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Tensor Algebra over Hilbert Space: Field Theory in Classical Phase Space

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Abstract It is shown using tensor algebras, namely Symmetric and Grassmann algebras over Hilbert Space that it is possible to introduce field operators, associated to the Liouville equation of classical statistical mechanics, which are characterized by commutation (for Symmetric) and anticommutation (for Grassmann) rules. The procedure here presented shows by construction that many-particle classical systems admit an algebraic structure similar to that of quantum field theory. It is considered explicitly tha case of n-particle systems interacting with an external potential. A new derivation of Schönberg's result about the equivalence between his field theory in classical phase space and the usual classical statistical mechanics is obtained as a consequence of the algebraic structure of the theory as introduced by our method.

1. INTRODUCTION

The use of HIlbert space in classical statistical mechanics (CSM) was proposed a long time ago by Koopman[1]. Some physical realisatlons of this Hilbert spáce have been presented by Della Riccia and Wiener[2] and by Schönberg[3]. In these realisations one considers the Hilbert space $\mathbb{H} = L_2(\Omega_n)$ of square-integrable functions $\Theta_n(p,q)$ defined on the phase space

$$\Omega_n(\vec{p}_1,\vec{p}_2,\ldots,\vec{p}_n,\vec{q}_1,\vec{q}_2,\ldots,\vec{q}_n) \equiv \Omega_n(p,q) \equiv \Omega_n .$$

The dynamical transformation group T^t on Θ_n then induces the one-parameter unitary group U_t in $\mathbf{H}$ such that

$$(U_t\,\Theta_n)(p,q) = \Theta_n(T^{-t}p,\ T^{-t}q).$$

The generator K_n of the group U_t is called the Liouville operator or Liouvillian of the system and we have

$$U_t = e^{-iK_nt} \quad (t \text{ real}) .$$

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It can be shown that if the classical Hamiltonian $H_n(p,q) \equiv H_n$ is a sufficiently smooth function (and it is assumed that this is always the case), then K_n is a self-adjoint and, in general, unbounded operator given by

$$K_n \Theta_n = i\{H_n,\Theta_n\}_n$$

where

$$\{H_n,\Theta_n\}_n = \sum_{k=1}^{n} \left(\frac{\partial H_n}{\partial \vec{p}_k} \cdot \frac{\partial \Theta_n}{\partial \vec{q}_k} - \frac{\partial H_n}{\partial \vec{q}_k} \cdot \frac{\partial \Theta_n}{\partial \vec{p}_k}\right) \equiv \{H_n, \}_n \; n$$

is the Poisson bracket. The Liouville equation is then written as

$$\frac{\partial \Theta_n}{\partial t} = i\, K_n\, \Theta_n \,, \quad \Theta_n \in L_2(\Omega_n) \;. \tag{1}$$

Thus, with the formulation in terms of Hilbert space the states of a classical system are represented by normalized functions Θ_n in $L_2(\Omega_n)$ with the inner product

$$<\Theta_n(p,q)\,,\Theta'_n(p,q)> = \int \Theta^*_n(p,q)\,\Theta'_n(p,q)\,dpdq \;; \tag{2}$$

the classical observables $A(p,q)$ of the system are represented by operators $\mathsf{A}(p,q)$ of multiplication by corresponding real-valued functions $A(p,q)$. As in quantum theory, the quantity

$$<\Theta_n(p,q),\; \mathsf{A}(p,q)\Theta_n(p,q)> \tag{3}$$

is interpreted as the expectation value of the observable $\mathsf{A}(p,q)$ in the state $\Theta_n(p,q)$. Since the observable $\mathsf{A}(p,q)$ is the operator of multiplication by the phase-space function $A(p,q)$, the expression (3) can be written as

$$\int A(p,q)\Theta^*_n(p,q)\Theta_n(p,q)\,dpdq \;; \tag{4}$$

Then, the quantity

$$f_n(p,q)\,dpdq = \Theta^*_n(p,q)\Theta_n(p,q)\,dpdq = |\Theta_n(p,q)|^2\; dpdq$$

can be interpreted as the probability of finding, at a time t, the mechanical system in a point $(p,q) \in \Omega_n$ in the interval

$$[(p,q) - (p+dp, q+dq)] \;.$$

Consequently the expression (4) represents the ensemble average of the physical quantity $A(p,q)$ in the Gibbs ensemble

$$f_n(p,q;t) = |\Theta_n(p,q;t)|^2 \;.$$

Furthermore, if Θ_n is a solution of the Liouville equation (1) then Θ_n^* and

$$f_n(p,q;t) = \Theta_n^*(p,q;t)\,\Theta_n(p,q;t)$$

satisfy tha same equation. Hence, the Hilbert space formulation of CSM is showed to be consistent with Gibbs statistical mechanics.

