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Dirac Particle on S^2

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Abstract The problem of a Dirac particle in stationary motion on S^2 - a two dimensional sphere embedded in Euclidean space E^3 - is discussed. It provides a particularly simple case of an exactly solvable constrained Dirac particle whose properties are here studied, with emphasis on its magnetic moment.

1. INTRODUCTION

The quantum mechanical problem of a nonrelativistic particle constrained to move on a N-sphere S^N is well-known to admit exact solutions. For the corresponding problem of a constrained Dirac particle on S^N, solutions, to our knowledge are not known for a general value of N, although the particular case of $N=2$ (the two-dimensional sphere S^2, embedded in Euclidean space E^3), as we shall see, admits exact solutions. This is a direct consequence of the embedding property of S^2 in E^3, which allow us to derive stationary solutions on S^2 as a restriction ($r=R$) imposed on the free motion parametrized by spherical coordinates (r,θ,ϕ) in E^3, R being the radius of the 2-sphere.

After discussing the energy spectrum of a Dirac particle on S^2 in section 2, we carry out, in section 3, perturbative calculations of the magnetic moment of the system. Section 4 is devoted to some final remarks on the problem.

2. THE ENERGY SPECTRUM OF THE FREE DIRAC PARTICLE ON S^2

Our starting point are the stationary solutions of a free Dirac particle in E^3, in spherical coordinates. Since this is a rather well-known subject we go directly to the main results to be used in the sequel.

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$$\mu_{\pm} = \pm \frac{Q}{2M}\left[1 + k - \frac{2k}{1 + \frac{(E+M)^2R^2}{(k+1)^2}}\right]\frac{k}{k \pm \frac{1}{2}} \tag{3.13}$$

Finally, it may be of interest to point out that for the magnetic moment calculated via the iterated Dirac equation, as the expectation value of

$$V = \frac{Q}{2E}(L_3 + \Sigma_3) \tag{3.14}$$

we obtained

$$\mu_{\pm} = \pm \frac{Q}{2E}\, k \begin{cases} 1 - \dfrac{R^2\, a^2}{k + \frac{1}{2}} & \text{if } k > 0 \\ 1 + \dfrac{R^2\, b^2}{k - \frac{1}{2}} & \text{if } k < 0 \end{cases} \tag{3.15}$$

where the expresslons for a and b are given by eqs. (2.8_3).

4. FINAL REMARKS

The solutions obtained are remarkably simple and are given in exact form for any value of k (or j). The energy spectrum as given by formula (2.8_3) is a discrete one and corresponds to parabolic Regge trajectories ($E^2 \propto j^2$). It also presents a double degeneracy in k, which may be interpreted as a parity doubling.

We remark that this degeneracy can be easily broken up - and exact solutions still obtained - if we introduce in the Dirac Hamiltonian an additional term of the form $F\,(K,J^2)$, F denoting an arbitrary function of the commuting angular operators K and J^2. Finally, a few words about the magnetic moment calculations are in order. In should be noted that magnetic moments calculated via (3.1) or via (3.14) give, in general, different results, the differences being of the order of 25%.

An exceptional case is that of the ground state ($E=M$) with $k=-1$. In this case, eq. (3.12) gives $\mu_{-1} = 0$ whereas, from eq. (3.15) the result $Q/2M$ is obtained. This may be easily understood having in mind that in this state $b=0$ and remarking that in the first case the magnetic moment is given by the product a b whereas in the second, it is given by a sum of two terms, one proportional to a^2, the other to b^2

Let us add a few words about the future outlook. This work can proceed in at least two directions: in the first, a generalization to N--spheres is envisaged. In the second, instead of spheres, the Dirac particle is constrained to move on a surface of different symmetry as, for instance, a circular cylinder of given length, a case also amenable to an exact treatment. We hope to return to these interesting topics at a later date.

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REFERENCES AND NOTES

1. For a recent discussion, see P.Leal Ferreira, Rev. Brasil. Fís., Special volume, p. 183, July (1984).
2. For the Dirac equation and related quantities we follow the notation and conventions of J.J.Sakurai, *Advanced Quantum Mechanics*, Addison--Wesley Publ. Co., Reading MA (1967).
3. This definitions is consistent with the secular equation resulting from the double degeneracy since the off-diagonal elements of the perturbation vanish in first order.
4. This fact was recognized long ago by H.Margenau, Phys. Rev. *57*, 383 (1940).

Resumo

Discute-se o problema de uma partícula de Dirac em movimento estacionário sobre S^2-uma esfera bidimensional imersa em espaço euclidiano E^3. Tem-se aqui um caso particularmente simples e exatamente solúvel de uma partícula de Dirac vinculada, cujas propriedades são estudadas com ênfase no momento magnético do sistema.