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Heat Flux Equation Applied to Thermal Conduction Problem

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Abstract In this paper we consider the distribution of the temperature function for the problem of a cylinder of radius a surrounded by an infinite medlum of another material. The heat flux Q at the surface $r=a$ is kept constant. The differential equations of heat flux are solved through the use of finite Hankel and Weber transforms, whose fundamental properties are given in the Appendix.

1. INTRODUCTION

Unsteady heat flux calculations arise in conection with a wide variety of technical processes. In some cases is required the temperature distribution troughout a body at a given time. In other cases the ternperature variation with the time at a given point in a body may be required as for example, in food processing, ceramlc manufacture, etc. Problems on conduction of heat in composite circular or hollow circular cilinders rectangular blocks, square plates, etc, are treated by Laplace transforrnation by Carslaw and Jaeger[1].

Here we consider a infinite cylinder of a material of conductivlty K_1, diffusivity k_1, densíty ρ_1 and speclfic heat c_1, which is surrounded by an infinite medium of a material of conductivity K_2, diffusivity k_2, and specific heat c_2. The cylinder radius is a and the heat flux Q at the surface $r=a$ is a constant.

In previous papers, Battig and Kalla[2], Battig, Kalla and Luccioni[3] have considered problems of thermal conduction using the Weber transform. Now, we apply the differential equations of heat flux which were derived by Battlg and Kalla[4]. The finite Hankel[5] and Weber[6] transforms are used to obtain the solutions of the differential equations.

2. THE SOLUTION OF THE PROBLEM

Since the heat flux in the boundary $r=a$ is given, the corresponding equations for the cylinder and the surrounding medium are used. These equations are[4]

$$\frac{\partial^2 f_1}{\partial r^2} + \frac{1}{r}\frac{\partial f_1}{\partial r} - \frac{1}{r^2} f_1 = \frac{1}{k_1}\frac{\partial f_1}{\partial t} \qquad 0 < r \leq a \tag{1}$$

$$\frac{\partial^2 f_2}{\partial r^2} + \frac{1}{r}\frac{\partial f_2}{\partial r} - \frac{1}{r^2} f_2 = \frac{1}{k_2}\frac{\partial f_2}{\partial t} \qquad a \leq r < \infty \tag{2}$$

Let us prescribe the initial and boundary conditions as

$$v_1(r,0) = V_1 \quad , \quad t = 0 \qquad r < a \tag{3}$$

$$v_2(r,0) = V_2 \quad , \quad t = 0 \qquad r > a \tag{4}$$

$$f_1(a,t) = f_2(a,t) = Q \; , \quad t > 0 \qquad r = a \tag{5}$$

$$v_1(a,t) = v_2(a,t) \quad , \quad t > 0 \qquad r = a \tag{6}$$

In eqs. (5) and (6) it is assumed that there is no contact resistence at the surface of separation between the two materials.

For the region $0 \leq r \leq a$, multiplying both sides of eq. (1) by $J_1(r\xi_i)$ integrating over the section of the cylinder and using the property (A3) we obtain

$$\frac{d\, \bar{f}_1(\xi_i,t)}{dt} + k_1 a\xi_i f_1(a) J_1'(a\xi_i) + k_1\xi_i^2 \bar{f}_1(\xi_i,t) = 0 \tag{7}$$

Applying the inversion formulae (A1) in eq. (7) with boundary condition (5) we obtain the solution

$$f_1(r,t) = \frac{2A}{a^2}\sum_{i=1}^{\infty} e^{-k_1\xi_i^2 t}\frac{J_1(r\xi_i)}{[J'(a\xi_i)]^2} + \frac{2f(a)}{a}\sum_{i=1}^{\infty}\frac{1}{\xi_i}\frac{J_1(r\xi_i)}{J'(a\xi_i)}\left[e^{-k_1\xi_i^2 t} - 1\right] \tag{8}$$

where A is a integration constant.

