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Multipole Stationary Soliton Solutions to the Einstein Equations

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Abstract The generation of multipole stationary solutions to the vacuum Einstein equations using the Belinsky-Zakharov inverse scattering method with a diagonal seed solution is studied.

One of the most powerful methods used to generate new stationary axially symmetric solutions to the vacuum Einstein equations from known solutions (seed solutions) is the inverse scattering method[1,2] (ISM). An example of new solution obtained using the ISM is the N Kerr-NUT solution . Also this solution can be obtained using the closely related method of Bäcklund Transformtions (BT)[4,5].

The purpose of this note is to use the Belinsky-Zakharov version of the ISM to generate a class of stationary solutions. The seed solution that we shall use is the particular class of Weyl solutions[6] that can be expanded in zonal harmonics. A similar class of solutions was studied by Harrison[7] and by Hoenselaers, Kinnersley and Xanthoupoulos[8] (HKX) using BT. The relation between the different new solution generating algorithms (NSGA) already mentioned is studied in Refs. 9-11. In particular we have that the double Harrison BT is equivalent to the Belinsky-Zakharov ISM with a scattering matrix with two simple poles modulo a coordinate transformation. Also the solutions obtained using HKX Bäcklund transformations and Belinsky-Zakharov ISM are closely related[9].

The Belinsky-Zakharov ISM presents severa1 advantages over the NSGA already mentioned: a) it is simpler to apply, b) gives the solution directly in terms of the metric potentials and c) it can be applied to a broader class of evolution equations[12]. The solution of the vacuum Einsteln equation using this method is based on the fact that for the metric

$$ds^2 = e^{\sigma}(dr^2 + dz^2) + \gamma_{ab}\, dx^a\, dx^b \tag{1}$$

with γ_{ab} and σ functions of z and r only, $(\phi,t) \equiv (x^3,x')$ and a and b running from 3 to 4, they can be written as

$$(r\,\gamma_r\,\gamma^{-1})_r + (r\,\gamma_z\,\gamma^{-1})_z = 0 \tag{2}$$

$$\det\gamma = -\,r^2 \tag{3}$$

$$\sigma_r = -\,(r)^{-1} + (4r)^{-1}\,\mathrm{Tr}(U^2-V^2) \tag{4}$$

$$\sigma_z = (2r)^{-1}\,\mathrm{Tr}(UV) \tag{5}$$

$$U \equiv r\,\gamma_r\,\gamma^{-1}\ ,\quad V \equiv r\,\gamma_z\,\gamma^{-1} \tag{6}$$

where γ is the 2×2 matrix associated to γ_{ab} and the subscripts r and z denote partial differentiation. The condition of integrability of σ, i.e., $\sigma_{rz} = \sigma_{zr}$, is exactly eq. (2), thus any solution to eq. (2) will give us a σ that can be obtained as a simple quadrature of eqs. (4) and (5).

Soliton solutions to eq.(2) are obtained by solving the *Schrödinger equations*

$$D_r\,\psi_0 = \frac{r\,U_0 + \lambda\,V_0}{\lambda^2 + r^2}\,\psi_0 \tag{7}$$

$$D_z\,\psi_0 = \frac{r\,V_0 - \lambda\,U_0}{\lambda^2 + r^2}\,\psi_0 \tag{8}$$

$$\psi_0\Big|_{\lambda=0} = \gamma_0 \tag{9}$$

$$D_r \equiv \partial_r + \frac{2\lambda r}{\lambda^2 + r^2}\,\partial_\lambda \tag{10}$$

$$D_z \equiv \partial_z - \frac{2\lambda^2}{\lambda^2 + r^2}\,\partial_\lambda \tag{11}$$

for the wave function ψ_0. This wave function is a 2x2 complex matrix function of z, r and the spectral parameter λ. U_0 and V_0 are obtained replacing γ in eq.(6) by a known solution to eq.(2), γ_0. The solution γ_0 is called the seed or background solution. The knowledge of ψ_0 allows us to find the new solution γ to eq. (2), given by[1]

$$\gamma_{ab} = (\gamma_0)_{ab} - \sum_{k,\ell} \frac{N_a^{(\ell)}\,(\Gamma^{-1})_{\ell k}\,N_b^{(k)}}{\mu_k\,\mu_\ell} \tag{12}$$

where

$$\Gamma_{k\ell} = \frac{m_a^{(k)} (\gamma_0)_{ab}\, m_b^{(\ell)}}{r^2 + \mu_k \mu_\ell} \qquad (13)$$

$$N_a^{(k)} = m_b^{(k)} (\gamma_0)_{ba} \qquad (14)$$

$$m_a^{(k)} = m_{0b}^{(k)} M_{ba}^{(k)} \qquad (15)$$

$$M^{(k)} = \psi_0^{-1}\Big|_{\lambda=\mu_k} \qquad (16)$$

$$\mu_k = \alpha_k - z \pm \left[(\alpha_k - z)^2 + r^2\right]^{1/2} \qquad (17)$$

where the summation convention on the indices a and b has been adopted. The indices k and R run from 1 to N, N being the number of solitons. $m_{ob}^{(k)}$ and α_k are sets of arbitrary constants.

In general (12) does not satisfy eq. (3). To remedy this problem we can defined the matrix[1]

$$\gamma^{Ph} = r\gamma/(-\det\gamma)^{1/2} \qquad (18)$$

that is also a solutlon of eq.(2) and satisfies eq.(3). For the solution (12), the evaluation of det y, as well as the integration of eq.(4) and eq. (5) can be made explicitely[1].