Some time ago Schönberg[3] proposed the application of second quantization methods to the Liouville equation. He considered a system of indistinguishable particles and defined field operators , and $\psi^\dagger(\vec{p},\vec{q})$ characterized by commutation and anti-commutation rules, that is,

$$[\psi(\vec{p},\vec{q}),\psi(\vec{p}',\vec{q}')]_\pm = 0 \tag{5a}$$

$$[\psi^\dagger(\vec{p},\vec{q}),\psi^\dagger(\vec{p}',\vec{q}')]_\pm = 0 \tag{5b}$$

$$[\psi(\vec{p},\vec{q}),\psi^\dagger(\vec{p}',\vec{q}')]_\pm = \delta(\vec{p}-\vec{p}')\,\delta(\vec{q}-\vec{q}') \;, \tag{5c}$$

where

$$[A,B]_\pm = A\,B \pm B\,A$$

and

$$\vec{p} \equiv (p_1,p_2,p_3)$$
$$\vec{q} \equiv (q_1,q_2,q_3) \;.$$

Schönberg also introduced the equation

$$i\frac{\partial}{\partial t}\chi(t) = K\,\chi(t) \tag{6}$$

where[3,5]

$$\chi(t) = \Theta_0\,\chi_0 + \sum_{n=1}^{\infty}\frac{1}{\sqrt{n!}}\int \Theta_n(p,q)\chi_n(p,q)\,dpdq$$

$$K = i\int \psi^\dagger(\vec{p},\vec{q})\;\{H,\psi(\vec{p},\vec{q})\}_1\;d\vec{p}d\vec{q}$$

$$\chi_n(p,q) = \psi^\dagger(\vec{p}_1,\vec{q}_1)\psi^\dagger(\vec{p}_2,\vec{q}_2)\dots\psi^\dagger(\vec{p}_n,\vec{q}_n)\ \chi_0$$

χ_0 single the vacuum defined in the phase-space Ω of a singlc-particle, and in the rules (5) the sign minus (plus) corresponds to classical bosons (fermions). Having done so, in order to establish a relatlonship between his theory and the usual CSM, he proceeded to show that eq. (6) leads to eq. (1) when the system has a fixed number n of particles. His demonstration follows the usual presentation of the quantum mechanical theory[4] to introduce the second quantization method for the study of the many-body problem; it is clear but does not adequately exhibit important algebraic aspects of that method.

In this communication we present an alternative procedure to obtain Schönberg's results. We will use tensor algebras over Hilbert spaces[6-9] in connection with quantum field theory; more specifically Symmetric (for classical bosons) and Grassmann (for classical fermions) algebras[10-13]. For simplicity we will consider the case of n-particle systems interacting with an external potantial. However, the method here developed may be in principle extended for a system with interacting particles, taking in account their interaction

$$\sum_{i<j} U\ (\vec{q}_i,\vec{q}_j)\ .$$

Our procedure is simple; it is closely related to the multilinear algebraic method applied to Schrödinger equation. The importance of this procedure is that it allows to show explicitly that the classical and quantum theory admit similar algebraic structures.

2. NOTATIONS AND PRELIMINARES

We will represent a complex Hilbert space of arbitrary dimension by

$$\mathbb{H} \equiv \{\theta,\gamma,\eta,\dots\} = L_2(\Omega)\ .$$

The Hilbert space

$$\mathbb{H} = \mathbb{H}\otimes\mathbb{H}\otimes\dots\otimes\mathbb{H}$$

(n factors) will be the n-fold product of $\mathbb{H}$ with itself; for n=l we will use

$$\mathbb{H}_1 \equiv \mathbb{H}\ ;$$

will denote the space of all complex numbers with the inner product $\langle a,b\rangle = a^* b$. We put $\mathbb{F}$ for the Hilbert space direct sum, i.e.,

$$\mathbb{F} = \overset{m}{\underset{n=O}{5}} \mathbb{H}_n$$

The canonical representation $\Gamma^{(n)} = \{[\sigma]\}$

of the symmetric group

$$S_n = \{\sigma\}$$

of degree n on the space $\mathbb{H}_n$ is determined by the condition that

$$[\sigma] \ (\gamma_1 \otimes \gamma_2 \ \otimes \dots \otimes \gamma_n) = \gamma_{\sigma 1} \otimes \gamma_{\sigma 2} \ 8 \ \dots \otimes \gamma_{\sigma n}$$

for arbitrary vectors $\gamma_1, \gamma_2, \dots, \gamma_n$ in $\mathbb{H}$. The operators

$$\hat{S} = (n!)^{-1} \sum_a [\sigma]$$

and

$$\hat{A} = (n!)^{-1} \sum_a \text{sign } (a) \ [\sigma]$$

are called symmetrizatlon and alternation, respectively, on $\mathbb{H}_n$. If $\phi \in \mathbb{H}_n$ and $h \in \mathbb{H}_m$ with $n + m = r$, their tensor product is a $(m+n)$-tensor denoted as @ 8 $\lambda \in \mathbb{H}_r$ and defined by

i) $\{\phi \otimes \lambda\} \ (\gamma_1, \gamma_2, \dots, \gamma_n, \gamma_{n+1}, \dots, \gamma_{n+m}) = \phi(\gamma_1, \gamma_2, \dots, \gamma_n) \lambda(\gamma_{n+1}, \dots, \gamma_{n+m})$

with $\gamma_i \in \mathbb{H}$

ii) $\phi \ 8 \ h = h \otimes \phi = (\text{constant } \phi) \ h$, if $\phi \in$ M,

