar{f}_1|\bar{f}_1) = \langle L\bar{f}_1|\delta\bar{f}_1\rangle)$ is that the linear operator L be symmetric with respect to the same scalar product. This condition could also become sufficient *only* if the functional domain $\{\bar{f}_1\}$ (domain of definition of the operator L) is convex. In practice, this means that one must look for an appropriate definition of the scalar product, if there exists, which fulfills such a condition of symmetry. Another possible approach consists in appropriately constraining the class $\{\bar{f}_1\}$ in such a wayas to obtain a non-convex domain. In this case, the previous condition of symmetry is no longer a sufficient condition for the existence of a variational formulation. Thus, one may look for a constrainedvariational formulation of a special nature, e.g., one for which the constraints are such that the class of admissible variations is not a convex domain. Here we shall give examples of both types of variational formulations for the drift Fokker-Planck equation (3).

As an example of an application of the first method, we adopt here an approach similar to that developed by Cercignani[4] in the case of the linearized Boltzmann equation in problems of rarified gas dynamics. Thus, denoting by P the parity operator in velocity space which exchanges the sign of $v_{\parallel}$, i.e., such that

$$Pf(\vec{r},\vec{v}_{\perp},v_{\parallel},t) = f(\vec{r},\vec{v}_{\perp},-v_{\parallel},t) \quad ,$$

with $\vec{v}_{\perp} = \vec{v}-\hat{n}v_{\parallel}$, we define the "inner" product of two arbitrary functions $g = \{g_1,\ldots,g_r\}$ and $h = \{h_1,\ldots,h_r\}$ belonging to $\{f_1\}$ and relative to the domain $\Omega = R^3_{\vec{v}} \times S$ (with $R^3_{\vec{v}}$ the velocity space and S an isobaric surface, i.e. a generic surface of equation π_0 = const. with

$$\pi_0 = 3\ \Sigma_{s=1,r} N_{0,s} T_{0,s})$$

as:

$$<g|h>_P = \Sigma_{s=1,r} <\int d^3v g_s\ Ph_s/f_{0,s}>_S \tag{7}$$

where in the case of a toroidal axisymmetric hydromagnetic equilibrium the surface average is defined as[8]:

$$<A>_S = \oint_C d\chi B\ A/B_P|\nabla\chi| / \int_C d\chi\ B/B_P|\nabla\chi| \tag{8}$$

Here C is the projection of a magnetic flux line on a poloidal plane (i.e. a plane belonging to the principal axis of the torus), χ is a curvilinear coordinate on C, B_P is the poloidal component of $\vec{B}$ (i.e., $B_P = \vec{B}.\hat{e}_\chi$ with $\hat{e}_\chi = \nabla\chi/|\nabla\chi|$) which is assumed always $\neq 0$ along C, and A is a furiction integrable in χ. It is obvious that such an' inner product is not a scalar product in the standard sense, in fact it exhibits all the correct properties, except that $<g|g>_P$ may have negative values too. However, it is readily seen that the linear operator L is indeed symmetric with respect to such a product; in fact,

$$<g|v_{\parallel}\,\hat{n}.\nabla h>_P = <v_{\parallel}\,\hat{n}.\nabla g|h>_P$$
$$<g|Ch>_P = <Cg|h>_P \qquad (9)$$

where we have introduced the notation $Ch = \{C_1(f_0|h),\ldots,C_r(f_0|h)\}$. On the other hand, it is obvious that $<g|PLg>$ may take both positive and negative values, hence, L is not positive (or negative) definite with respect to such a product.

Thanks to eqs. (9) a variational principle can be constructed without difficulty. In fact, defining the functional $W_1(\bar{f}_1|\bar{f}_1)$:

$$W_1(\bar{f}_1|\bar{f}_1) = <\bar{f}_1|L\bar{f}_1-2F>_P \qquad (10)$$

we obtain a variational principle for the drift equation (3) if we impose Euler's equation:

$$\delta_s W_1(\bar{f}_1|\bar{f}_1) = 0 \qquad (\forall s = 1,r) \qquad (11)$$

Here the variations are made with respect to $\delta\bar{f}_{1,s}^{(D)}$ and $\delta\bar{f}_{1,s}^{(P)}$, ($\bar{f}_{1,s}^{(D)}$ being the odd, and $\bar{f}_{1,s}^{(P)}$ the even part of $\bar{f}_{1,s}$ with respect to $v_{\parallel}$) which are considered linearly independent $\forall s = 1,r$. Similarly, a variational principle for the integral equation (6) is furnished by eq. (11) by imposing, in addition, a constraint on the class of admissible variations, i.e.:

$$\hat{n}.\nabla\delta\bar{f}_{1,s}^{(D)} = 0$$
$$\hat{n}.\nabla\delta\bar{f}_{1,s}^{(P)} = 0 \qquad (12)$$

