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Analytic Determination of Hornoclinic Points in Chaotic Maps

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Abstract Poincaré sections reduce the flow of Hamiltonian systems with two degrees of freedom to area preserving maps. If the map is chaotic, separatrices emanating from unstable fixed points intersect in isolated homoclinic (or heteroclinic) points. We verify the numerical precision of the Birkhoff normal form in calculating homoclinic points of the family of quadratic maps. A simple asymptotic formula is derived for two infinite sequences of homoclinic points, as they approach the fixed point for positive or negative iterations of the map.

1. INTRODUCTION

The flow of autonomous Hamiltonian systems of one degree of freedom is always *integrable*, that is, the orbits lie in invariant curves (level of the Hamiltonian). This is not the case for general differentiable area preserving *mappings* of the phase plane onto itself, such as obtained by taking a *Poincaré section* of Hamiltonian systems with two degrees of freedom (see e.g. Berry[1]). It is then often found that *chaotic orbits* fill densely some areas in the plane, in the limit of infinite iterations.

Consider a one-parameter family of maps, with the property that as the parameter $a \to 0$, the mapping becomes infinitesimal, i.e. a Hamiltonian flow. If for $\alpha=0$ the map has any unstable fixed points, the separatrices that emanate from them must either reenter the same point or some other fixed point such as in Fig. la or b.

All points on the separatrix have the property that in both limits, when the iteration number $n \to \infty$ and $n \to -\infty$, they arrive at an unstable fixed point. This property defines *homoclinic points*,

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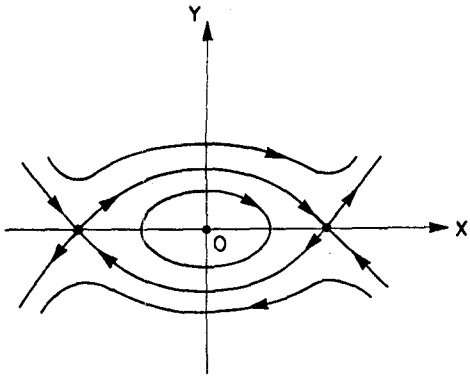
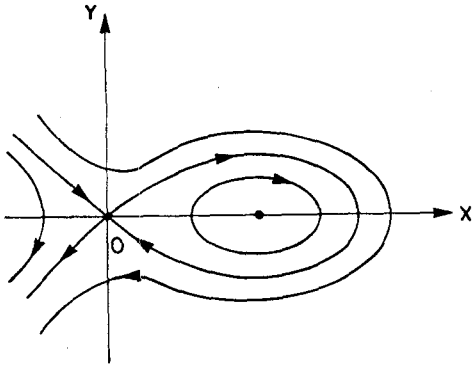


(a)



(b)

Fig. 1 - The separatrices connected to unstable equilibrium points of a Hamiltonian flow are level curves of the Hamiltonian. (a) Level curves of the Hamiltonian of the pendulum. (b) Level curves of a Hamiltonian with cubic potential energy. The invariant curves of the quadratic map (8) have this configuration as $\alpha \to 0$.

if the same fixed point is approached when both limits are taken, and *heteroclinic points* otherwise.

For generic families of maps the picture changes for any $\alpha \neq 0$. Then the rule for separatrices is to intersect transversely in *isolated* homoclinic (heteroclinic) points. Since the images of a homoclinic point under the mapping are also homoclinic points, the separatrices wiggle in a complicated way. The region swept out by the separatrices cannot contain smooth closed invariant curves. There typical trajectories are chaotic.

It is notoriously difficult to calculate properties of a chaotic motion directly from numerical iterations of the map, because of the

exponential instability of the orbits with respect to any uncertainty of the initial conditions. *In* section 2 we examine a *simple analytical* method for calculating homoclinic and heteroclinic points based on *Birkhoff normal foms*[2]. Numerical verification of the method for general quadratic maps is presented in section 3. It is foundthat beyond a certain parameter threshold convergence of the homoclinic points is obtained within the numerical accuracy.