With $\phi \in \mathbb{H}_n$ fixed, the correspondence

$$\lambda \in \mathbb{H}_m \to \phi \otimes \lambda = \hat{P}(\phi) \ \lambda \in \mathbb{H}_r \tag{7}$$

is linear mapping which depends on $\phi \in \mathbb{H}_n$. With $\phi \in \mathbb{H}_n$ and $h \in \mathbb{H}_m$ we can define the contractlon of λ wlth respect to ϕ, $\hat{C}(\phi) \ h$, such that

i) $\hat{C}(\phi) \ \lambda = 0 \in \mathbb{H}_0$, if $n > m$ (8a)

ii) $\hat{C}(\phi) \ \lambda = \langle\phi, \lambda\rangle = \lambda(\phi) \in \mathbb{H}_0$, if $n = m$ (8b)

iii) $\{\hat{C}(\phi)\lambda\} \ (\gamma_1, \gamma_2, \dots, \gamma_{m-n}) = \lambda(\phi \otimes \gamma_1 \otimes \gamma_2 \otimes \dots \otimes \gamma_{m-n})$, if $n < m$ (8c)

iv) $\hat{C}(\phi) \ \lambda = (\text{constant } \phi) \ \lambda$, if $\phi \in \mathbb{H}_0$; (8d)

it is a mapping which is antilinear in ϕ.

It is interesting to notice that if $\phi \in \mathbb{H}$ and $\lambda \in \mathbb{H}_m$, then

$$\hat{P}(\phi) : \mathbb{H}_m \to \mathbb{H}_{m+1}$$

$$\hat{C}(\phi) : \mathbb{H}_m \to \mathbb{H}_{m-1}$$

An element of the Hilbert space direct sum $\mathbb{F}$ will be denoted by

$$0 \equiv (\phi_n) \equiv (\phi_0, \phi_1, \ \dots \ , \phi_n, \dots)$$

with $\phi_n \in \mathbb{H}_n$ such that

$$\sum_{n=0}^{\infty} ||\ \phi_n\ ||^2 < \infty$$

If we have

$$0 \equiv (\phi_n) \quad , \quad A \equiv (h,) \quad \in \mathbb{F}$$

we define

$$\langle\Phi, \Lambda\rangle = \sum_{n=0}^{w} \langle\phi_n, \lambda_n\rangle \quad .$$

The operators $\hat{P}(\lambda_n)$ and $\hat{C}(\lambda_n)$ are defined on $\mathbb{F}$, respectively, by

$$\hat{P}(\lambda_n)\Phi = \hat{P}(\lambda_n)\ (\phi_0, \phi_1, \dots, \phi_i, \dots)$$

$$= (\underbrace{0,0,0,\dots 0}_{(n-1)\text{-fold}}, \lambda_n \otimes \phi_0, \lambda_n \otimes \phi_1, \dots) \qquad (9)$$

and

$$\hat{C}(\lambda_n)\Phi = \hat{C}(\lambda_n)\ (\phi_0, \phi_1, \dots, \phi_i, \dots)$$

$$= (\hat{C}(\lambda_n)\ \phi_n, \ \hat{C}(\lambda_n)\ \phi_{n+1}, \ \dots) \qquad (10)$$

where $\Phi \in \mathbb{F}$ and $\lambda_n \in \mathbb{H}_n$. It follows[8] that

$$\langle\hat{P}(\lambda_n)\ \Phi, \ \Phi'\ \rangle = \langle\Phi, \ \hat{C}(\lambda_n)\Phi'\ \rangle$$

that is, $\hat{P}(\lambda_n)$ is the adjoint operator of $\hat{C}(\lambda_n)$.

The symmetrization and alternation operators $\hat{S}$ and $\hat{A}$ are defined on $\mathbb{F}$ by

$$\hat{S}\ \Phi = \hat{S}(\phi_n) = (\hat{S}\ \phi_0, \ \hat{S}\ 6,, \dots, \ \hat{S}\ \phi_n, \ \dots)$$

$$\hat{A}\ \Phi = \hat{A}(\phi_n) = (\hat{A}\ \phi_0, \ \hat{A}\ \phi_1, \dots, \ \hat{A}\ \phi_n, \ \dots)$$

with $\hat{A}\ \phi_0 = \hat{S}\ \phi_0 = \phi_0$, $\Phi \in \mathbb{F}$, $\phi_n \in \mathbb{H}_n$. Thus, one obtains the Hilbert spaces

$$\mathbb{F}^{-}(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \hat{S}\,\mathbb{H}_n \quad , \quad \mathbb{F}^{+}(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \hat{A}\,\mathbb{H}_n$$

called the Symmetric algebra and the Grassmann algebra, respectively. In fact, given $\Phi' = (\phi_n)$, $\Lambda' = (\lambda_n)$ with $\phi_k = 0$ snd $\lambda_k = 0$ except for a finite number of indices k (Φ' and $\Lambda' \in \mathbb{F}' \subset \mathbb{F}$), It is possible to define the symmetric product by

$$\Phi' \odot \Lambda' = \hat{S}(\Phi' \otimes \Lambda') \tag{11a}$$

and the exterior product by

$$\Phi' \wedge \Lambda' = \hat{A}(\Phi' \otimes \Lambda') \ . \tag{11b}$$

These products (11a, 11b) are associative. linear with respect to their factors, and define on $\mathbb{F}'^{(-)}(\mathbb{H}) = \hat{S}\,\mathbb{F}'(\mathbb{H})$ and $\mathbb{F}'^{(+)}(\mathbb{H}) = \hat{A}\,\mathbb{F}'(\mathbb{H})$, respectively, algebra structures. In what follows we will just $\mathbb{F}^{-}(\mathbb{H})$ and $\mathbb{F}^{+}(\mathbb{H})$ to specify these algebras, thuç omiting the prime syperscript.