Let us next consider a variational formulation of the second type, i.e., a constrained variational formulation recently proposed by the author[13]. We define initially the scalar product:

$$<g|h> = \Sigma_{s=1,r} <\int d^3v g_s h_s/f_{0,s}>_S \qquad (13)$$

and we notice that by writing L =D-C (with C = $\{C_1,...,C_r\}$ and C_i = = $v_{\shortparallel}\hat{n}.\nabla$) the linear operators D and C result antisymmetric and symmetric relative to such a product, respectively. The above statement can be expressed as:

$$<g|Lh> = - <Lg|h> \qquad <g|Ch> = <Cg|h> \tag{14}$$

At thesame time, the operator C is semi-negative definite, i.e., $<g|Cg> \leq 0$, being = 0 when

$$g_s = f_{0,s}(G_s(v) + v_{\shortparallel}v_0/v^2_{th,s}) \tag{15}$$

where

$$G_s(v) = \frac{N_{1,s}}{N_{0,s}} + \frac{T_1}{T_{0,s}}(x_s^2 - \frac{3}{2}) \;,\; x_s = v/v_{th,s} \tag{16}$$

$N_{1,s}$, T_1, v_0 are density, temperature and velocity perturbations to the maxwellian distribution $f_{0,s}(\vec{r},\vec{v},t)$, respectively. Thus, the operator C (or $-C$) is negative (positive) definite in an appropriate functional class $\{\bar{f}_1\}'$ from which functions of type (15) have been subtracted.

In order to construct a variational principle with such a definition of the scalar product we recall[8,11] that a variational formulation for the integral Eq. (6) is provided by Euler's equation:

$$\delta_s W_2(\bar{f}_1|\bar{f}_1) = 0 \qquad (\forall s = 1,r) \tag{17}$$

where the functional $W_2(\bar{f}_1|\bar{f}_1)$ may be defined, for example, as:

$$W_2(\bar{f}_1|\bar{f}_1) = \Sigma_{s=1,r} <\bar{f}_1|C\bar{f}_1+2F> \tag{18}$$

and the variations are performed with respect to $\delta\bar{f}^{(D)}_{1,s}$ and $\delta\bar{f}^{(P)}_{1,s}$, and are subject to the constraints (12). Thus, it is natural do attempt to obtain, in analogy with the previous Eq. (11), a variational principle for the drift equation (3) in terms of Eq. (17) by appropriately constraining the class of admissible variations. It seems obvious that the simplest constraints can be constructed in terms of Eq. (3) itself since they are clearly fulfilled by its solutions. Thus, we require, for example:

$$Q_1(\bar{f}_1|\bar{f}_1) = <\bar{f}_1^{(D)}|D\bar{f}_1^{(P)} - C\bar{f}_1^{(D)} - F_1^{(D)}> = 0 \qquad (19)$$

$$Q_2(\bar{f}_1|\bar{f}_1) = <\bar{f}_1^{(P)}|D\bar{f}_1^{(D)} - C\bar{f}_1^{(P)} - F_1^{(P)}> = 0 \qquad (20)$$

where $F_{1,s}^{(D)}$ and $F_{1,s}^{(P)}$ are the odd and even parts of $F_{1,s}$ with respect to $v_{\parallel}$, respectively.

It is immediate to prove that Eq. (17) with such constraints on the class of variations (now it is sufficient to perform the variations with respect to $\delta\bar{f}_{1,s}^{(P)'}$ and $\delta f_{1,s}^{(P)'}$, where $\bar{f}_{1,s} = <\bar{f}_{1,s}>_S + \bar{f}'_{1,s})$, is indeed a variational principle for Eq. (3), and, moreover, that it results in a minimum variational principle (see Appendix B). From Eqs. (17) and the constraints (19) and (20), it follows, in particular, that another possible choice for the functional $W_2(\bar{f}_1|\bar{f}_1)$ (obviously, there is an infinity of them) is just:

$$W_2(\bar{f}_1|\bar{f}_1) = -<\bar{f}_1|C\bar{f}_1> \quad . \qquad (18')$$

Thus, if $\bar{f}_{1,s} \in \{\bar{f}_1\}'$ (i.e. the functional class from which functions of the type (15) have been excluded), the functional $W_2(\bar{f}_1|\bar{f}_1)$ is, in this case, positive definite. It is interesting to remark that the constrained variational principle (17) is analogous to that reported in ref. 5, from which, however, it differs on the choice of the variational functional $W_2(\bar{f}_1|\bar{f}_1)$, which in their case is given by:

$$W'_2(\bar{f}_1|\bar{f}_1) = J_D^2/(J_+ + J_B^2/J_-) \qquad (21)$$

where

$$\begin{aligned} J_D &= -<Ph\,|\,L(Dh + v_{\parallel} Gf_0\, E_{\parallel}^{\text{rot}} + g_s^{(D)})> \quad , \\ J_+ &= -<Ph|CPh> \quad , \\ J_- &= -<Dh|CHh> \quad , \\ J_B &= <Dh|LPh> \quad , \end{aligned} \qquad (22)$$

and we have introduced the positions:

$$h_s = \bar{f}_{1,s} - v_{\shortparallel}\, G_s\, f_{0,s}\, E_{\shortparallel}^{\mathrm{rot}} \tag{24}$$

with G_S being the solution of the Spitzer-Harm equation:

$$C_s(f_0 | v_{\shortparallel}\, f_0 G) = - \frac{e_S}{T_{0,s}}\, v_{\shortparallel}\, f_{0,s} \tag{25}$$