2. THE BIRKHOFF NORMAL FORMAL

A linear canonical change of coordinates will take any analytical mapping, with an unstable fixed point at the origin into the form

$$q' = \lambda\left(q + \sum_{k=2}^{\infty} \sum_{\ell=0}^{k} q'_{k\ell}\, q^{k-\ell}\, p^{\ell}\right)$$
$$p' = \lambda^{-1}\left(p + \sum_{k=2}^{\infty} \sum_{\ell=0}^{k} p'_{k\ell}\, q^{k-\ell}\, p^{\ell}\right) \quad . \tag{1}$$

It was proved by Birkhoff[2] that this map can be transformed into the *normal form*

$$Q' = U(QP)\,Q$$
$$P' = (U(QP))^{-1} P \quad , \tag{2}$$

by the formal nonlinear transformation

$$q = Q + \sum_{k=2}^{\infty} \sum_{\ell=0}^{k} q_{k\ell}\, Q^{k-\ell}\, P^{\ell}$$
$$p = P + \sum_{k=2}^{\infty} \sum_{\ell=0}^{k} p_{k\ell}\, Q^{k-\ell}\, P^{\ell} \quad . \tag{3}$$

The fact that U is a function only of the product QP,

$$U(QP) = \lambda\left(1 + \sum_{k} U_{2k}\,(QP)^{k}\right) \quad , \tag{4}$$

leads immediately to

$$Q'P' = QP \quad . \tag{5}$$

Thus in the normal coordinates (Q,P) all orbits lie on invariant curves -hyperbolae, just as in the linear approximation of (1). The difference is that in the normal form the rate of expansion and contraction along the principal directions varies between the hyperbolae.

The coordinate transformation (3) is qualified as formal because its convergence is not guaranteed. Indeed one can easily seethat it cannot converge indefinitely far from the origin if the separatrix has a transverse homoclinic crossing with another separatrix. As previously discussed, the separatrix will then bend into infinitely tight wiggles, which behaviour cannot be reproduced by a convergent series.

Normal forms for stable fixed points do not converge anywhere, because of the small denominator problem[3] for the coefficients of the transformation. This is not present in the normal form transformation for unstable fixed points, as made evident by the recurrence relations for the coef'ficients,

$$\lambda U_{k-1}\delta_{k,2\ell+1} + (\lambda^{k-2\ell}-\lambda)\, q_{k\ell} = -\frac{\lambda}{4}\, q'_{k\ell} - \lambda^{k-2\ell} \sum_{j=1} q_{k-2j,\ell-j}\,(U^{k-2\ell})_{2j}$$

$$\lambda^{-1}(U^{-1})_{k-1}\delta_{k,2\ell+1} + (\lambda^{k-2\ell}-\lambda^{-1})p_{k\ell} = \frac{\lambda^{-1}}{4}\, p'_{k\ell} - \lambda^{k-2\ell} \sum_{j=1} p_{k-2j,\ell-j}\,(U^{k-2\ell})_{2j}, \tag{6}$$

where we have defined the auxiliary set of coefficients $(U^n)_j$ from the binomial expansion

$$(U(QP))^n = \lambda^n \left[1 + n \sum_{j=1}^{\infty} U_{2j}(QP)^j + \ldots\right] = \lambda^n \sum_{j=0}^{\infty} (U^n)_{2j}(QP)^k . \tag{7}$$

(The nontrivial content of Birkhoff's theorem is exemplified by the compatibility of the series for U and U^{-1} obtained from (6)).

The parametric equations for the separatrices are given by (3) with either Q or P set equal to zero. It is thus only necessary to compute the coefficients q_{k0}, p_{k0}, q_{kk} and p_{kk} in (6), which can be calculated independently from the other coefficients and the U_{2k}. The equations for the separatrices are non-linear, so they may cross at an approximate homoclinic point. Heteroclinic points need the calculation of normal forms for both relevant fixed points. The approxi-

mate separatrices obtained from a finite truncation of the normal form will have a finite number of oscillations and hence we can only approximate a finite number of homoclinic points directly. However, it is sufficient that a single homoclinic point be well approximated, since the others will be its forward and backward images in the normal form map. It is the possibility of finding such a point which we investigate numerically in the following section.