On the Hilbert space $\mathbb{F}$ one can define the operator $\hat{h}(N)$ such that

$$\hat{h}(N)\Phi = (h(0)\ \phi_0,\ h(1)\ \phi_1, \ldots, h(n)\ \phi_n, \ldots)$$

where $h(n)$ is a function Jefined on the set of non-negative integers.The operators $\hat{h}(N)$ and the mappings $\hat{P}(\phi)$ and $\hat{C}(\phi)$ given by eqs.(9,10) allow us to define, on the Grassmann aigebra $\mathbb{F}^{+}(\mathbb{H})$, the operators

$$b^{+}(\eta) = \sqrt{N}\ \hat{A}\ \hat{P}(\eta) = \hat{A}\ \hat{P}(\eta)\ \sqrt{N+1} \tag{12a}$$

$$b^{-}(\eta) = \hat{C}(\eta)\ \hat{A}\ \sqrt{N} = \sqrt{N+1}\ \ \hat{C}(\eta)\ \hat{A} \quad ; \quad \eta \in \mathbb{H} \equiv L_2(\Omega) \tag{12b}$$

and, on the Symmetric algebra $\mathbb{F}^{-}(\mathbb{H})$ the operators

$$a^{+}(\eta) = \sqrt{N}\ \hat{S}\ \hat{P}(\eta) = \hat{S}\ \hat{P}(\eta)\ \sqrt{N+1} \tag{13a}$$

$$a^{-}(\eta) = \hat{C}(\eta)\ \hat{S}\ \sqrt{N} = \sqrt{N+1}\ \hat{C}(\eta)\ \hat{S} \quad ; \quad \eta \in \mathbb{H} \equiv L_2(\Omega) \tag{13b}$$

It follows that in $\mathbb{F}^{+}(\mathbb{H})$, we have

$$[b^{+}(\eta),\ b^{+}(\gamma)]_{+} \subseteq 0 \tag{14a}$$

$$[b^{-}(\eta),\ b^{-}(\gamma)]_{+} \subseteq 0 \tag{14b}$$

$$[b^{-}(\eta),\ b^{+}(\gamma)]_{+} \subseteq \langle \eta, \gamma \rangle\ \hat{A}\ ;\ \eta, \gamma \in \mathbb{H} \tag{14c}$$

and, correspondently, in $\mathbb{F}^-(\mathbb{H})$,

$$[a^+(\eta),\ a^+(\gamma)]_- \subseteq 0 \tag{15a}$$

$$[a^+(\eta),\ a^-(\gamma)]_- \subseteq 0 \tag{15b}$$

$$[a^+(\eta), a^+(\gamma)]_- \subseteq <\eta,\gamma> \hat{S};\quad \eta\ \ \gamma \in \mathbb{H} \tag{15c}$$

where the symbol $\subseteq$ indicates the inclusion of the domains of the operators and the equality of the operators where the domalns coincide. The field operators will be defined by using the operators $b^-(\eta)$ and $a^{\pm}(\eta)$ given by eqs. (12a,b) and (13a,b), respectively.

3. GREEN'S FUNCTION OF THE LIOUVILLE EQUATION

The Green's function of the Liouville equation has been studied by Andrews[14] and by Balescu[15]. We consider the Liouvllle equation

$$-i \frac{\partial}{\partial t} \eta(\vec{p},\vec{q},t) = K_1\ \eta(\vec{p},\vec{q},t) \tag{16}$$

with

$$H_1 = \sum_{j=1}^{3} \frac{p_j^2}{2m} + V(q_1,q_2,q_3,t);\quad \vec{q} \equiv (q_1,q_2,q_3),\quad \vec{p} \equiv (p_1,p_2,p_3)\ .$$

The function $V(\vec{q},t)$ represents an external potential. The Green's function of eq. (16) we denote by $\Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')$; it is a solution of eq. (16) in the variables $(\vec{p},\vec{q},t)$ and takes for $t = t'$ the value $\delta(\vec{p}-\vec{p}')\delta(\vec{q}-\vec{q}')$. Consequently, we have

$$-i \frac{\partial}{\partial t} \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t') = i \sum_{j=1}^{3} \left(\frac{p_j}{m}\frac{\partial}{\partial q_j} - \frac{\partial V}{\partial q_j}\frac{\partial}{\partial p_j}\right) \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')$$

with

$$\Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t) = \delta(\vec{p}-\vec{p}')\delta(\vec{q}-\vec{q}')\ .$$

It follows that $\Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')$ has the following properties

i) $$-i \frac{\partial}{\partial t'} \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t') = -i \sum_{j=1}^{3} \left(\frac{p_j'}{m}\frac{\partial}{\partial q_j'} - \frac{\partial V}{\partial q_j'}\frac{\partial}{\partial p_j'}\right) \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')$$

ii) $$\eta(\vec{p},\vec{q},t) = \int d\vec{p}'\ d\vec{q}'\ \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')\eta(\vec{p}',\vec{q}',t') \tag{17}$$

iii) $<\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t), \quad \Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p}\,',\vec{q}\,',t')> = \Gamma(\vec{p},\vec{q},t;\vec{p}\,',\vec{q}\,',t')$. (18)

4. FIELD OPERATORS IN PHASE SPACE

In what follows we consider classical bosons only. Hence we will work with the Symmetric algebra $\mathbb{F}^{-}(\mathbb{H})$, $\mathbb{H} \equiv L_2(\Omega)$. The same development may be applied to classical fermions, by only using the Grassmann algebra $\mathbb{F}^{+}(\mathbb{H})$ and its correspondent operators instead of the Symmetric algebra.