It is immediate to show that the variational principle:

$$\delta_s W'(\bar{f}_1 | \bar{f}_1) = 0 \tag{17'}$$

with the constraints:

$$Q_1'(f_1|f_1) = \langle h^{(D)} | D h^{(P)} \rangle - \langle h^{(D)} | C h^{(D)} \rangle = 0 \tag{19'}$$

$$Q_2'(f_1|f_1) = \langle h^{(P)} | D h^{(D)} \rangle - \langle h^{(P)} | C h^{(P)} \rangle - \langle h^{(P)} | D v_{\shortparallel} G f_0 E_{\shortparallel}^{\mathrm{rot}} \rangle + \langle h^{(P)} | \vec{v}_D \cdot \nabla f_0 \rangle = 0 \tag{20'}$$

leads to a variational formulation for the equation:

$$D(h_s + v_{\shortparallel} G_s f_{0,s} E_{\shortparallel}^{\mathrm{rot}}) = - \vec{v}_{D,s} \cdot f_{0,s} + C_s(f_0 | h) \tag{21}$$

stemming from Eq. (3), upon substitution of the solution of Eq. (25), which in this case is assumed to be known 'a priori' (contrary to the variationel formulation previously described). It can easily be proven that (17') with the constraints (19') and (20') is, contrary to the previous case, a mini-max variational principle (being minimum with respect to $\bar{f}_{1,s}^{(D)}$ and maximum with respect to $\bar{f}_{1,s}^{(P)}$).

4. RELATIOIYSHIPS WITH THE MATERIAL FLUXES

An important feature of the present approach is that the previous variational functionals $W_1(\bar{f}_1 | \bar{f}_1)$ and $W_2(\bar{f}_2 | \bar{f}_2)$ are related to the material fluxes (of particle and kinetic energy) as well astothe parallel (electric) current density J,, flowing parallel (or antiparallel) to $\vec{B}$. In fact, substituting the solution of Eq. (3) into Eq. (10), one finds:

$$W_1(\bar{f}_1|\bar{f}_1) = <\bar{f}_1|\vec{v}_D.\nabla f_0>_P + <\bar{f}_1|\frac{e}{T_0} v_{\shortparallel} f_0 E_{\shortparallel}^{\mathrm{rot}}>_P \tag{26}$$

or, from Eq. (7):

$$W_1(\bar{f}_1|\bar{f}_1) = -\Sigma_{s=1,r}\{A_{1s}\Gamma_{1s} + A_{2s}\Gamma_{2s}\} - <A_3 J_{\shortparallel}/T_{0,s}>_S \tag{27}$$

where we have defined the thermodynamic forces A_{is} (i=1,3; s=1,r):

$$A_{1s} = -\frac{\partial}{\partial\pi_0}\ln N_{0,s}(1-\frac{3}{2}\eta_s)$$
$$A_{2s} = \frac{\partial}{\partial\pi_0}\ln T_{0,s} \tag{28}$$
$$A_{3s} = A_3 = E_{\shortparallel}^{\mathrm{rot}}$$

(with $\eta_s = \partial\ln N_{0,s}/\partial\ln T_{0,s}$), while Γ_{1s} and Γ_{2s} are, the so-called geometrical fluxes of particle and kinetic energy across an isobaric surface[8,11] respectively, and they may be written in the form:

$$\Gamma_{1s} = -<\nabla\pi_0 . \int d^3v\, \vec{v}_{D,s}\bar{f}_{1,s}\,_S>$$
$$\Gamma_{2s} = -<\nabla\pi_0 . \int d^3v\, \frac{m_s v^2}{2}\vec{v}_{D,s}\bar{f}_{1,s}>_S \tag{29}$$

and

$$J_{\shortparallel} = \Sigma_{s=1,r}\, e_s \int d^3v\, v_{\shortparallel}\bar{f}_{1,s} \; .$$

Similarly, it results for $W_2(f_1|f_1)$:

$$W_2(\bar{f}_1|\bar{f}_1) = -\Sigma_{s=1,r}\{A_{1s}\Gamma_{1s} + A_{2s}\Gamma_{2s}\} + <A_3 J_{\shortparallel}/T_{0,s}>_S \tag{30}$$

Thus, in particular, $W_2(\bar{f}_1|\bar{f}_1)$ i.e., its extremal value, coincides with the surface-average of the local entropy production rate due to $\bar{f}_{1,s}$ ($\dot{S}(\bar{f}_1|\bar{f}_1) = -<f_1|Cf_1>$):

$$W_2(\bar{f}_1|\bar{f}_1) = \dot{S}(\bar{f}_1|\bar{f}_1) \tag{31}$$

while, from Eqs. (27) and (30), it obviously results:

$$W_1(\bar{f}_1|\bar{f}_1) = \dot{S}(\bar{f}_1|\bar{f}_1) - 2<A_3 J_{\shortparallel}/T_{0,s}>_S \; . \tag{32}$$

Notice that $\dot{S}(\bar{f}_1|\bar{f}_1)$ is related to the *total* surface-averaged entropy production rate by the simple relationship:

$$\dot{S}(f_1|f_1) = \dot{S}(\bar{f}_1|\bar{f}_1) + \dot{S}(\tilde{f}_1|\tilde{f}_1) \tag{33}$$

where

$$\dot{S}(\tilde{f}_1|\tilde{f}_1) = - <\tilde{f}_1|C\tilde{f}_1>$$

and $\tilde{f}_{1,s}$ is defined by Eq. (2).