We shall specialize the class of seed solutions to the classof dlagonal matrlces γ_0

$$\gamma_0 = \mathrm{diag}\left[r^2 \exp(-\Phi),\ -\exp\Phi\right] \qquad (19)$$

i.e., to tho usual Weyl solution[6]. In this case eq. (2) reduces to the Laplace eqution

$$\Phi_{rr} + \Phi_r/r + \Phi_{zz} = 0 \qquad (20)$$

Since γ_0 is diagonal,we can assume that ψ_0 is also a diagonal matrix. From eqs. (7)-(9), (15), (16) and $(\psi_0)_{34} = (\psi_0)_{43} = 0$ we find that there is no loss of generallty in setting

$$\psi_0 = \mathrm{diag}\left[(r^2 - \lambda^2 - 2\lambda z)\exp(-F),\ -\exp F\right] \qquad (21)$$

From eqs.(21), (10) and (11) weget that the systemof equations (7) and (8) together with the boundary condition (9) is equivalent to the scalar equations

$$(r\partial_r - \lambda\partial_z + 2\lambda\partial_\lambda)F = r\,\Phi_{,r} \tag{22}$$

$$(r\partial_z + \lambda\partial_r) = r\,\Phi_{,z} \tag{23}$$

and

$$F\Big|_{\lambda=0} = \Phi \tag{24}$$

Thus, the integration of eq.(2) for a diagonal seed solutions reduces to the Integration of eqs. (22)-(24).

Particularly interesting solutions to eq.(20) are those that can be expanded In zonal harmonics, i.e.,

$$\Phi = \sum_{n=0}^{\infty} (A_n\,\Phi_n^{ext} + B_n\,\Phi_n^{int}) \tag{25}$$

where A_n and B_n are arbitrary constants and

$$\Phi_n^{ext} = R^{-(n+1)}\,P_n(\cos\theta) \tag{26}$$

$$\Phi_n^{int} = R^n\,P_n(\cos\theta) \tag{27}$$

R and θ are related to r and z by $r = R\sin\theta$ and $z = R\cos\theta$. The metric associated to $A_0\,\Phi_0^{ext}$ is the Chazy-Curzon metric[13,14]. Diagonal metrics with higher multipole moments are studied in Ref. 15. In general, metrics associated with $\Sigma\, A_n\,\Phi_n^{ext}$ are asymptotically flat, since

$$\Phi_n^{ext}\Big|_{R\to\infty} = 0$$

Let us denote by F_n^{ext} and F_n^{int} the functions F, solutions to eqs. (22)-(24) with Φ replaced by Φ_n^{ext} and Φ_n^{int} respectively. Due to the linearity of eqs. (20), (22)-(24) the function F associated to eq.(25) is

$$F = \sum_{n=0}^{\infty} A_n\,F_n^{ext} + B_n\,F_n^{int} \tag{28}$$

One can easily verify that the solution F assosiated to $\Phi_0^{ext} = 1/R$ is

$$F_0^{ext} = \frac{z + R}{(z + R + \lambda)R} \tag{29}$$

To find the rest of the F_n we notice that eqs. (20) and (22) - (24) are invariant under a translation in the z variable and that the zonal harmonics genarating function is $\Phi_0^{ext}(z-\zeta,r)$, where ζ is a parameter. Thus

$$F_n^{ext}(z,r) = \frac{(-1)^n}{n!} (\partial_z)^n F_0^{ext}(z,r) \tag{30}$$

The stationary even number soliton solution in eqs.(12)-(18) obtained from the seed $\Sigma A_n \Phi_n^{ext}$ is also asymptotically flat since in this case the ISM preserves the seed solution asymptotic properties[9].

The functions F_n^{int} can be computed in a similar way, i.e., by making a Taylor expansion in the variable $1/\zeta$. The fact that Φ_n^{int} is an homogeneous function of degree n in the variables z and r can be used to find F_n^{ext} In an explicit way. The function Φ_n^{int} can be written as

$$\Phi_n^{int}(z,r) = \sum_{k=0}^{[n/2]} \frac{n!(-1)^k z^{n-2k} r^{2k}}{(n-2k)!\,(k!)^2\, 2^{2k}} \tag{31}$$

The Legendre pollnomlals alone can be also written in a similar way[16].

From eqs. (31) and (22)-(24) we find[17]

$$\begin{aligned} F_n^{int}(z,r) = & \sum_{\ell=0}^{n} \frac{(-1)^\ell \binom{n}{\ell}}{2^\ell \lambda^\ell} (z + \tfrac{1}{2}\lambda)^{n-\ell} r^{2\ell} \\ & - \sum_{\ell=1}^{n} \sum_{k=0}^{[(n-\ell)/2]} \frac{(-1)^{k+\ell} n!\, r^{2(k+\ell)} z^{n-\ell-2k}}{(n-\ell-2k)!\, k!(k+\ell)!\, 2^{2k+\ell} \lambda^\ell} \end{aligned} \tag{32}$$

The metrica assoclated to Φ_n^{int} are not asymptotically flat. They describe the interior of a hollow body. The two-sollton solution associated to $B_1 \Phi_1^{int} + B_2 \Phi_2^{int}$ is a particular case of the one studied in Ref. 18.

Finally we want to indicate that the two-soliton solution associated to

$$\Phi = A_0 \Phi_0^{ext}(z,r) + A_0' \Phi_0^{ext}(z-\zeta,r) \tag{33}$$

i.e., to a two center Chazy-Curzon metric, is similar to the bipolar metric obtained by the application of a double rank-zero HKX transformation to eq. (33) recently studied by Dietz and Hoenselaers[19].

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Resumo

A geração de soluções estacionãrias multipolares das equações de Einstein para o vácuo é estudada usando o método de espalharnento inverso de Belinsky e Zakharov com uma solução somente diagonal.