The one particle Hilbert space is

$$\mathbb{H} \equiv L_2(\Omega) \equiv \{\eta(\vec{p}\,',\vec{q}\,',t)\}$$

with the inner product given by eq. (2). We will work with the classical Heisenberg Picture[16].

The Symmetric algebra is

$$\mathbb{F}^{-}(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \hat{S}\, \mathbb{H}_n$$

where $\mathbb{H}_n$ is the n-fold tensor product of $\mathbb{H}$, i.e.,

$$\mathbb{H}_n \equiv \{\eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n\}$$

with $\eta_i \in \mathbb{H}$.

We define the field operator $\hat{\psi}^{-}(\vec{p},\vec{q},t)$ and its hermitean conjugate $\hat{\psi}^{+}(\vec{p},\vec{q},t)$ by uçing the Green's function $\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)$ and the operators (13a, 13b), that is,

$$\hat{\psi}^{-}(\vec{p},\vec{q},t) = a^{-}\{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t) \tag{19a}$$

$$\hat{\psi}^{+}(\vec{p},\vec{q},t) = a^{+}\{\Gamma(\vec{\pi},\vec{\xi},\tau;p,q,t) \tag{19b}$$

where $(\vec{\pi},\vec{\xi},\tau)$ are the independent variables of the function $\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t) \in \mathbb{H}$ and $(\vec{p},\vec{q},t)$ are considered parameters; hence the field operators depend on $(\vec{p},\vec{q},t)$ as independent variables. The operator $\hat{\psi}^{-}(\vec{p},\vec{q},t)$ $(\hat{\psi}^{+}(\vec{p},\vec{q},t))$ is the annihilatlon (creation) operators of one particle in the state $\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)$.

Frorn relations (15a, b, c) and property (18) of Γ, we have

$$[\hat{\psi}^-(\vec{p},\vec{q},t),\ \hat{\psi}^-(\vec{p}',\vec{q}',t')]_- = [\hat{\psi}^+(\vec{p},\vec{q},t)\ \hat{\psi}^+(\vec{p}',\vec{q}',t')]_- = 0 \tag{20}$$

$$[\hat{\psi}^-(\vec{p},\vec{q},t),\ \hat{\psi}^+(\vec{p}',\vec{q}',t')]_- = \langle\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t),\ \Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p}',\vec{q}',t')\rangle$$

$$= \Gamma(\vec{p},\vec{q},t;\vec{p}',\vec{q}',t')\ . \tag{21}$$

And for $t = t'$

$$[\hat{\psi}^-(\vec{p},\vec{q},t);\ \ \hat{\psi}^+(\vec{p}',\vec{q}',t)]_- = \delta(\vec{p}-\vec{p}')\delta(\vec{q}-\vec{q}')\ . \tag{22}$$

Relations (20) and (22) are corresponding to those imposed by Schönberg[3] on the field operators in Classical Schrödinger Picture. In the present formulation they follow from the definition of $a^{\pm}(\eta)$ together with the use of Symmetric algebra structure of $\mathbb{F}^-(\mathbb{H})$ and properties of the Green's function Γ.

We obtain from the definitions (19a, b), (8) and (7) that

$$i\frac{\partial}{\partial t}\hat{\psi}^-(\vec{p},\vec{q},t) = i\frac{\partial}{\partial t}a^-\{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}$$

$$= a^-\{-i\frac{\partial\Gamma}{\partial t}(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}$$

$$= a^-\{i\sum_{j=1}^{3}\left(\frac{p_j}{m}\frac{\partial}{\partial q_j}-\frac{\partial V}{\partial q_j}\frac{\partial}{\partial p_j}\right)\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}$$

$$= -K_1\ \ \hat{\psi}^-(\vec{p},\vec{q},t) \tag{23}$$

and, correspondently, for the operator $\hat{\psi}^+(\vec{p},\vec{q},t)$

$$i\frac{\partial}{\partial t}\hat{\psi}^+(\vec{p},\vec{q},t) = a^+\ \{i\frac{\partial}{\partial t}\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}$$

$$= -\ K_1\ \hat{\psi}^+(\vec{p},\vec{q},t) \tag{24}$$

that is, the Liouville equatlon is verified for both

$$\hat{\psi}^-(\vec{p},\vec{q},t)\quad\text{and}\quad\bar{\psi}^+(\vec{p},\vec{q},t)$$