Considering the initial condition (3) the temperature $v_1(r,t)$ is obtained from the relation

$$v_1(r,t) = -\frac{1}{K_1}\int f_1(r,t)\,dr \tag{9}$$

$$= V_1 \sum_{i=1}^{\infty} e^{-k_1\xi_i^2 t} + \frac{2f(a)}{aK_1}\sum_{i=1}^{\infty}\frac{1}{\xi_i^2}\frac{J_0(r\xi_i)}{J_1'(a\xi_i)}\left[e^{-k_1\xi_i^2 t} - 1\right]$$

For the region $r > a$, multiplyingeq. (2) by $r.Z_1(r\lambda)$, integrating with respect to r from a to ∞ and using the property (A5) we obtain

$$\frac{d\, f_2^*(\lambda,t)}{dt} + k_2\lambda^2\, f_2^*(\lambda,t) + \frac{2k_2}{\pi} f_2(a) = 0 \tag{10}$$

Applylng the inverslon formulae (A4) in eq. (10) with boundary condltlon (5) we obtain the following solution

$$f_2(r,t) = \int_0^{\infty} B\, e^{-k_2\lambda^2 t}\, F_1(\lambda,r)\lambda d\lambda + \int_0^{\infty} \frac{2f(a)}{\pi\lambda^2}\left(e^{-k_2\lambda^2 t} - 1\right) F_1(\lambda,r)\lambda d\lambda$$

where

$$F_1(\lambda,r) = \frac{J_1(\lambda,r)Y_1(\lambda,a) - Y_1(\lambda,r)J_1(\lambda,a)}{J_1^2(\lambda,a) + Y_1^2(\lambda,a)}$$

Considering the inltial condltlon (4), the temperature $v_2(r,t)$ Is obtalned from the relation

$$v_2(r,t) = -\frac{1}{K_2}\int f_2(r,t)\, dr = V\, \frac{\int_0^{\infty} e^{-k_2\lambda^2 t} F(\lambda,r)d\lambda}{\int_0^{\infty} F(\lambda,r)d\lambda} \tag{11}$$

$$\frac{2f(a)}{K_2\pi}\int_0^{\infty}\left(e^{-k_2\lambda^2 t} - 1\right) F(\lambda,r)\,\frac{d\lambda}{\lambda^2}$$

where

$$F(\lambda,r) = \frac{J_0(\lambda,r)Y_1(\lambda,a) - Y_0(\lambda,r)J_1(\lambda,a)}{J_1^2(\lambda,a) + Y_1^2(\lambda,a)}$$

3. VERIFICATION OF THE SOLUTIONS (9) AND (11)

At $t=0$, the solutions given by eqs. (9) and (11) obviously are satisfled

$$v_1(r,0) = V_1$$

$$v_2(r,0) = V_2$$

At $r=a$, from the solution (9) we obtain

$$- K_1 \frac{\partial v_1}{\partial r}\bigg|_{\substack{r=a\\t>0}} = \frac{2}{a} f_1(a) \sum_{i=1}^{\infty} \frac{1}{\xi_i} \frac{J_1(a\xi_i)}{J_1'(a\xi_i)} \left[e^{-k_1\xi_i^2 t} - 1\right]$$

Considering the result (A2) this expression is reduced to

$$- K_1 \frac{\partial v_1}{\partial r}\bigg|_{\substack{r=\\t>0}} = f_1(a)$$

Likewise, from the solution (11) we get

$$- K_2 \frac{\partial v_2}{\partial r}\bigg|_{\substack{r=a\\t>0}} = K_2 V_2 \frac{\int_0^\infty e^{-k_2\lambda^2 t} F_1(\lambda,a)\lambda d\lambda}{\int_0^\infty F_1(\lambda,a)\lambda d\lambda} + \frac{2f(a)}{\pi}\int_0^\infty \left[e^{-k_2\lambda^2 t} - 1\right] F_1(\lambda,a)\frac{d\lambda}{\lambda}$$

where

$$F_1(\lambda,a) = \frac{J_1(\lambda,a)Y_1(\lambda,a) - Y_1(\lambda,a)J_1(\lambda,a)}{J_1^2(\lambda,a) + Y_1^2(\lambda,a)}$$

Considering the expression (A6) we obtain

$$- K_2 \frac{\partial v_2}{\partial r}\bigg|_{\substack{r=a\\t>0}} = f_2(a)$$

The solutions (9) and (11) also satisfy the condition (6)

$$v_1(a,t) = v_2(a,t) \quad , \quad t > 0$$

This can be demonstrated using the directional derived definition of a function

$$dv = f(r).dr$$

At $r=a$ we have $dv = 0$, and then

$$v(a,t) = \text{const.}$$

Taking into account the condition (5) we conclude that

$$v_1(a,t) = v_2(a,t)$$

4. COMMENTS

From the solutions (9) and (11) we can observe that they are related by the parameter $Q = f_1(a) = f_2(a)$, from which we conclude that $v_1(r,t)$ and $v_2(r,t)$ are mutually dependent.