Hence, in order to determine the fluxes (29) and the parallel current density $J_{\parallel}$, it suffices to compute directly either $W_1(\bar{f}_1|\bar{f}_1)$ or $W_2(\bar{f}_1|\bar{f}_1)$ in terms of the thermodynamic forces. Their knowledge is, in turn, equivalent to that of the surface-average total entropy production rate $(\dot{S}(\bar{f}_1|\bar{f}_1)$ may be computed[8]). In fact, defining the transport coefficients $L_{ij}^{(sk)}$ $(i,j = 1,3;\ s,k = 1,r)$ in terms of $\dot{S}(f_1|f_1)$ as:

$$\dot{S}(\bar{f}_1|\bar{f}_1) = \Sigma_{s,k=1,r}\ \Sigma_{i,j=1,3} <L_{ij}^{(sk)} A_{is}A_{jk}>_S \tag{34}$$

one obtains the following constitutive relations for the fluxes and the current density:

$$\Lambda_{1s}-\Gamma_{1Es} = -\Sigma_{k=1,r}\ \Sigma_{j=1,3} <L_{1j}^{(sk)} A_{jk}>_S$$

$$\Lambda_{2s}-\Gamma_{2Es} = -\Sigma_{k=1,r}\ \Sigma_{j=1,3} <L_{2j}^{(sk)} A_{jk}>_S\ T_{0,s} \tag{35}$$

$$<J_{\parallel s}A_3>_S = -\Sigma_{k=1,r}\ \Sigma_{j=1,3} <L_{3j}^{(sk)} A_{jk}A_3>_S\ T_{0,s}$$

provided the reciprocity relations:

$$L_{ij}^{(sk)} = L_{ji}^{(ks)}\, T_{0,k}/T_{0,s} \tag{36}$$

are fulfilled. Notice that in the previous equations Λ_{is} $(i=1,2)$ are the *total* particle and kinetic energy fluxes across an isobaric surface, while Γ_{iEs} $(i=1,2)$ are the so-called eletric-drift fluxes[8,11]. They are related by the following equation:

$$\Lambda_{is} = \Gamma_{is} + \Gamma_{iEs} + \Gamma_{iCs} \qquad (i=1,2) \tag{37}$$

with

$$\Lambda_{1s} = - \langle \nabla \pi_0 \cdot \int d^3v\, \vec{v}\, f_s(\vec{r},\vec{v},t) \rangle_S$$
$$\Lambda_{2s} = - \langle \nabla \pi_0 \cdot \int d^3v\, \vec{v}\, \frac{m_s v^2}{2} f_s(\vec{r},\vec{v},t) \rangle_S \tag{38}$$

$$\Gamma_{1s} = - c v_{th,s}^{-2} \langle \kappa d^3v\, v_\perp\, f_{0,s} \rangle_S$$
$$\Gamma_{2s} = - c v_{th,s}^{-2} \langle \kappa \int d^3v\, \frac{m_s v^2}{2} v_\perp f_{0,s} \rangle_S \tag{39}$$

(with $r = E_{\parallel}^{rot} B_P^2 R/(BB_T)$ and $B_T = (B^2 - B_P^2)^{1/2}$) where the Λ_{iCs} are the classical fluxes reported in ref. 8 (Eqs. (2.10).

In conclusion, both variational formulations previously obtained exhibit the required property of allowing the computation ofthe physically relevant quantities (the material fluxes) in terms of the functionals $W_1(\bar{f}_1|\bar{f}_1)$ and $W_2(\bar{f}_1|\bar{f}_1)$. Such a property is, evidently, an important advantage for the purpose of obtaining at the same time good numerical accuracy and mathematical simplicity for the actual application of the previous methods in collisional transport theory.

5. AN APPROXIMATE SOLUTION METHOD

As an application of the variational formulations previously reported, we shall examine in the sequel the possibility of adopting the well known techniques of the "energy method"[15]. In particular, we shall propose an approximate solution technique which is founded on the constrainedvariational formulation given by Eqs. (17), (19) and (20) for the drift Fokker-Planck equation. We remark, first of all, that the search for approximate solutions of Eq. (3) can actually he limited to *weakly convergent* sequences $\{u_{n,s}$, with $n \in N$ and s=I,$r\}$ in the sence that

$$\lim_{n\to\infty} \left|\, \|\bar{f}_1\|_C - \|u_n\|_C \right| = 0 \tag{40}$$

where we have defined the norm in energy (or "energy")*:

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* Notice that an energy norm of this type cannot be defined in terms of the inner product (7), since in this case, the operator $-C$ is not positive definite with respect to the same functional class $\{\bar{f}_1\}'$.

$$||u_n-\bar{f}_1||^2_C = - \langle u_n-\bar{f}_1|C(u_n-\bar{f}_1)\rangle \tag{41}$$

It follows immediately that if a sequence $\{u_{n,s};\ n\in N,\ s=1,r\}$ converges in energy, i.e., in the sense:

$$\lim_{n\to\infty} ||u_n-\bar{f}_1||_C = 0 \tag{42}$$

then, thanks to the triangle inequality:

$$|\,||\bar{f}_1||_C - ||u_n||_C\,| \leq ||\bar{f} - u_n||_C \tag{43}$$

it also converges weakly in the sense (40).