5. THE VARIATIONAL PRINCIPLE AND THE LIOUVILLE OPERATOR

It is shown in ref. 3 that in each sub-space $\hat{S}\,\mathbb{H}_n \subset \mathbb{F}(\mathbb{H})$ one obtains a Liouville equation for n-particle probability amplitude Θ_n. We now proceed to show how to use our formulation to express this result using tensor algebra. Equations (23) and (24) can be derived from the variational principle

$$6 \int L \, d\vec{p} d\vec{q} \, dt = 0$$

with the Lagrangian denslty L given by

$$L = i \, \bar{\psi}^{+}(\vec{p},\vec{q},t) \frac{\partial}{\partial t} \hat{\psi}^{-}(p,q,t) + i \sum_{j=1}^{3} \hat{\psi}^{+}(\vec{p},\vec{q},t) \frac{p_j}{m} \frac{\partial}{\partial q_j} \hat{\psi}^{-}(\vec{p},\vec{q},t)$$

$$- i \sum_{j=1}^{3} \hat{\psi}^{+}(\vec{p},\vec{q},t) \frac{\partial V}{\partial q_j} \frac{\partial}{\partial p_j} \hat{\psi}^{-}(\vec{p},\vec{q},t) \,. \tag{25}$$

It follows that the conjugated momenta are

$$\hat{\Omega}^{-}(\vec{p},\vec{q},t) = \frac{\partial L}{\partial \left(\frac{\partial \hat{\psi}^{-}}{\partial t}\right)} = i \, \hat{\psi}^{+}(\vec{p},\vec{q},t) \tag{26a}$$

$$\hat{\Omega}^{+}(\vec{p},\vec{q},t) = \frac{\partial L}{\partial \left(\frac{\partial \hat{\psi}^{+}}{\partial t}\right)} = 0 \tag{26b}$$

and we can define a Hamlltonian density

$$\hat{K} = \hat{\Omega}^{-}(\vec{p},\vec{q},t) \frac{\partial}{\partial t} \hat{\psi}^{-}(\vec{p},\vec{q},t) + \hat{\Omega}^{+}(\vec{p},\vec{q},t) \frac{\partial}{\partial t} \hat{\psi}^{+}(\vec{p},\vec{q},t) - L$$

or, wlth eqs. (25) and (26a, b)

$$\hat{K} = \sum_{j=1}^{3} \frac{p_j}{m} \hat{\Omega}^{-} \frac{\partial}{\partial q_j} \hat{\psi}^{-} - \sum_{j=1}^{3} \frac{\partial}{\partial q_j} \hat{\Omega}^{-} \frac{\partial}{\partial p_j} \hat{\psi}^{-} . \tag{27}$$

Hence, the Hamiltonlan operator acting on the Symmetric algebra

$$\mathbb{F}^{-}(\mathbb{H}) = \bigoplus_{n=0}^{m} \hat{S}\,\mathbb{H}_n$$

wlll be

$$\hat{K}(t) = i \int \left(\sum_{j=1}^{3} \left[\frac{p_j}{m} \hat{\Omega}^- \frac{\partial}{\partial q_j} \hat{\psi}^- - \frac{\partial V}{\partial q_j} \hat{\Omega}^- \frac{\partial}{\partial p_j} \psi^- \right] \right) d\vec{p}\, d\vec{q}. \quad (28)$$

This operator (28) corresponds to the "second quantized" Liouville operator K introduced by Schönberg[3]. Thus, the result of Ref. 3 referred above, in the context of the present formulation means that the Hamiltonian operator (28) in each sub-space $\hat{S}\, \mathbb{H}_n$ is reduced to the usual Liouville operator for n-particle systems (see eq. (1)), i.e.,

$$\{H_n, \ \}_n = i \sum_{\ell=1}^{n} \sum_{j=1}^{3} \left(\frac{p_{\ell j}}{m} \frac{\partial}{\partial q_{\ell j}} - \frac{\partial V}{\partial q_{\ell j}} \frac{\partial}{\partial p_{\ell j}} \right) = \sum_{\ell=1}^{n} K_1^{\ell} \equiv K_n .$$

In fact from the definitions (19a,b) of the operators $\hat{\psi}^{\pm}(\vec{p},\vec{q},t)$ we have, for $\phi \equiv \phi_n \in \hat{S}\, \mathbb{H}_n$, i.e.,

$$\phi = \sum_{\sigma \in S_n} \eta_{\sigma 1}(\vec{\pi}_1,\vec{\xi}_1,\tau_1) \otimes \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \quad (29)$$

that

$$\hat{\psi}^-(\vec{p},\vec{q},t)\phi = \sqrt{n} \sum_{\sigma \in S_n} \eta_{\sigma 1}(\vec{p},\vec{q},t) \cdot \eta_{\sigma 2}(\pi_2,\xi_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \quad (30)$$

and

$$\hat{\psi}^+(p,q,t)\{\eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n)\} = \sqrt{n} \sum_{\sigma \in S_n} \Gamma(\vec{\pi}_1,\vec{\xi}_1,\tau_1;\vec{p},\vec{q},t) \otimes$$

$$\otimes\, \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \quad . \quad (31)$$

where we have used the definitions (7), (8) and the property (17).