3. COMPUTATIONAL VERIFICATION FOR QUADRATIC MAPS

The linearization of a map with an unstable fixed pointatthe origin characterizes its neighbourhood. Introducing the next term in the Taylor series (1) we obtain quadratic maps which can be reduced through a linear change of coordinate to the form[4]

$$\begin{aligned} x' &= \cosh(\alpha)\, x + \sinh(\alpha)\, (y-x)^2 \\ y' &= \sinh(\alpha)\, x + \cosh(\alpha)\, (y-x)^2 \quad , \end{aligned} \tag{8}$$

where the single parameter is the eigenvalue a. A rotation of $\pi/4$ puts this map in the form of eyuation (1). For $a < a_c \simeq 1.76$ there exists also a single stable fixed point. In this range (8) can be transformed into the area preserving Henon map[5]. Beyond a_c the stable fixed point undergoes an infinite sequence of period doubling bifurcations equivalent to those studied by Bountis[6]. Iterations of the map for $a = 0.9$, for orbits in the right quarter-plane which contains the stable fixed point, are shown in Fig. 2. The majority of the orbits fall on invariant closed curves but the outer ones break up with the formation of islands surrounding stable periodic points and chaotic regions around the separatrices. As a reaches a_c, the whole family of closed invariant curves disappears.

Considering the linearization of (8) we determine the direction of separatrices at the origin. So, if we choose many points on the outgoing separatrix near the origin and iterate the map we obtain points approximately on the *nonlinear* outgoing separatrix. Here exponential instability is no problem since the stretching direction is exactly along the separatrix so that points near it are actually driven onto it. In the samemanner, we can trace out the incorning

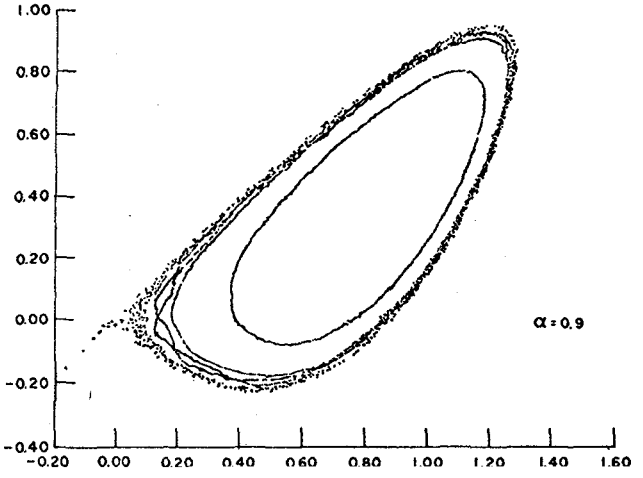


Fig. 2 - Iterations of four orbits of the quadratic map (8) for the parameter $\alpha = 0.9$. The two innermost orbits lie at least approxirnately on invariant closed curves which surround the stablefixedpoint. The third orbit is excluded from a set of islands surrounding stable periodic points. The outerrnost orbit ultimately escapes beyond the separatrices, rnaking an angle of $\pm \pi/4$ with the origin, after filling fairly evenly the accessible region.

separatrix by taking points on the incoming separatrix near the origin and iterating the inverse map. Again, exponential instability is on our side. Separatrices so obtained are shown in Fig. 3a. If a sufficient number of orbits is computed along the separatrix, then the homoclinic points can be obtained by linearly interpolating between successive points surrounding the crossing. The coordinates of the firsthornoclinic points are shown in the first two columns of Table 1. It can be shown that for $a \to \infty$ the coordinates of the first homoclinic points are (2,2).