Let us now suppose that a sequence $\{u_{n,s};\ n\in N,\ s=1,r\}$ converges weakly to $\bar{f}_{1,s}$. Then, according to Eq.(18') provided both $\bar{f}_{1,s}$ and $u_{n,s}$ have finite energy, this implies that

$$\lim_{n\to\infty} |W_2(u_n|u_n) - W_2(f_1|f_1)| = 0 \tag{44}$$

with $W_2(u_n|u_n) = ||u_n||^2_C$ and $W_2(\bar{f}_1|\bar{f}_1) = ||\bar{f}_1||^2_C$. However, since $W_2(u_n|u_n)$ and $W_2(\bar{f}_1|\bar{f}_1)$ are (positive definite) quadratic forms in terms of the thermodynamics forces (see previous Section), Eq. (44) implies in turn:

$$\lim_{n\to\infty} |\Gamma_{is}(u_n) - \Gamma_{is}(\bar{f}_1)| = 0 \qquad (i=1,2;s=1,r) \tag{45}$$

$$\lim_{n\to\infty} |\langle A_3\{J_{11}(u_n) - J_{11}(\bar{f}_1)\}\rangle_S| = 0 \tag{46}$$

(and analogously for the transport coefficients $L^{(sk)}_{ij}$, $i,j = 1,3$ and s,k $= 1,r$), i.e., the sequence converges also in the mean, and the approximate fluxes $\Gamma_{is}(u_n)$ (as well as $\langle A_3J_{11}(u_n)\rangle_S$) converge to their exact values. It is easy to see that the weak convergence (40) does not imply energy convergence. To show this explicitly, notice that from Eq. (18) it follows that

$$W_2(u_n|u_n) = -||u_n||^2_C + 2\langle u_n|F\rangle, \tag{47}$$

and taking into account Eq. (3) one arrives at

$$W_2(u_n|u_n) = -||u_n-\bar{f}_1||^2_C + ||\bar{f}_1||^2_C + 2\langle u_n|D\bar{f}_1\rangle \tag{48}$$

On the other hand, since from Eqs. (19) and (20) it follows that

$$W_2(u_n|u_n) = ||u_n||_C^2 ,$$

weak convergence (40) does not imply energy convergence unless:

$$\lim_{n\to\infty} \langle u_n|D\bar{f}_1\rangle = 0 \tag{49}$$

holds too*. It is therefore quite natural to look for approximate solutions of Eq. (3) in the form of weakly convergent (or energy convergent) sequences.

A possible realization of a weakly convergent sequent can be obtained in the form of an extremal sequence. Introducing the notations $u_{n,s} = u_{n,s}^{(D)} + u_{n,s}^{(P)}$, $u_{n,s}^{(D)}$ and $u_{n,s}^{(P)}$ being, respectively, the odd and even parts of $u_{n,s}$ with respect to $v_{\parallel}$, and:

$$W_2(u_n^{(D)}|u_n^{(D)}) = \|u_n^{(D)}\|_C^2$$
$$W_2(u_n^{(P)}|u_n^{(P)}) = \|u_n^{(P)}\|_C^2 \tag{50}$$

we say that $\{u_{n,s}; n \in N, s = 1,r\}$ is an *extremal* sequence for $W_2(u_n|u_n)$ if $u_{n,s}^{(D)}$ and $u_{n,s}^{(P)}$ are such that:

$$\lim_{n\to\infty} \left|\,||u_n^{(D)}||_C - m\right| = 0$$
$$\lim_{n\to\infty} \left|\,||u_n^{(P)}||_C - M\right| = 0 \tag{51}$$

where $m = \|\bar{f}^{(D)}\|_C$ and $M = \|\bar{f}^{(P)}\|_C$, and $\bar{f}_{1,s}$ is a solution of Eq. (3). Thus, m and M are the extrema (minima) of $W_2(g^{(D)}|g^{(D)})$ and $W_2(g^{(P)}|g^{(P)})$, according to the definitions (18) and (50) (see also Appendix B).

In order to construct a extremal sequence for $W_2(u_n|u_n)$, we adopt here a technique which, in a sense, is analogous to the well known Rayleigh-Ritz direct variational solution method (actually developed originally for symmetric and positive-definite operators). We introduce two sets of *coordinate* functions $\{\phi_{i,s}^{(D)}(\vec{r},\vec{v}), i \in N\}$ and $\{\phi_{i,s}^{(P)}, i \in N\}$, which are, respectively odd and even functions with respect to $v_{\parallel}$, and are defined on the functional class $\{\bar{f}_1\}'$. We assume that

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* This is a consequence of the fact that the operator L is not positive definite. Hence, a norm in energy with respect to L, in analogy with the definition (41), cannot be defined.

