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Magnetic Dipole Field in a Schwarzschild Metric with Non-Minimal Coupling

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Abstract The influence of a non-minimal coupling term of electromagnetism and gravity is studied for a magnetic dipole field in the Schwarzschild metric. It is found that the new coupling term changes the magnetic dipole moment even for small masses.

1. INTRODUCTION

A.R. Prasanna[1] suggested a non-minimal coupling of electromagnetism and gravity writing the total Lagrangian

$$L = \sqrt{-g}\ \{R + \alpha F_{ab}F^{ab} + \beta R_{abcd}F^{ab}F^{cd}\} \tag{1.1}$$

which yields the field equations

$$\partial_j\left[\alpha\sqrt{-g}\ F^{ij} + \beta\sqrt{-g}\ R^{pqij}\ F_{pq}\right] = 0 \tag{1.2}$$

and

$$R_{ij} - \frac{1}{2} g_{ij} R = -2\alpha\left[F_{ia} F_j^{\ a} - \frac{1}{4} g_{ij} F_{ab} F^{ab}\right] \tag{1.3}$$

$$+ \beta\left[\frac{1}{2} g_{ij} R_{amsq} F^{am} F^{sq} - 3R_{(i}^{\ \ msq} F_{j)m} F_{sq} + 2(F^m_{\ \ (i} F_{j)}^{\ \ s})_{;sm}\right]$$

where ( ) denotes symmetrization of the índices i,j.
The constant a is related to Einstein's constant a = $-\frac{1}{2}\kappa$ =-1.04.10^{-43} kg^{-1}m^{-1}s^2 (or in natural units with $\hbar = c = 1$, $\alpha = -3.3.10^{-69}$ m^2) and β is a new coupling constant with dimension m^4 (in natural units). As β has dimension, and a dimension different from that of a, there is no a-priori criterion to estimate its value. The only natural value would be β=0, invoking the *equivalence principle*. However, the equivalence principle is a physical one and hence it has to be checked by experiments. The main value of a model like the one given by eq. (1.1) is to provide us with specific experimental tests of the equivalence principle.

The non-minimal coupling term in eq. (1.1) represents an interesting deviation from the equivalence principle because it respects the geometrical nature of gravity and the gauge symmetry of electromagnetism. Moreover, the classical experiments testing the equivalence principle such as the Eötvös experiment are performed in space time regions of small curvature and they are not specific for couplings of this kind.

In this paper we study the influence of the gravitational field with the non-minimal coupling term on a magnetic dipole field. We neglect the back reaction of the magnetic field on the spacetime metric i.e. we consider the right hand side of eq. (1.3) equal to zero.

2. A PERTURBATIVE ANALYSIS OF MAXWELL'S EQUATIONS

We will consider Maxwell's equations (1.2) in the Schwarzschild metric

$$g_{tt} = -(1 - \frac{2M}{r}) \;,\quad g_{rr} = (1 - \frac{2M}{r})^{-1} \;,\; g_{\theta\theta}=r^2 \;,\; g_{\phi\phi}=r^2\sin^2\theta \tag{2.1}$$

where 2M is the Schwarzschild radius. For small M the metric can be written as

$$g_{ab} = n_{ab} + Mm_{ab} + O(M^2)$$

with

$$n_{tt} = -1 \;,\quad n_{rr} = 1 \;,\quad n_{\theta\theta} = r^2 \;,\quad n_{\phi\phi} = r^2 \sin^2\theta \tag{2.2}$$

and

$$m_{tt} = m_{rr} = \frac{2}{r} \;,\; m_{\theta\theta} = m_{\phi\phi} = 0 \tag{2.3}$$

We also define

$$\tilde{n}^{ab} n_{bc} = \delta^a_c$$

and

$$\tilde{m}^{ab} = \tilde{n}^{ac} \tilde{n}^{bd} m_{cd}$$

Let ∇ denote the covariant derivative defined with the Christofell connection calculated from n_{ab}. Then eq. (1.2) to first order in M reads

$$\alpha\nabla_j F^{ij} + \beta\nabla_j (R^{pqij} F_{pq}) = 0 \tag{2.4}$$

where we used the fact that for the Schwarzschild metric the trace Γ^n_{kn} is the same as the one calculated from n_{ab}, The electromagnetic tensor will be written as

$$F_{ab} = f_{ab} + Mh_{ab} + O(M^2) \tag{2.5}$$

where f_{ab} satisfies

$$\nabla_j(\tilde{n}^{ia}\ \tilde{n}^{jb}\ f_{ab}) = 0 \tag{2.6}$$

Inserting (2.5) into (2.4) and using (2.6) we obtain

$$\nabla_j(\tilde{n}^{\ell a}\ \tilde{n}^{jb}\ h_{ab}) = J^{\ell} \tag{2.7}$$

with

$$J^{\ell} = \lim_{M\to 0} \nabla_j\left[(\tilde{m}^{\ell p}\ \tilde{n}^{jq} + \tilde{n}^{\ell p}\ \tilde{m}^{jq} - \frac{\beta}{\alpha M}\ R^{pq\ell j}\ f_{pq}\right] \tag{2.8}$$