The coefficirnts of the Birkhoff normal form are given explicitly by

$$q_{kk} = (1/4)\,(1-\lambda^{-k-1})^{-1}\,\left(2t_{k-1,k-1} + \sum_{i=2}^{k-2} t_{k-i,k-i}\, t_{i,i}\right)$$

$$p_{kk} = (1/4)\,(\lambda^{-k+1}-1)^{-1}\,\left(2t_{k-1,k-1} + \sum_{i=2}^{k-2} t_{k-i,k-i}\, t_{i,i}\right)$$

$$q_{k0} = (1/4)(1-\lambda^{k-1})^{-1} (2t_{k-1,0} + \sum_{i=2}^{k-2} t_{k-i,0}\, t_{i,0})$$

$$p_{k0} = (1/4)\, (\lambda^{k+1} - 1)^{-1} (2t_{k-1,0} + \sum_{i=2}^{k-2} t_{k-i,0}\, t_{i,0})$$

$$t_{k\ell} \equiv q_{k\ell} + p_{k\ell} \tag{9}$$

Approximate separatrices are shown in Figs. 3b. It is in the context of the normal form that we can make precise the idea of the first homoclinic point as that with the least sum $P_0 + Q_0$ of the normal form coordinates. The (q_0, p_0) coordinates of the first homoclinic points, for the same parameter values as used for the direct calculation, are shown in the third and fourth columns of Table 1.

Table 1 - (x_a, y_a) are the coordinates of the first homoclinic point from the computation of the separatrices as in Fig. 3a. (x_b, y_b) arethe coordinates according to the Eirkhoff normal form as in Fig. 3b. n is the order of the normal form for which the numerical convergence to the numbers displayed was obtained.

α	x_a	y_a	x_b	y_b	n
1.4	1.872831	1.753758	1.872848	1.753780	20
1.6	1.941903	1.885523	1.941697	1.885339	20
1.8	1.972095	1.944593	1.972109	1.944608	18
2.0	1.986109	1.972317	1.986111	1.972320	16
2.5	1.997332	1.994668	1.997333	1.994670	14
3.0	1.999411	1.998861	1.999450	1.998900	10
3.5	1.999851	1.999734	1.999882	1.999765	8
4.0	1.999984	1.999958	1.999974	1.999948	8
4.5	1.999991	1.999985	1.999994	1.999988	6
5.0	1.999998	1.999997	1.999998	1.999997	6
6.0	1.999998	1.999998	2.000000	2.000000	6
7.0	2.000004	2.000004	1.999999	1.999999	6

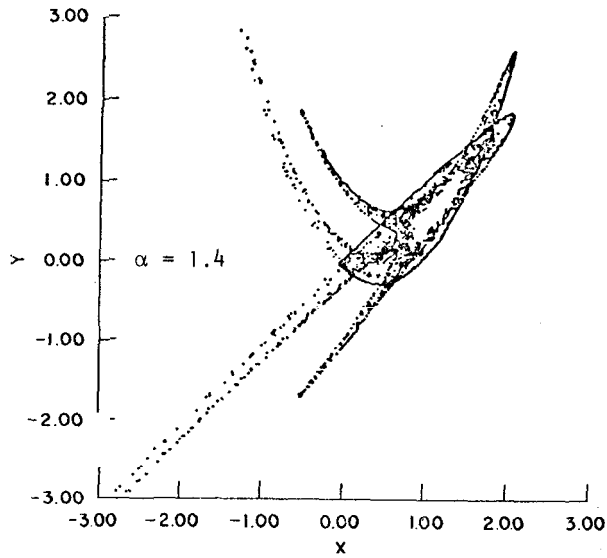


Fig.3 (a)

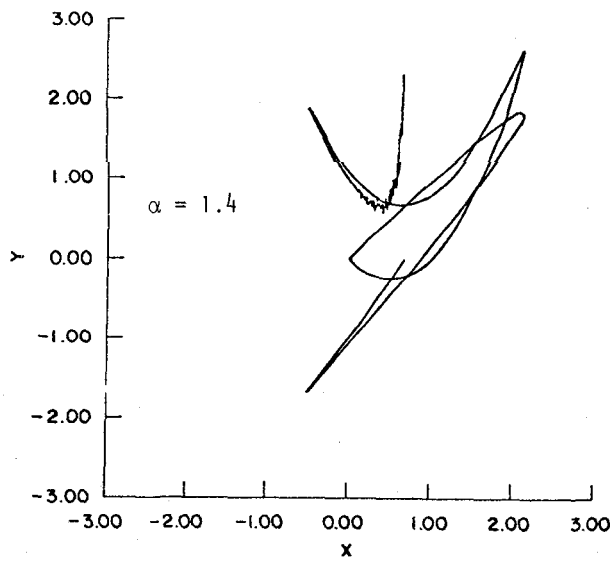


Fig.3 (b)