$\phi_{i,s}^{(D)}$, (for $i \in N$) and $\phi_{i,s}^{(P)}(\vec{r},\vec{v})$ are independent, orthonormal in energy in the sense:

$$<\phi_i^{(P)} \phi_j^{(P)}> = \delta_{ij} \qquad \forall i, j \in N \tag{52a}$$

$$<\phi_i^{(D)} \phi_j^{(D)}> = \delta_{ij} \qquad \forall i, j \in N \tag{52b}$$

$$<\phi_i^{(P)} \phi_j^{(D)}> = 0 \qquad \forall i, j \in N$$

and complete in the sense of energy convergence, i.e., there are an integer k* and constants a_{Pi}, a_{Di} $(i=1,k)$ such that, for $k \geq k^*$, one finds:

$$|| g - \Sigma_{i=1,k}\, a_{Pi}\, \phi_i^{(P)} - \Sigma_{i=1,k}\, a_{Di}\, \phi_i^{(D)} ||_C < \varepsilon \tag{53}$$

for an arbitrary $g_s\{\bar{f}_1\}'$ and $\varepsilon > 0$.

Then, an extrema1 sequence can simply be constructed by introducing the sequences:

$$u_{n,s}^{(D)} = \Sigma_{i=1,n}\, a_{Di}\, \phi_{i,s}^{(D)}$$

$$u_{n,s}^{(P)} = \Sigma_{i=1,n}\, a_{Pi}\, \phi_{i,s}^{(P)} \tag{54}$$

whero the Fourier coefficients $a_{Di} = <\phi_i^{(D)} | u_n^{(D)} >$ and $a_{Pi} = <\phi_i^{(P)} | u_n^{(P)} >$ are chosen in such a way that the functional $W_2(u_n|u_n)$ has a conditional extremum, i.e., $W_2(u_n^{(D)} | u_n^{(D)})$ and $W_2(u_n^{(P)} | u_n^{(P)})$ are minima under the constraints:

$$Q_1(u_n|u_n) = 0 \tag{55}$$

$$Q_2(u_n|u_n) = 0 \tag{56}$$

Hence, a solution for the Fourier coefficients a_{Pi}, a_{Di} $(i=1,n)$ can be determined, for example, by the method of Lagrange multipliers by solving the set of algebraic equations:

$$\frac{\partial}{\partial a_{Di}} \{W_2(u_n|u_n) + \lambda_1 Q_1(u_n|u_n) + \lambda_2 Q_2(u_n|u_n)\} = 0 \tag{57}$$

$$\frac{\partial}{\partial a_{Di}} \{W_2(u_n|u_n) + \lambda_1 Q_1(u_n|u_n) + \lambda_2 Q_2(u_n|u_n)\} = 0 \quad (58)$$

with

$$\hat{n}.\nabla a_{Di} = \hat{n}.\nabla a_{Pi} = 0 \quad (59)$$

and Eqs. (55) and (56). The following result can then be proven:

<u>*Theorem 1*</u> - *If Eq. (3) has a solution with finite energy, a solution of the set of algebraic equations (55)-(58) gives, in terms of the definitions given by Eqs. (54), an extremal sequence for* $W_2(u_n|u_n)$ +.

A sketch of the proof can be given by adopting a method similar to that of Ref. 15 (see pages 88-90), so, only some guidelines shall be given here. Let us consider, as an illustration, the discussion for $u^{(D)}_{n,s}$ (the same is true for $u^{(P)}_{n,s}$). If $u^{(S)}_{n,s} \in \{\bar{f}_1\}'$ then by definition:

$$W_2(u^{(D)}_n|u^{(D)}_n) \geq m^2 \ . \quad (60)$$

On the other hand, m^2 is an exact lower bound for $W_2(g^{(D)}|g^{(D)})$, under the constraints (19) and (20), for $g^{(D)}_s \in \{\bar{f}_1\}'$. Thus, there exists a funstion $v^{(D)}_s(v) \in \{\bar{f}_1\}'$ sush that:

$$||v^{(D)}||_C < m + \frac{\varepsilon}{2} \quad (61)$$

for an arbitrary positive ε. Since the basis $\{\phi^{(D)}_{i,s}(r,v),\ i \in N\}$ is orthonormal and complete in energy, it follows, furthermore, that there exist $k^* \in N$ and constants b_{Di} $(i=1,n)$ such that for $n > k^*$:

$$|\ ||v^{(D)}_n||_C - ||v^{(D)}||_C| < \frac{\varepsilon}{2}$$

$$v^{(D)}_{n,s} = \Sigma_{i=1,n}\, b_{Di}\, \phi^{(D)}_{i,s} \ .$$

+ This result is thus completely analogous to that obtained by the classical Rayleigh-Ritz direct variational solution method. See, for example, Ref. 15.

It follows that

$$m \leq \| v_n^{(D)} \|_C \leq \| v^{(D)} \|_C + \frac{\varepsilon}{7} \leq m + \varepsilon \tag{63}$$

However, since evidently $\| u_n^{(D)} \|_C \leq \| v_n^{(D)} \|_C$ with $u^{(D)}$ given by Eqs. (54) - (58), by letting $\varepsilon \to 0$, we infer that $\| u_n^{(D)} \|_C \to m$. Hence, the sequence $u_{n,s}^{(D)}$ is minimal for $W_2(u_n^{(D)} | u_n^{(D)})$.