Furthermore,

$$\frac{\partial}{\partial q_j} \hat{\psi}^-(\vec{p},\vec{q},t)\phi = \frac{\partial}{\partial q_j} a^-\{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}\phi = \quad (32)$$

$$= \sqrt{n} \sum_{\sigma \in S_n} \frac{\partial}{\partial q_j} \eta_{\sigma 1}(\vec{p},\vec{q},t) \cdot \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n)$$

and

$$\frac{\partial}{\partial p_j} \hat{\psi}^-(\vec{p},\vec{q},t)\phi = \frac{\partial}{\partial p_j} a^-\{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)\}\phi =$$

$$= \sqrt{n} \sum_{\sigma \in S_n} \frac{\partial}{\partial p_j} \eta_{\sigma 1}(\vec{p},\vec{q},t) \, . \, \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \, . \tag{33}$$

Thus with the expression of $\hat{K}(t)$ given by eq. (28) we obtain from eqs. (30-33)

$$\hat{K}(t)\phi = n \sum_{\sigma \in S_n} \Bigg[i \int \sum_{j=1}^{3} \left(\frac{p_j}{m} \Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t) \frac{\partial}{\partial q_j} \eta_{\sigma 1}(\vec{p},\vec{q},t) \right) d\vec{p}\, d\vec{q}$$

$$- i \int \sum_{j=1}^{3} \left(\frac{\partial V}{\partial q_j} \Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t) \frac{\partial}{\partial p_j} \eta_{\sigma 1}(\vec{p},\vec{q},t) \right) d\vec{p}\, d\vec{q} \Bigg] \otimes$$

$$\otimes \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \, . \tag{34}$$

If we denote in eq. (34) the expression within braces by $\gamma_{\sigma,t}(\vec{\pi},\vec{\xi},\tau)$ we have

$$\hat{K}(t)\phi = n \sum_{\sigma \in S_n} \gamma_{\sigma,t}(\vec{\pi},\vec{\xi},\tau) \otimes \eta_{\sigma 2}(\vec{\pi}_2,\vec{\xi}_2,\tau_2) \otimes \ldots \otimes \eta_{\sigma n}(\vec{\pi}_n,\vec{\xi}_n,\tau_n) \, .$$

Evaluating γ $(\vec{\pi},\vec{\xi},\tau)$ with the result that

$$\Gamma(\vec{\pi},\vec{\xi},t;\vec{p},\vec{q},t) = \delta(\vec{\pi}-\vec{p}) \; \delta(\vec{\xi}-\vec{q})$$

we obtain

$$\gamma_{\sigma,t}(\vec{\pi},\vec{\xi},t) = i \sum_{j=1}^{3} \left(\frac{\pi_j}{m} \frac{\partial}{\partial \xi_j} - \frac{\partial V}{\partial \xi_j} \frac{\partial}{\partial \pi_j} \right) \eta_{\sigma 1}(\vec{\pi},\vec{\xi},t)$$

$$= K_1 \, \eta_{\sigma 1}(\vec{\pi},\vec{\xi},t) \, .$$

Then, we can write from eq. (34)

$$\hat{K}(t)\phi = n \sum_{\sigma \in S_n} (K_1 \eta_{\sigma 1}) \otimes \eta_{\sigma 2} \otimes \ldots \otimes \eta_{\sigma n}$$

$$= \sum_{\sigma \in S_n} (K_1 \eta_{\sigma 1}) \otimes \eta_{\sigma 2} \otimes \ldots \otimes \eta_{\sigma n} +$$

$$+ \sum_{\sigma \in S_n} \sum_{\ell=2}^{n} \eta_{\sigma \ell} \otimes \eta_{\sigma 2} \otimes \ldots \otimes \eta_{\sigma(\ell-1)} \otimes (K_1 \eta_{\sigma 1})$$

$$\otimes \eta_{\sigma(\ell+1)} \otimes \ldots \otimes \eta_{\sigma n}$$

$$= \sum_{\sigma \in S_n} \sum_{\ell=1}^{n} \eta_{\sigma 1} \otimes \eta_{\sigma 2} \otimes \ldots \otimes \eta_{\sigma(\ell-1)} \otimes (K_1 \eta_{\sigma \ell}) \otimes \eta_{\sigma(\ell+1)} \otimes$$

$$\otimes \ldots \otimes \eta_{\sigma n} \tag{35}$$

i.e., in the sub-space $\hat{S}\mathbb{H}_n$ we have, for $\hat{K}(t)$, the expression

$$i \sum_{\ell=1}^{n} \sum_{j=1}^{3} \left(\frac{\pi_{\ell j}}{m} \frac{\partial}{\partial \xi_{\ell j}} - \frac{\partial V}{\partial \xi_{\ell j}} \frac{\partial}{\partial \pi_{\ell j}} \right) = K_n \ .$$