Equation (2.7) has the form of ordinary flat spacetime inhomogeneous Maxwell equations with a source J^{R}.

3. MAGNETIC DIPOLE FIELD

We will consider the magnetic field created by a distribution of electric currents on the surface of a sphere of radius R and mass m (equivalent to the Swarzschild radius $2M$). This distributlon is such that it has a magnetic dipole $\vec{\mu}$ parallel tothe z direction, where $z = r\cos\theta$. The current J calculated from (2.8) reads

$$J = \left(\frac{A}{r^6} + \frac{B}{r^8}\right)\partial_{\phi} \tag{3.1}$$

where

$$A = -6\ |\vec{\mu}|\ ,\quad B = -18\ \frac{\beta}{\alpha}\ |\vec{\mu}| \tag{3.2}$$

Instead of solvind eq. (2.7) to find the first order correction to the magnetic field (h_{ab}), we will look for the corresponding first order correction $\delta\vec{\mu}$ to the magnetic dipole moment. $\delta\vec{\mu}$ can be obtained by the relation

$$\delta\vec{\mu} = \frac{M}{8\pi}\int (\vec{r}\times\vec{J})\,d^3\vec{r} \tag{3.3}$$

Writing

$$\vec{r} = r\ \vec{e}_r$$

and

$$\vec{J} = (\frac{A}{r^6} + \frac{B}{r^8})\, r \sin\theta\, \vec{e}_\phi$$

we obtain

$$\vec{r}\times\vec{J} = -\,(\frac{A}{r^6} + \frac{B}{r^8}) r^2 \sin\theta\, \vec{e}_\theta \tag{3.4}$$

The first order correction of the magnetic dipole moment in the z direction is then

$$\delta\mu = \frac{M}{8\pi} \int_R^\infty \int_0^\pi \int_0^{2\pi} (\frac{A}{r^6} + \frac{B}{r^8})\, r^4 \sin^3\theta\, d\phi\, d\theta\, dr \tag{3.5}$$

where the integration is performed outside the matter distribution. $\delta\mu_x$ and $\delta\mu_y$ vanish because of the symmetry of the current distribution. Performing the integral eq. (3.5) we find

$$\delta\vec{\mu} = -M(\frac{2}{R} + \frac{2\beta}{\alpha R^3})\vec{\mu} \tag{3.6}$$

The first term of eq. (3.6) represents the first order correction due to the minimal coupling of gravity and electromagnetism. For a given mass density $\rho = 3M/4\pi R^3$ this contribution is proportional to R^2. The second term is the contribution of the non-minimal coupling and it is proportional only to the mass density. So in principle the effect of the non-minimal coupling could be observed with small masses, provided $|\beta|$ is sufficiently large. This dependence on the density suggests to study the effect with matter of nuclear density. For a neutron ($R_n \approx 0.8\times10^{-15}$m, $m_n \approx 1.7\times10^{-27}$kg, $M_n \approx 1.2\times10^{-54}$m) we get

$$|\frac{\delta\mu}{\mu}| = |\frac{\beta}{\alpha}|\ 4.8\times10^{-9}\ \text{m}^{-2} \tag{3.7}$$

As the theoretical predictions of the magnetic moments of nucleons based on the quark model reach a precision of a few percent, we may at least conclude thst $|\frac{\delta\mu}{\mu}| < 1$. This gives for $|\beta|$ with a = -3.3×10^{-69} m^2

$$|\beta| < 10^{-60}\ \text{m}^4 \tag{3.8}$$

This upper bound for $|\beta|$ is by a factor 10^9 stronger than the upper bound available from measurements of deflection of light[2].

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Resumo

A influência de um termo de acoplamento não-mínimo entre eletromagnetismo e gravitação é estudada para um campo de dipolo magnético na métrica de Schwarzschild. Encontra-se que o novo termo de acoplamento muda o momento de dipolo magnetico, mesmo para massas pequenas.