The method used here is useful to tackle other problems of heat conduction, like: I) $f(r,t) = 0$; ii) $f(r,t)$ = constant or a function of the time; iii) $f(r,t) = h.v$ (linear heat transfer at the surface).

APPENDIX

If $f(r)$ satisfies Dirichlet's conditions in the interval $(0,a)$ and if its finite transform in that range is defined as[6]

$$H_1\left[f(r);\xi_i\right] = \bar{f}(\xi_i) = \int_0^a r\, f(r) J_1(r,\xi_i)\, dr$$

which is called a finite Hankel transform of first kind; ξ_i are the positive roots of the transcendental equation

$$J_1(a,\xi_i) = 0$$

then at any point $(0,a)$ at which the function is continuous we have

$$H_1^{-1}\left[\bar{f}(\xi_i);r\right] = f(r) = \frac{2}{a^2}\sum_{i=1}^{\infty} \bar{f}(\xi_i)\,\frac{J_1(r,\xi_i)}{\left[J_1'(a,\xi_i)\right]^2} \tag{A1}$$

If $f(r) = r$, then

$$\bar{f}(\xi_i) = \frac{a^2}{\xi_i} J_2(a,\xi_i)$$

Applying the inversion formulae (A1), we get

$$-a = 2\sum_{i=1}^{\infty} \frac{1}{\xi_i}\,\frac{J_1(a,\xi_i)}{J_0(a,\xi_i)} \tag{A2}$$

Furthermore we have the following property[6]

$$H_1\left[\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{1}{r^2}f;\ \xi_i\right] = -a\,\xi_i\,f(a)J_1'(a,\xi_i) - \xi_i^2\,\bar{f}(\xi_i) \qquad \text{(A3)}$$

We shall denote the Weber transform of a function $f(r)$ as[5]

$$W_\nu[f(r)] = f^*(\lambda) = \int_a^\infty r\,f(r)\,Z_\nu(\lambda,r)\,dr$$

where

$$Z_\nu(\lambda,r) = J_\nu(\lambda,r)Y_\nu(\lambda,a) - Y_\nu(\lambda,r)J_\nu(\lambda,a)$$

J_ν and Y_ν are Bessel functions of first and second kind respectively of order ν and λ is a root of the equation

$$Z_\nu(\lambda,a) = 0$$

The inversion formulae for the Weber transform is

$$W_\nu^{-1}[f^*(\lambda)] = f(r) = \int_0^\infty f^*(\lambda)\,\frac{Z_\nu(\lambda,r)}{J_\nu^2(\lambda,a)+Y_\nu^2(\lambda,a)}\,\lambda d\lambda \qquad \text{(A4)}$$

We shall use the following property of this transform[3]

$$W_\nu\left[\frac{\partial^2 f}{\partial r^2}+\frac{1}{r}\frac{\partial f}{\partial r}-\frac{1}{r^2}f\right] = -\frac{2}{\pi}f(a) - \lambda^2 f^*(\lambda) \qquad \text{(A5)}$$

Finally we have ref.7, p.352 (16)

$$\int_0^\infty \frac{J_\nu(\lambda,r)Y_\nu(\lambda,a) - J_\nu(\lambda,a)Y_\nu(\lambda,r)}{\lambda\left[J_\nu^2(\lambda,a)+Y_\nu^2(\lambda,a)\right]}\,d\lambda = -\frac{\pi}{2}\left(\frac{a}{r}\right)^\nu \qquad 0 < a < r \qquad \text{(A6)}$$

In the eqs. (1) and (2) we have $\nu = 1$.

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Resumo

No presente artigo estudamos a distribuição da função da temperatura para o problema de um cilindro de raio a rodeado por um meio infinito de outro materlal. O fluxo de calor Q através da superfície r = a é mantido constante. As equações diferenciais do fluxo de calor são resolvidas pela utilização das transformadas finitas de Hankel e Weber, cujas propriedades fundamentais são todas no Apêndice.