6. CONCLUIDING REMARKS

Various possible variational formulations for the drift Fokker-Planck equation have been discussed, which are based either on "constrained" variational principles, where the class of admissible variations is appropriately limited in terms of "ad hoc" constraint equations, or on a variational principle based on the definition of an appropriate inner product (in terms of which the linear operator appearing in the given integro-differential equation is symmetric). We have shown that in the cases presented above the variational functionals are related to physically meaningful dynamical variables (material fluxes across an isobaric surface and the parallel electric current density), and therefore such variational formulations appear potentially useful for an accurate determination of such dynamical variables. An approximate method of solution based on a "constrained" variatiorial principle has also been described. In particular, we have shown how an approximate solution can be constructed in the form of an extremal sequence, proving, in addition, its weak convergence. The present results seem useful in order to allow açcurate transport calculations for collisional or even strongly collisional magnetoplasmas, in the sense that either $\nu_{s,eff}/\omega_{b,s} \sim 1$ or $\nu_{s,eff}/\omega_{b,s} \gg 1$. In fact, for such types of plasmas an accurate and systematic investigation, based on rigorous mathematical methods is in the author's view, still largely missing. In particular, it seems potentially useful to ascertain the accuracy of previous transport calculations performed by various authors and based on different approximation techniques. Selected applications and comparisons with other previous work shall be the object of a forthcoming paper.

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APPENDIX A

In order to define the adimensional parameter $\rho_s = \nu_{s,eff}/\omega_{b,s}$, it is convenient to write the drift Fokker-Planck equation (see eq.(21), defined in Sec.3) in terms of adimensional variables and the case $E_{\parallel}^{rot}=0$. Eq. (21) can then be written in the form:

$$\frac{\partial}{\partial \chi} g_s = \frac{\nu_{s,eff}}{\omega_{b,s}} D_s(h) \tag{A.1}$$

where

$$\nu_{s,eff} = \frac{B}{v_{\parallel} B_p |\nabla\chi|} \nu_s \omega_{b,s} \tag{A.2}$$

and

$$\nu_s = 4\pi e_s^4 N_{0,s} \ln \Lambda / (m_s^2 v_{th,s}^3) \tag{A.3}$$

is the Spitzer self-collision frequency. The linear integro-differential operator D_s, appearing in Eq. (A.l) is related to the Fokker-Planck collision operator (i.e. $D_s(h) = C_s(f_0|h)/\nu_s$), and χ denotes an appropriate angle-like curvilinear coordinate along the close path C. Recalling the representation of the linearized Fokker-Planck collision operator in the Landau form[8,11] in terms of the v-space coordinates (v,λ,ζ) with $v = |\vec{v}|$ and $\lambda = 2\mu/v^2$ one obtains (for $T_{0,s} = T_{0,k}$ for all species):

$$D_s(h) = H_s^{(0)}(h_s) + H_s^{(1)}(h_s) + H_s^{(2)}(h) \tag{A.4}$$

where

$$H_s^{(0)}(h_s) = \hat{U}_s(v)\ \frac{(1-\lambda B)^{\frac{1}{2}}}{B}\frac{\partial}{\partial\lambda}\{\lambda(1-\lambda B)^{\frac{1}{2}}\frac{\partial}{\partial\lambda}h_s\}$$

$$H_s^{(0)}(h_s) = \frac{1}{2v^2}\ \frac{\partial}{\partial v}\ \{v^2\frac{\partial}{\partial v}(\hat{U}_s(v))\ \exp\{\alpha_s(v)\}\ \frac{\partial}{\partial v}\ (h_s\ \exp\{-\alpha_s(v)\})\}$$

$$H_s^{(2)}(h) = \frac{1}{2}\Sigma_{k=1,r}\ \frac{ekv^3_{th,s}}{e_s^2 N_{0,s}}\ \frac{\partial}{\partial\vec{v}}\cdot\int d^3v'\frac{\partial^2 u}{\partial\vec{v}\partial\vec{v}}\{h_k\frac{\partial}{\partial\vec{v}}f_{0,s} - \frac{m_s}{m_k}f_{0,s}\frac{\partial}{\partial\vec{v}'}h_k\}$$

(A.5)

and

$$\hat{U}_s(v) = \frac{2}{\pi^{\frac{1}{2}}}\Sigma_{k=1,r}\frac{e_k^2 N_{0,k}}{e_s^2 N_{0,s}}\frac{1}{x_s^3 x_k}\{(\exp\{-x_k^2\} + 2x_k(1-\frac{1}{2x_k^2})\int_0^{x_k} dt\ \exp -t^2\}$$

where **(A.6)**

$$\alpha(v) = -x_s^2$$

Notice that the linearized Fokker-Planck collision operator $C_s(f_0|h)$ may also be written in the form:

$$C_s(f_0|h) = \Sigma_{k=1,r}\ \{C_{sk}(f_{0,k}|h_s) + C_{sk}(h_k|f_{0,s})\} \qquad \textbf{(A.7)}$$

where

$$C_{sk}(f_{0,k}|k_s) = q_{sk}\frac{\partial}{\partial\vec{v}}\cdot\int d^3v'\ \frac{\partial^2 u}{\partial\vec{v}\partial\vec{v}}\cdot\{f_{0,k}\frac{\partial}{\partial\vec{v}}h_s - \frac{m_s}{m_k}h_s\frac{\partial}{\partial\vec{v}'}f_{0,k}\}$$

(A.8)

$$C_{sk}(h_k|f_{0,s}) = q_{sk}\frac{\partial}{\partial\vec{v}}\cdot\int d^3v'\ \frac{\partial^2 u}{\partial\vec{v}\partial\vec{v}}\cdot\{h_k\frac{\partial}{\partial\vec{v}}f_{0,s} - \frac{m_s}{m_k}f_{0,s}\frac{\partial}{\partial\vec{v}'}h_k\}$$

(A.9)

with $\vec{u} = \vec{v}-\vec{v}'$ $\quad q_{sk} = 2\pi e_s^2 e_k^2\ \ell n\ (\Lambda_{sk}/m_s^2)$ and $\ell n\,\Lambda_{sk}$ the Coulomb logarithm.