Concluding this section we observe that the operators $\bar{\bar{R}}^{-}(\vec{p},\vec{q},t)$ and $\hat{\psi}^{-}(\vec{p},\vec{q},t)$ constitute a canonical (Hamiltonian) formalism for the Liouville field or super-classical field $\psi(\vec{p},\vec{q},t)$ as it is called by Schönberg[17]. In fact, we have from eqs. (20,22,26a)

$$[\hat{\Omega}^{-}(\vec{p},\vec{q},t),\ \hat{\Omega}^{-}(\vec{p}\,',\vec{q}\,',t')]_{-} = [\hat{\psi}^{-}(\vec{p},\vec{q},t),\ \hat{\psi}^{-}(\vec{p}\,',\vec{q}\,',t')]_{-} = 0 \tag{36}$$

$$[\hat{\Omega}^{-}(\vec{p},\vec{q},t),\ \hat{\psi}^{-}(\vec{p}\,',\vec{q}\,',t)]_{-} = i\ \delta(\vec{p}-\vec{p}\,')\ \delta(\vec{q}-\vec{q}\,') \tag{37}$$

and it is easily verified, as a consequence of eq. (35), that

$$\frac{\partial \hat{\psi}^{-}(\vec{p},\vec{q},t)}{\partial t} = -\ i[\hat{K}(t),\ \hat{\psi}^{-}(\vec{p},\vec{q},t)]_{-} \tag{38}$$

$$\frac{\partial \hat{\Omega}^{-}(\vec{p},\vec{q},t)}{\partial t} = -\ i[\hat{K}(t),\ \hat{\Omega}^{-}(\vec{p},\vec{q},t)]_{-} \tag{39}$$

where the operators are acting on Symmetric algebra $\mathbb{F}^{-}(\mathbb{H})$. It follows that equations (23, 24) are obtained from relations (38, 39), using eqs.

(26a, 28), relations (36, 37) and considering the identity

$$[\hat{F}\hat{G},\hat{B}]_- = [\hat{F},\hat{B}]_- \hat{G} + \hat{F}[\hat{G},\hat{B}]_-$$

6. CQNCLUSIONS

Considering tensor algebra over Hilbert space, we have presented an alternative procedure to shcwn the possibility of expressing classical statistical results using field operators characterized by cornmutation and anticornrnutation rules as in quantum field theory. We have considered explicitly the case of classical bosons and a Liouville operators given by

$$K_n = i \sum_{\ell=1}^{n} \sum_{j=1}^{3} \left(\frac{p_j}{m} \frac{\partial}{\partial q_{\ell j}} - \frac{\partial V}{\partial q_{\ell j}} \frac{\partial}{\partial p_{\ell j}}\right)$$

where $V(q)$ is an externa1 potential. The Symmetric algebra over Hilbert space $\mathbb{H} = L_2(\Omega)$, (Ω = phase space of a single particle), $\mathbb{F}^-(\mathbb{H})$, is the natural aigebraic structure to study this case. We have defined field operators acting on $\mathbb{F}^-(\mathbb{H})$ by

$$\hat{\psi}^-(\vec{p},\vec{q},t) = a^- \{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)$$

$$\hat{\psi}^+(\vec{p},\vec{q},t) = a^+ \{\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)$$

that is, $\hat{\psi}^-(\vec{p},\vec{q},t)$ $(\hat{\psi}^+(\vec{p},\vec{q},t))$ is the annihilation (creation) operator for one particle in the state $\Gamma(\vec{\pi},\vec{\xi},\tau;\vec{p},\vec{q},t)$ which is a solution of the Liouville equation such that for $\tau = t$ it becomes $\delta(\vec{\pi}-\vec{p})\delta(\vec{\xi}-\vec{q})$. With these cperators we have determined the "second quantized" Liouville operator. Hence a new derivation of Schönberg's result about the equivalente between such a field theory in classical phase space and the usual formalism of classical statistical mechanics has been obtained exploring algebraic aspects of the procedure here presented. The procedure also shows by construction that the classical and quantum theory admit similar algebraic structures, a result which is important from the standpoint of the foundations of Physics[18,19]. In fact, in last two decades some physicists and mathernatlcians have given attention[20-29] to the problem of expressing the quantum mechanical mean values as classical sverages over phase-space distribution functions; in the present paper we have raised the problem of expressing statistical classical results with the mathematical structure of the quantum theory for many-particle systems (field operators, tensor algebra, commutators, anti-ccmmutators, etc.). For this purpose the second quantized formulation of statistical classical mechanics formulated by Schönberg and Koopman has been re-visited by exploring algebraic aspects of the second quantization method; this description of statistical mechanics in terms of Hilbert space can be considered a generalization of the framework introduced by Prigogine and coworkers[30] once that i) with the use of the Hilbert space direct sum $\mathbb{F}(\mathbb{H})$, we can introduce in a natural way the concept of *grand ensemble*; ii) the use of symmetric or antisynnnetrical Hilbert space direct sum $\mathbb{F}^{\mp}(\mathbb{H})$ allows us to introduce the indistinguishability of particles within the context of classical mechanics.

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Resumo

Mostra-se usando álgebras tensoriais, a saber, a álgebra Simétrica e a álgebra de Grassmann, que é possível introduzir operadores de campos associados à equação de Liouville da mecânica estatística clássica; esses operadores são caracterizados por relações de comutação (no caso da álgebra Simétrica) e por relações de anti-comutação (no caso da álgebra de Grassmann). Com o método apresentado mostra-se por construção que sistemas clássicos de muitas partículas admitem uma estrutura algébrica similar àquela da teoria quântica de campos. Considera-se explicitamente o caso de sistemas de n-partículas interagindo com um potencial externo. O resultado de Schönberg sobre a equivalência entre sua teoria de campos no espaço de fase clássico e a mecânica estatística clássica usual é obtido de um ponto de vista novo, explorando aspectos algébricos do método aqui apresentado.