Fig. 3 - (a) Points on the incoming and outgoing separatrices obtained by positive and negative iterations on the linear approximation of the separatrices very near the origin, for $\alpha=1.4$. (b) The two separatrices calculated fromthe Birkhoff normal form up to twentieth order.

Within the numerical precision we find that where the series converge they give correctly the first homoclinic point. For high values of the parameter α numerical convergence is fast - we need only the first five or so coefficients. Close to the limit of the range of convergence we found it is necessary to calculate up to twenty coefficients. It is noteworthy that convergent values for the homoclinic coordinates were obtained even for parameter values for which the map was not globally chaotic and still contained a family of closed invariant curves.

4. CONCLUSION

We have verified the power of the Birkhoff normal form near a stable fixed point of an area preserving map in the region where the coordinate transformation is not one-to-one. The direct calculation of the first homoclinic point is not difficult since exponential instability is not a problem, but it is not easy to calculate successive homoclinic points directly. One choice is to iterate the map far beyond what is necessary to obtain the first homoclinic point, and then use the successive intersections of the separatrices. This process cannot be carried through indefinitely for a finite number of orbits. The other choice is to iterate the forward and backward orbit of the first homoclinic point. But here exponential instability is a hindrance. As the fixed point is approached any error, no matter how small, in the original evaluation will eventually push the iterations off the separatrix - the orbit will diverge from the origin.

Successive homoclinic points can however be easily calculated using the normal form. Indeed, along the separatrices $U(QP) = h = \exp(\alpha)$, so that the normal form coordinates are

$$P_n = e^{n\alpha} P, \quad , \quad Q_n = e^{-n\alpha} Q, \quad . \tag{10}$$

Asymptotically close to the origin, where direct calculation fails, we can use the linear part of the transformation (3) giving

$$p_n = e^{n\alpha} P_0 \quad , \quad q_n = e^{-n\alpha} Q_0 \ . \tag{11}$$

The coordinates of the homoclinic points which converge onto the origin of the quadratic map (8) are thus given explicitly by

$$(x_n, y_n) = e^{-n\alpha} (Q_0, Q_0)$$
$$(x_n, y_n) = e^{n\alpha} (P_0, -P_0) \tag{12}$$

REFERENCES

1. M.V. Berry, Topics in non-linear mechanics, Ed. S. Jorna (Am. Inst. Phys. Proc. *46*), 16 (1978).
2. G.D. Birkhoff, Acta Math. ***43***, 1 (1920).
3. J.K. Moser, *Stable and Random Motions* (Princeton Univ. Press, Princeton, 1973).
4. T.J.S.B. Coutinho, Tese de Mestrado - UNICAMP (1984).
5. M. Henon, Quarterly of Applied Math. *27*, 291 (1969).
6. T.C. Bountis, Physica ***3D***, 577 (1981).

Resumo

Pode-se reduzir o estudo do fluxo de um sistema Hamiltoniano de dois graus de liberdade ao de uma aplicação do plano que conserva área, por meio de seções de Poincaré. Se a aplicação for caótica, as separatrizes que emanam de um ponto fixo instável se intersectam em pontos homoclínicos (ou heteroclínicos) isolados. Verificamos a precisao numérica do cálculo de pontos homoclínicos para a família de aplicações quadráticas, a partir da forma normal de Birkhoff. Derivamos uma simples fórmula assintótica para duas sequências infinitas de pontos homoclínicos que se aproxima do ponto fixo para iterações positivas ou negativas da aplicação.