APPENDIX B

For completeness we introduce here, the proof of the following result:

Theorem (*theorem of* the *minimal functional* $W_2(\bar{f}_1|\bar{f}_1)$).

The functional $W_2(g|g)$, *according* to *the definition (18), has* a *minimum in* the *class* $\{\bar{f}_1\}'$ *in which* the *variations* are *subject* to the *constraint equations (19) and (20),* for $\bar{f}_{1,s}$ *solution of Eq.(3).*

The proof can be obtained by noticing that $W(g|g)$ has a minimum both with respect to $<\bar{f}_{1,s}>_S$ and $\bar{f}'_{1,s}$ (where $\bar{f}_{1,s} = <\bar{f}_{1,s}>_S + \bar{f}'_{1,s}$) as well as with respect to $\bar{f}^{(D)}_{1,s}$ and $\bar{f}^{(P)}_{1,s}$, separately. It is convenient to introduce the following parametrization:

$$<g_s^{(\alpha)}>_S = <\bar{f}_{1,s}>_S + \alpha_s<\delta\bar{f}_{1,s}>_S \tag{B.1}$$

Adopting the method of Lagrange multipliers, we obtain from Eqs. (17), (19) and (20), from the first variation with respect to the variations of type (B.1):

$$\frac{\partial}{\partial\alpha_s}\{W_2(g^{(\alpha)}|g^{(\alpha)}) + \mu_1 Q_1(g^{(\alpha)}|g^{(\alpha)}) + \mu_2 Q_2(g^{(\alpha)}|g^{(\alpha)})\}|_{\alpha_s=0} = 0 \tag{B.2}$$

which leads to an integral equation depending on the Lagrange multipliers μ_1 and μ_2. On the other hand, Eqs. (19) and (20) imply:

$$Q_1(\bar{f}_1|\bar{f}_1) + Q_2(\bar{f}_1|\bar{f}_1) = -<\bar{f}_1|C\bar{f}_1 + F_1> = 0 \tag{B.3}$$

From the definition (18) for $W_2(\bar{f}_1|\bar{f}_1)$, it results immediately, for compatibility with such an equation, that $\mu_1 = \mu_2 = 0$, while evidently, the second variation gives:

$$\frac{\partial^2}{\partial\alpha_s^2} W_2(g^{(\alpha)}|g^{(\alpha)}) > 0 \tag{B.4}$$

Similarly, we consider the variations with respect to $\delta f^{(D)'}_{1,s}$ and $\delta f^{(P)'}_{1,s}$, adopting for convenience the parametrizations:

$$g_s^{(D)}(\beta^D) = \bar{f}_{1,s}^{(D)\,'} + \beta_s^D \delta \bar{f}_{1,s}^{(D)\,'} \qquad (B.5)$$

$$g_s^{(P)}(\beta^P) = \bar{f}_{1,s}^{(P)} + \beta_s^P \delta \bar{f}_{1,s}^{(P)\,'} \qquad (B.6)$$

In analogy with Eq. (B.3), one finds:

$$\frac{\partial}{\partial \beta_s^m}\{W_2(g(\beta)|g(\beta)) + \lambda_1^m Q_1(g(\beta)|g(\beta)) + \lambda_2^m Q_2(g(\beta)|g(\beta)\}|_{\alpha_s^m=0} = 0 \qquad (B.7)$$

for $m=D,P$ and $s=1,r$. The multipliers λ_1^m and λ_2^m are again determined by taking into account Eqs. (19) and (20). Using for $W_2(\bar{f}_1|\bar{f}_1)$ the definition (18), it results $\lambda_1^D = \lambda_2^P = 0$ and $\lambda_2^D = \lambda_1^P = 2$. We find again:

$$\frac{\partial^2}{\partial (\alpha_s^m)^2}\{W_2(g(\beta)|g(\beta)) + \lambda_1^m Q_1(g(\beta)|g(\beta)) + \lambda_2^m Q_2(g(\beta)|g(\beta))\} > 0 \qquad (B.8)$$

and thus the searched result is reached. It is interesting to point out that if adopt instead the definitions (18') for the functional $W_2(g|g)$, we immediately obtain that $W_2(g|g)$ has, on the contrary a *maximum* for $\bar{f}_{1,s}$, solution of Eq.(3).

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Resumo

Procedimentos variacionais diferentes são discutidos no caso do problema do transporte colisional em um plasma confinado magneticamente e métodos aproximados de solução são indicados. Em particular, é proposto um procedimento análogo ao de Rayleigh e Ritz. Aplicações selecionadas para a investigação de magnetoplasmas onde o efeito colisional é arbitrário são discutidos brevemente.